



SZÁSZ-KANTOROVICH OPERATORS IN SYMMETRIC INTERVALS

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Abstract

In approximation theory, linear positive operators of general additive form are generally studied on the positive half-axis. In this study, the Szász–Kantorovich–Chlodovsky type operator is defined on intervals extending over the entire real axis. Certain parts of this work, which were previously presented at a conference, have been used in the present research only as needed. The approximation properties are examined by means of a new Lipschitz-type function [XX]. The operator is also studied using Peetre’s K-functional. Approximation and the rate of approximation in weighted spaces are investigated. Graphs of two different functions are plotted for two different values of b_s , and the corresponding numerical tables are prepared.

Keywords: Approximation, Szász -Mirakyan-Kantorovich Operators, Chlodovsky, weighted approximation, Symmetric Intervals, Voronovskaya type theorem, Lipschitz maximal function.

I. Introduction

Euler, one of the most important mathematicians of the 18th century, worked on minimizing distance errors in Russian maps (1777). Laplace’s 1843 work on determining the best-fitting ellipsoid for the Earth, as well as the contributions of many mathematicians up to Weierstrass, were all directed toward reducing approximation errors. Weierstrass brought this development to its culmination with a theorem stating that every function h continuous on a closed and bounded (compact) interval $[\alpha, \beta]$ can be approximated to any desired degree of accuracy by a sequence of polynomials on the same interval. In 1912, Bernstein [VII] introduced the polynomials that now bear his name and used the following sequence of polynomials to prove Weierstrass’ theorem:

$$B_s(h, u) = \sum_{j=0}^s \varphi_j^s(u) h\left(\frac{j}{s}\right) \quad (1.1)$$

where $\varphi_j^s(v) = \binom{s}{j} u^j (1-u)^{s-j}$, $u \in [0,1]$.

Let $X \subseteq \mathbb{R}$ and

$$B(X) = \{g: X \rightarrow \mathbb{R} / |g(t)| \leq K_g \text{ for all } t \in X\}$$

$$C(X) = \{g: X \rightarrow \mathbb{R} : g \text{ is continuous}\}$$

$$\hat{C}(X) = \{g: X \rightarrow \mathbb{R} : g \text{ is uniformly continuous}\}$$

$$C_b(X) = B(X) \cap C(X), \hat{C}_b(X) = B(X) \cap \hat{C}(X)$$

$$C_b^r(X) = \{g \in C_b(X) : g^{(r)} \in C_b(X), r = 0, 1, 2, 3, \dots\}$$

The standard norm of the space $B(X)$ is

$$\|g\|_b = \sup_{t \in X} |f(t)|.$$

In this case, K_g changes only depending on g . Bernstein polynomials are among the polynomials that have been most extensively generalized and modified. One of these modifications is due to Chlodovsky. In 1937, Chlodovsky [IX] modified the Bernstein polynomials on moving subintervals as follows:

$$C_s(h, u) = \sum_{j=0}^s \varphi_j^s\left(\frac{u}{b_s}\right) h\left(\frac{j}{b_s}\right) \quad (1.2)$$

where (b_s) is a sequence with positive terms and has the following properties

$$\lim_{s \rightarrow \infty} b_s = \infty \text{ ve } \lim_{s \rightarrow \infty} \frac{b_s}{s} = 0$$

Thus, when $s \rightarrow \infty$ the interval $[0, b_s]$ expands to $[0, \infty)$. Another modification is the sequence of Szász [XXII] polynomials defined on $[0, \infty)$. These are

$$S_r(h, u) = \sum_{j=0}^r q_j^r(u) h\left(\frac{j}{r}\right). \quad (1.3)$$

where $q_j^r(u) = e^{-ru} \frac{(ru)^j}{j!}$ and $u \in [0, \infty)$.

Similar to Kantorovich's [XVI] modifications of the Bernstein polynomials, Kantorovich-type versions of the Szász polynomials were also defined in [XXIV], and many researchers have subsequently studied them (see, for example, [XXIII, XIX, III, XVII, IV]).

Çilo [X] was the first to generalize Bernstein polynomials to a symmetric interval in 2012 (see also [XI]), and this generalization is given below,

$$C_r(h; u) = \frac{1}{2^r} \sum_{j=0}^r \binom{r}{j} (1+u)^j (1-u)^{r-j} h\left(2\frac{j}{r} - 1\right) \quad (1.4)$$

where $u \in [-1, 1]$.

Considering the studies mentioned above, in this work the Szász–Kantorovich operators are defined by the following operator sequence, which will be studied in detail.

$$R_s(h, u) = \frac{s}{b_s} \sum_{j=0}^{\infty} q_j^s \left(\frac{u}{b_s} + 1 \right) \int_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} h(v) dv \quad (1.5)$$

where $u \in [-b_s, \infty)$, $q_j^s \left(\frac{u}{b_s} + 1 \right) = e^{-s \left(\frac{u}{b_s} + 1 \right)} \frac{\left[s \left(\frac{u}{b_s} + 1 \right) \right]^j}{j!}$ [XXV, XXVI]. Throughout this study, the conditions set by Chlodovsky on (b_s) will be used as follows, and the compact subintervals of $[-b_s, b_s]$ that expand along the entire real axis will be used.

$$\lim_{s \rightarrow \infty} b_s = \infty \text{ ve } \lim_{s \rightarrow \infty} \frac{b_s^2}{s} = 0 \quad (1.6)$$

Different versions of Kantorovich-type polynomials are used in smoothing two- or three-dimensional images and in neural networks; only two articles are cited here as examples; there may be many more studies on this subject [IV, V].

II. Preliminaries

Let $e_j(\tau) = \tau^j$ ve $\theta_j(u) = (\tau - u)^j$, $j = 0, 1, 2, 3, \dots$

Lemma 2.1. [XXV, XXVI] The following equations hold for the integral in operator (1.5).

$$\begin{aligned} \text{i)} \quad & \int_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} dv = \frac{b_s}{s} \\ \text{ii)} \quad & \int_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} v dv = \frac{1}{2} v^2 I_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} = \frac{b_s}{2s} \frac{2j-2s+1}{s} b_s = \frac{b_s}{2s} \left(2 \left(\frac{j}{s} - 1 \right) b_s + \frac{b_s}{s} \right) \\ \text{iii)} \quad & \int_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} v^2 dv = \frac{v^3}{3} I_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} = \frac{b_s^3}{s^3} \left(j^2 + j(1-2s) + s^2 - s + \frac{1}{3} \right) \\ \text{iv)} \quad & \int_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} v^3 dv = \frac{v^4}{4} I_{\frac{j-s}{s}b_s}^{\frac{j-s+1}{s}b_s} = \frac{b_s^4}{4s^4} [4j^3 + (6 - 12s)j^2 + (12s^2 - 12s + 4)j] \end{aligned}$$

Lemma 2.2 [XXV, XXVI] The following equations are satisfied for (1.5)

$$\begin{aligned} \text{i)} \quad & R_s(e_0, u) = 1 \\ \text{ii)} \quad & R_s(e_1, u) = u + \frac{b_s}{2s} \\ \text{iii)} \quad & R_s(e_2, u) = u^2 + \frac{2b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2} \\ \text{iv)} \quad & R_s(e_3, u) = u^3 + \frac{9b_s u^2}{2s} + \frac{(6s+7)b_s^2 u}{2s^2} + \frac{(10s+1)b_s^3}{4s^3} \\ \text{v)} \quad & R_s(e_4, u) = u^4 + \frac{8b_s u^3}{s} + \frac{3(2s+5)b_s^2 u^2}{s^2} + \frac{2(8s+3)b_s^3 u}{s^3} + \frac{(15s^2+25s+1)b_s^4}{5s^4} \end{aligned}$$

Lemma 2.3. [XXV, XXVI] The following equations regarding the central moments of operator (1.5) are true.

$$\begin{aligned} \text{i)} \quad & R_s(\theta_0, u) = 1 \\ \text{ii)} \quad & R_s(\theta_1, u) = \frac{b_s}{2s} \\ \text{iii)} \quad & R_s(\theta_2, u) = \frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2} \end{aligned}$$

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$$\begin{aligned} \text{iv)} \quad R_s(\theta_4, u) &= \frac{-2b_s^2 u^2}{s^2} + \frac{(6s+15)b_s^3 u}{s^3} + \frac{(15s^2+25s+1)b_s^4}{5s^4} \\ \text{v)} \quad \frac{s}{b_s^2} R_s(\theta_2, u) &= \frac{u}{b_s} + \frac{3s+1}{3s} \\ \text{vi)} \quad \frac{s^2}{b_s^4} R_s(\theta_4, u) &= -\frac{2u^2}{b_s^2} + \frac{(6s+15)u}{sb_s} + \frac{(15s^2+25s+1)}{5s^2} \end{aligned}$$

The expression

$$\omega(h, \sigma) = \sup_{\substack{|u-t| \leq \sigma \\ u, t \in [a, b]}} |h(u) - h(t)| \quad (2.1)$$

is known as the first modulus of continuity.

The properties of the modulus of continuity are available in many sources, one of which is [II].

If the function h satisfies the condition

$$|h(t) - h(u)| \leq M|t - u|^\mu \quad (2.2)$$

for every $t, u \in [-A, A]$, then the function h is said to be of the Lipschitz class, and this class is denoted by $Lip^\mu_M([-A, A])$. Here A is a positive real number.

Ozarslan and Aktuglu [XX] gave a new definition of Lipschitz, which is given below.

$$|w(y) - w(t)| \leq C \frac{|y-t|^\mu}{(y+\sigma_1 t + \sigma_2 t^2)^{\mu/2}} \quad (2.3)$$

where $w \in C[0, \infty)$, $\sigma_1, \sigma_2, C > 0$ are any constants, $y, t \in [0, \infty)$, and $0 < \mu \leq 1$. The function class that satisfies this condition is denoted by $Lip_C^{\sigma_1, \sigma_2}(\mu)$.

1. The theory of approximation by linear positive operators was first independently established by Bohman [VIII] and Korovkin [XVIII]. The test functions appearing in the Bohman–Korovkin theorem will be computed, and the corresponding moments will be calculated using them.

Theorem 2.1 [XIII, XIV]. Let $(N_s)_{s=1}^\infty$ be a sequence of linear positive operators such that

$N: C[\alpha, \beta] \rightarrow B[\alpha, \beta]$. Here $h \in C[\alpha, \beta]$ satisfies the condition $|h(t)| \leq M$ for every $t \in \mathbb{R}$. If this sequence satisfies the following conditions, then the sequence $(N_s h)_{s=1}^\infty$ converges uniformly to the function h .

$$\begin{aligned} 1) \lim_{s \rightarrow \infty} N_s(1; u) &= 1, & 2) \lim_{s \rightarrow \infty} N_s(v; u) &= u, & 3) \lim_{s \rightarrow \infty} N_s(v^2; u) &= u^2 \\ &(\text{uniformly}) \end{aligned}$$

III. Main Results.

The following theorem will consider the smooth approximation of the operator (1.5) on the interval $[-A, A]$, where $A > 0$ is a fixed constant.

Theorem 3.1.[XXV] Let $h \in C_b[-A, A]$.

Then

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$$\lim_{s \rightarrow \infty} R_s(h, u) = h(u) \text{ (uniformly)}$$

is implemented correctly.

Proof.

$$\lim_{s \rightarrow \infty} \|R_s(e_0, u) - 1\|_{C[-A, A]} = \lim_{s \rightarrow \infty} \sup_{-A \leq u \leq A} |R_s(e_0, u) - 1| = 0$$

$$\lim_{s \rightarrow \infty} \|R_s(e_1, u) - u\|_{C[-A, A]} = \lim_{s \rightarrow \infty} \sup_{-A \leq u \leq A} |R_s(e_1, u) - u|$$

$$= \lim_{s \rightarrow \infty} \left| \frac{b_s}{2s} \right| = 0$$

$$\lim_{s \rightarrow \infty} \|R_s(e_2, u) - u^2\|_{C[-A, A]} = \lim_{s \rightarrow \infty} \sup_{-A \leq u \leq A} |R_s(e_2, u) - u^2|$$

$$= \lim_{s \rightarrow \infty} \sup_{-A \leq u \leq A} \left| \frac{2b_s u}{s} + \frac{b_s^2}{3s} \right|$$

$$= \lim_{s \rightarrow \infty} \left| \frac{2b_s A}{s} + \frac{b_s^2}{3s} \right| = \lim_{s \rightarrow \infty} \frac{b_s}{\sqrt{s}} \left(\frac{2A}{\sqrt{s}} + \frac{b_s}{3\sqrt{s}} \right) = 0$$

Considering the conditions on (b_s) and Theorem 2.1, the proof is finished.

Theorem 3.2. Let $h \in C_b[-A, A]$. Then the following inequality holds

$$|R_s(h; u) - h(u)| \leq (1 + \sqrt{3})\omega\left(h; \frac{b_s}{\sqrt{s}}\right)$$

Proof. Using the properties of the continuity modulus, we can write that,

$$|h(v) - h(u)| \leq \left(1 + \frac{|v - u|}{\delta_s}\right) \omega(h; \delta_s)$$

$$\begin{aligned} |R_s(h; u) - h(u)| &\leq |R_s(h(v) - h(u); u)| \\ &\leq R_s(|h(v) - h(u)|; u) \\ &\leq R_s\left(\left(1 + \frac{|v - u|}{\delta_s}\right) \omega(h; \delta_s); u\right) \\ &\leq \omega(h; \delta_s) \left\{R_s(1, u) + \frac{1}{\delta_s} R_s(|v - u|, u)\right\} \end{aligned}$$

when the Cauchy-Schwarz-Bunyakovsky inequality is used,

$$|R_s(h; u) - h(u)| \leq \omega(h; \delta_s) \left\{1 + \frac{1}{\delta_s} \sqrt{R_s((v - u)^2, u)}\right\}$$

Since Lemma 2.3(iii) and $u \in [-b_s, b_s]$,

$$u \leq b_s \Rightarrow \frac{b_s u}{s} \leq \frac{b_s^2}{s} \Rightarrow \frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2} \leq \frac{(6s+1)b_s^2}{3s^2} \leq \frac{9sb_s^2}{3s^2} = \frac{3b_s^2}{s}. \quad (*)$$

(The expression $(*)$ is an upper bound for the second moment.)

Thus, by using the properties of the modulus of continuity, we get

$$\begin{aligned} |R_s(h; u) - h(u)| &\leq \omega(h; \delta_s) \left\{ 1 + \frac{1}{\delta_s} \sqrt{\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2}} \right\} \\ &\leq \omega(h; \delta_s) \left\{ 1 + \frac{1}{\delta_s} \sqrt{\frac{3b_s^2}{s}} \right\}. \end{aligned}$$

The proof ends by choosing $\delta_s = \frac{b_s}{\sqrt{s}}$.

Theorem. 3.3. If $h \in Lip^\mu_M([-b_s, b_s])$ then

$$|R_s(h, u) - h(u)| \leq 3^{\mu/2} M \left(\frac{b_s^2}{s} \right)^{\frac{\mu}{2}}$$

Proof. Since $R_s(h, u)$ is a positive linear operator, combining (2.2), the Cauchy–Schwarz inequality, and Lemma 2.3 (iii), we obtain

$$\begin{aligned} |R_s(h; u) - h(u)| &\leq R_s(|h(v) - h(u)|; u) \\ &\leq R_s(M|v - u|^\mu; u) \\ &\leq M[R_s((v - u)^2; u)]^{\mu/2} \\ &\leq M \left[\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2} \right]^{\frac{\mu}{2}} \end{aligned}$$

Since $u \in [-b_s, b_s]$

$$|R_s(h; u) - h(u)| \leq 3^{\mu/2} M \left(\frac{b_s^2}{s} \right)^{\frac{\mu}{2}}$$

Theorem. 3.4. Let $h \in Lip_C^{\sigma_1, \sigma_2}(\mu)$ and $0 \neq u \in [-b_s, b_s]$ Then

$$|R_s(h; u) - h(u)| \leq C \left(\frac{\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2}}{\sigma_1 u + \sigma_2 u^2} \right)^{\frac{\mu}{2}}$$

Proof. Let's first prove it for $\mu = 1$.

Let $u \neq 0$ and $u \neq -\frac{\sigma_1}{\sigma_2}$ then $\frac{|v-u|}{(v+\sigma_1 u+\sigma_2 u^2)^{\frac{1}{2}}} < \frac{|v-u|}{(\sigma_1 t+\sigma_2 t^2)^{\frac{1}{2}}}$.

Thus, by Lemma 2.3(iii) and the Cauchy–Schwarz–Bunyakovsky inequality, we have

$$\begin{aligned} |R_s(h; u) - h(u)| &\leq R_s(|h(v) - h(u)|, u) \\ &\leq C R_s \left(\frac{|v-u|^\mu}{(v+\sigma_1 u+\sigma_2 u^2)^{\mu/2}}, u \right) \\ &\leq C R_s \left(\frac{|v-u|}{(\sigma_1 u+\sigma_2 u^2)^{1/2}}, u \right) \end{aligned}$$

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$$\begin{aligned} &\leq \frac{C}{(\sigma_1 u + \sigma_2 u^2)^{1/2}} R_s(|v - u|, u) \\ &\leq \frac{C}{(\sigma_1 u + \sigma_2 u^2)^{1/2}} (R_s((v - u)^2, u))^{1/2} \\ &\leq C \left(\frac{\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2}}{\sigma_1 u + \sigma_2 u^2} \right)^{1/2} \end{aligned}$$

Now let $0 < \mu < 1$. When Hölder's inequality is used by taking $p = \frac{2}{\mu}$ and $p = \frac{2}{2-\mu}$

$$\begin{aligned} |R_s(h, u) - h(u)| &\leq C (R_s(|h(v) - h(u)|^{\frac{2}{\mu}}, u))^{\frac{\mu}{2}} \\ &\leq C R_s \left(\left(\frac{|v-u|^\mu}{(v + \sigma_1 u + \sigma_2 u^2)^{\frac{\mu}{2}}} \right)^{\frac{2}{\mu}}, u \right)^{\frac{\mu}{2}} \\ &= C (R_s \left(\frac{|v-u|^2}{(v + \sigma_1 u + \sigma_2 u^2)}, u \right))^{\frac{\mu}{2}} \\ &\leq C \left(M_r^* \left(\frac{(v-u)^2}{\sigma_1 u + \sigma_2 u^2}, t \right) \right)^{\frac{\mu}{2}} \leq C \left(\frac{\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2}}{\sigma_1 u + \sigma_2 u^2} \right)^{\frac{\mu}{2}} \end{aligned}$$

Thus, the proof is finished.

The second modulus of continuity is given as follows,

$$\omega_2(h; \sigma) = \max_{\substack{v, v+2h \in A \\ |\theta| \leq \sigma}} |h(u + 2\theta) - 2h(u + \theta) + h(u)| \quad (3.1)$$

The K-Peetre's functional definition is given by

$$K_2(h; \sigma) = \inf_{g \in C_b^2(-\infty, \infty)} \left\{ \|h - g\|_{C_b(-\infty, \infty)} + \sigma \|g''\|_{C(-\infty, \infty)} \right\} \quad (3.2)$$

by J. Peetre in 1968 [XXI] where $g', g'' \in C_b(-\infty, \infty)$. There exists at least one constant C such that the inequality $K_2(h; \sigma) \leq C \omega_2(h; \sqrt{\sigma})$ holds [XII, Theorem 3.1.2]

Theorem 3.5. Let $h \in \hat{C}[-b_s, \infty)$, then for $u \in [-b_s, \infty)$, the following inequality is provided:

$$|R_s(h, u) - h(u)| \leq 4C \omega_2 \left(h; \frac{b_s}{\sqrt{s}} \right) + \omega \left(h; \frac{b_s}{2s} \right).$$

Proof. Through Taylor's series,

$$g(v) - g(u) = (v - u)g'(u) + \int_u^v (v - z)g''(z)dz \quad (3.3)$$

By applying the operator $R_s(h, u)$ to the equality (3.3), we get

$R_s(y - t, t) = \frac{b_s}{2s} \neq 0$ and Let's assign an auxiliary R_s operator from $R_s^*(h, t)$ in this way,

$$R_s^*(h, u) = R_s(h, u) - h(R_s(v, u)) + h(u) \quad (3.4)$$

By applying $R_s^*(h, u)$ to both sides of (3.3), we get

$$R_s^*(g, u) - g(u) = R_s(v - u, u)g'(u) + R_s^*\left(\int_u^v (v - z)g''(z)dz, u\right)$$

Considering (3.4)

$$\begin{aligned} R_s^*(g, u) - g(u) &= R_s\left(\int_u^v (v - z)g''(z)dz, u\right) \\ &\quad - \left(\int_u^{R_s(v, u)} (R_s(v, u) - t)g''(z)dz, u\right) + \int_u^u (u - z)g''(z)dz \\ |R_s^*(g, u) - gu| &\leq \left|R_s\left(\int_u^v (v - z)g''(z)dz, u\right)\right| + \left|\int_u^{R_s(v, u)} (R_s(v, u) - z)g''(z)dz\right| \\ &\leq R_s\left(\int_u^v |v - z||g''(z)|dz, u\right) + \int_u^{R_s(v, u)} |R_s(v, u) - z||g''(z)|dz \\ &\leq \frac{\|g''\|_b}{2} (R_s((v - u)^2, u) + (M_r^*(v, u))^2) \end{aligned}$$

Using Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} |R_s^*(g, u) - g(t)| &\leq \frac{\|g''\|_b}{2} \left(\left(\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2}\right) + \left(\frac{b_s u}{s}\right)^2 \right) \\ &\leq \frac{\|g''\|_b}{2} \left(\frac{(9s+1)b_s^2}{3s^2} + \left(\frac{b_s^2}{s}\right)^2 \right) \\ &\leq \frac{\|g''\|_b}{2} \left(\frac{b_s^2}{s} \frac{3b_s^2 + (9s+1)}{3s} \right) \end{aligned}$$

Since $\frac{b_s^2}{s} \rightarrow 0$ as $s \rightarrow \infty$, it follows that $\frac{3b_s^2 + (9s+1)}{3s} \leq 5$, then

$$|R_s^*(g, u) - g(t)| \leq \frac{5\|g''\|_b}{2} \frac{b_s^2}{s} \quad (3.5)$$

From equality (3.4) and $h \in C_b[-b_s, \infty)$, we have

$$|R_s^*(h, u)| \leq |R_s(h, u)| + \left|h\left(\frac{b_s}{2s}\right)\right| + |h(u)| \leq 3\|h\|_b \quad (3.6)$$

Considering equality (3.4)

$$\begin{aligned} |R_s(h, u) - h(u)| &\leq |R_s^*(h - g, u) - (h - g)(u)| + |R_s^*(h, u) - h(u)| \\ &\quad + \left|h\left(\frac{b_s}{2s}\right) - h(u)\right| \end{aligned} \quad (3.7)$$

When (3.1) and (3.4-3.7) are considered together, we obtain

$$|R_s(h, u) - h(u)| \leq 4\|h - g\|_b + \frac{5\|g''\|_b b_s^2}{2s} + \omega\left(h; \frac{b_s}{2s}\right)$$

If we take the infimum over g on the right-hand side

$$\begin{aligned} |R_s(h, u) - h(u)| &\leq 4K_2\left(h; \frac{b_s^2}{s}\right) + \omega\left(h; \frac{b_s}{2s}\right) \\ &\leq 4C\omega_2\left(h; \frac{b_s}{\sqrt{s}}\right) + \omega\left(h; \frac{b_s}{2s}\right). \end{aligned}$$

Thus the proof ends.

IV. Asymptotic Approximation.

When studying approximation by linear positive operators, the goal is to determine the functional expression, in terms of the derivatives of the function, that describes the asymptotic behaviour of the error.

In this section, we will calculate the limit of the difference between the function and its image under the operator, multiplied by a sequence that tends to infinity:

$R_s(h, u) - h(u)$ for the function $h \in C^2(X)$.

Theorem 4.1. Let $h \in C^2([-b_s, b_s])$, then the following equation holds

$$\lim_{s \rightarrow \infty} \frac{s}{b_s} \{R_s(h, u) - h(u)\} = \frac{1}{2!} h''(u)$$

Proof. We will use the Taylor expansion given below.

$$h(v) = h(u) + (v - u)h'(u) + \frac{1}{2}(v - u)^2 h''(u) + K_m(v, u)(v - u)^2$$

$$h(v) - h(u) = (v - u)h'(u) + \frac{1}{2}(v - u)^2 h''(u) + K_m(v, u)(v - u)^2. (*)$$

Here, K_m is the remaining term, and it is continuous,

$$K_m(v, u) = \frac{1}{3!} h'''(u)(v - u)^3 + \frac{1}{4!} h^{(4)}(u)(v - u)^4 + \dots$$

also $\lim_{v \rightarrow u} K_m(v, u) = 0$. When the R_s operator is applied to the (*) expression

$$\begin{aligned} R_s(h(v); u) - h(u) &= R_s((v - u); u)h'(u) + \frac{1}{2!} R_s((v - u)^2; u) h''(u) \\ &\quad + R_s(K_m(v, u)(v - u)^2, u) \\ &= \frac{b_s}{2s} h'(u) + \frac{1}{2} \left(\frac{b_s u}{s} + \frac{(3s+1)b_s^2}{3s^2} \right) h''(u) \\ &\quad + R_s(K_m(v, u)(v - u)^2, u) \\ \frac{s}{b_s^2} \{R_s(h(v); u) - h(u)\} &= \frac{1}{2b_s} h'(u) + \frac{1}{2} \left(\frac{u}{b_s} + \frac{(3s+1)}{3s} \right) h''(u) \\ &\quad + \frac{s}{b_s^2} R_s(K_m(v, u)(v - u)^2, u) \end{aligned}$$

If we take the limit of both sides of the above equation

$$\begin{aligned}\lim_{s \rightarrow \infty} \frac{s}{b_s^2} \{R_s(h(t); u) - h(u)\} &= \frac{1}{2!} h''(u) + \lim_{s \rightarrow \infty} \frac{s}{b_s^2} R_m(K_m(v, u)(v - u)^2, u) \\ \lim_{s \rightarrow \infty} \frac{s}{b_s^2} R_s(K_m(v, u)(v - u)^2) &\leq \lim_{s \rightarrow \infty} \sqrt{R_s(K_m^2(v, u))} \sqrt{\frac{s^2}{b_s^4} R_s((v - u)^4, u)} \\ &\leq \lim_{s \rightarrow \infty} \sqrt{R_s(K_m^2(v, u))} \sqrt{-\frac{2}{b_s^2} u^2 + \frac{(6s+15)}{sb_s} u + \frac{(15s^2+25s+1)}{5s^2}} \\ &\leq \sqrt{3} \lim_{s \rightarrow \infty} \sqrt{R_s(K_m^2(v, u))}\end{aligned}$$

Since $K_m(v, u) \rightarrow 0$ when $v \rightarrow u$, the result of the limit is zero.

V. Approximation in Weighted Space

In spaces of bounded functions, approximation by linear positive operators can be studied relatively easily on compact intervals. In contrast, when the underlying function space consists of functions that may grow at infinity (such spaces are called weighted spaces, and the controlling function is called the weight function), a different norm is required that is defined over the whole real line.

The weight function plays a role similar to the heavy concrete counterweights used at one end of a tall crane to balance varying loads.

Let

$\emptyset: \mathbb{R} \rightarrow \mathbb{R}^+, \emptyset(t) \geq 1, \lim_{|t| \rightarrow \infty} \emptyset(t) = \infty, g: \mathbb{R} \rightarrow \mathbb{R}$ and $|h(u)| \leq F_g \emptyset(u)$ where F_h is a constant that varies depending on h .

$B_\emptyset(-\infty, \infty) = \{h: |h(u)| \leq K_h \emptyset(u), h \text{ is a function on } (-\infty, \infty)\}$

$C(-\infty, \infty) = \{h: h \text{ is a continuous function on } (-\infty, \infty)\}$

$C_\emptyset(-\infty, \infty) = C(-\infty, \infty) \cap B_\emptyset(-\infty, \infty), C_\emptyset^k[-b_s, \infty) = \left\{h \in C_\emptyset: \lim_{|x| \rightarrow \infty} \frac{|h(x)|}{\emptyset(x)} = K_h\right\}$.

These function classes are used with the $\|h\|_\emptyset = \sup_{u \in (-\infty, \infty)} \frac{|h(u)|}{\emptyset(u)}$ norm.

Here, the weight function $\emptyset(u) = 1 + u^2$ given in Gadjev's [XIII],[XIV] work and Korovkin's theorem will be used.

Lemma 6.1. Let (R_s) be a sequence of linear positive operators that maps $C_\emptyset[-b_s, b_s]$ into $B_\emptyset[-b_s, b_s]$

Proof. It is enough to show $\lim_{i \rightarrow \infty} \|R_s(\emptyset(v), u)\|_{\emptyset, [-b_s, b_s]} = \emptyset(u)$, (uniformly) here.

$$\begin{aligned}R_s(\emptyset(v), u) &= 1 + u^2 + \frac{2b_s}{s} u + \frac{(3s+1)b_s^2}{3s^2} \\ &= \emptyset(u) + \frac{2b_s}{s} u + \frac{(3s+1)b_s^2}{3s^2}.\end{aligned}$$

Since

$$\lim_{s \rightarrow \infty} \sup_{u \in [-b_s, b_s]} \left(\frac{2b_s}{s} u + \frac{(3s+1)b_s^2}{3s^2} \right) = \lim_{r \rightarrow \infty} \left(\frac{(9s+1)b_s^2}{3s^2} \right) = 0,$$

then the proof is finished.

Theorem 6.1 Let $h \in C_\emptyset^k[b_s, \infty)$. Then $\lim_{s \rightarrow \infty} \|R_s(h, u) - u\|_{\emptyset, [-b_s, b_s]} = 0$.

Proof. It is sufficient to show that the conditions of the Korovkin theorem are satisfied in the weighted spaces given in [XIII–XIV].

The first condition is obvious. We now verify the second and third conditions.

$$\begin{aligned} \lim_{s \rightarrow \infty} \|R_s(e_1(v), u) - u\|_{\emptyset, [-b_s, b_s]} &= \lim_{s \rightarrow \infty} \sup_{u \in [-b_s, b_s]} \left| \frac{b_s}{2s} \right| \\ &= \lim_{r \rightarrow \infty} \frac{b_s}{2s} = 0. \end{aligned}$$

$$\lim_{s \rightarrow \infty} \|R_s(e_2(v), u) - u^2\|_{\emptyset, [-b_s, b_s]} = \lim_{s \rightarrow \infty} \sup_{u \in [-b_s, b_s]} \left| \frac{5b_s^2}{s} + \frac{b_s^2}{3s^2} \right| = 0$$

The proof ends.

Various moduli of continuity have been defined for weighted spaces. In this study, the following moduli of continuity given in [XV] will be used.

$$\varpi(h, \sigma_s) = \sup_{\substack{|\mu| \leq \sigma_s \\ u \in [0, b_s]}} \frac{|h(u+\mu) - h(u)|}{(1+\mu^2)(1+u^2)} \quad (5.1)$$

(see also [XIV]). Some properties of $\varpi(h, \sigma)$ can be found in [XXI]:

$$\lim_{r \rightarrow \infty} \varpi(h, \sigma_s) = 0 \text{ and}$$

$$|h(v) - h(u)| \leq 2(1 + \sigma_s^2)(1 + u^2)\varpi(h, \sigma_s)\gamma_s(v, u)$$

$$\text{Where } \gamma_r(v, t) = \left(1 + \frac{|v-t|}{\sigma_r}\right)(1 + (v-t)^2)$$

$$\gamma_s(v, u) \leq \begin{cases} 2(1 + \sigma_s^2), & |v - u| \leq \sigma_s \\ 2(1 + \sigma_s^2) \frac{(y-u)^4}{\sigma_s^4}, & |v - u| \geq \sigma_s \end{cases} \quad (5.2)$$

Theorem 6.2 Let $h \in C_\emptyset^k[-b_s, \infty)$. Then, for all sufficiently large s 's

$$\|R_s(h, u) - h(u)\|_{\emptyset, [-b_s, b_s]} \leq 496\varpi\left(h, \sqrt{\frac{b_s^2}{s}}\right)$$

$$\text{Proof. } |R_s(h, u) - h(u)| \leq R_s(|h(y) - h(u)|, u)$$

$$\leq 2(1 + \sigma_s^2)(1 + u^2)\varpi(h, \sigma_s)R_s(\gamma_s(v, u), u).$$

Considering (5.2) $\gamma_s(v, u) \leq 2(1 + \sigma_r^2)[1 + \frac{(v-u)^4}{\sigma_t^4}]$, then for all $v, u \in [-b_s, b_s]$ we have

$$|R_s(h, u) - h(u)| \leq 4(1 + \sigma_s^2)(1 + u^2)R_s(1 + \frac{(v-u)^4}{\sigma_s^4}, u)\varpi(h, \sigma_s)$$

$$\leq 4(1 + \sigma_s^2)(1 + u^2)[1 + \frac{61\frac{b_s^4}{s^2}}{\sigma_s^4}] \varpi(h, \sigma_s)$$

$$\frac{|R_s(h, u) - h(u)|}{(1 + u^2)} \leq 4(1 + \sigma_s^2)[1 + \frac{61\frac{b_s^4}{s^2}}{\sigma_s^4}] \varpi(h, \sigma_s)$$

If $\sigma_s = \sqrt{\frac{b_s^2}{s}}$ is chosen, then $\|R_s(h, u) - h\|_{\emptyset, [-b_s, b_s]} \leq 496\varpi\left(h, \sqrt{\frac{b_s^2}{s}}\right)$

VI. Graphs and Numerical Value Tables

In this section, graphs and the corresponding numerical tables are provided for different functions and various values of b_s

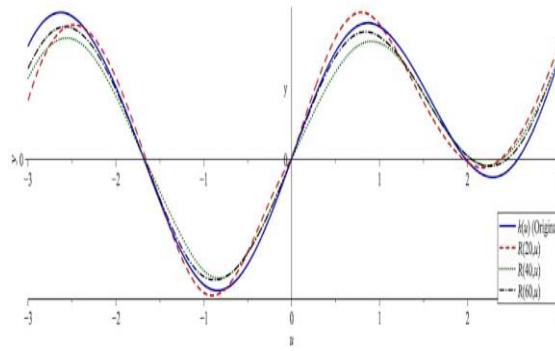


Fig. 1. $\left(u + \frac{3}{2}\right)(u - 2)\sin \frac{\pi u}{4}$, $b_s = \ln(1 + s)$

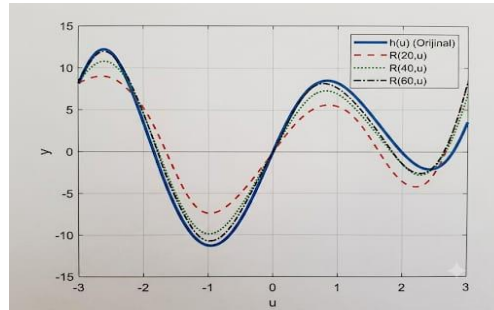


Fig. 2. $h(u) = \left(u + \frac{3}{2}\right)(u - 2)\sin \frac{\pi u}{4}$, $b_s = s^{1/3}$

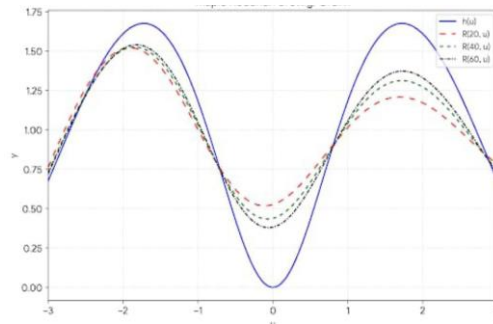


Fig. 3. $h(u) = \ln(1 + u^2)(1 + \cos(\frac{\pi u}{4}))$, $b_s = \ln(1 + s)$

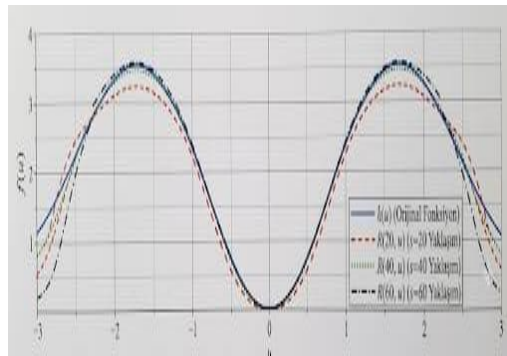


Fig. 4. $h(u) = \ln(1 + u^2)(1 + \cos(\frac{\pi u}{4}))$, $b_s = s^{1/3}$

In the graphs, the function h is plotted in blue, the operator for $s = 20$ is shown in red, for 40 in green, and for $s = 60$ in black.

The following tables have been prepared for $|R_s(h, u) - h(u)|$

Table 1: for Figure 2

s	$u = -2.5$	$u = 1$		s	$u = -2.7$	$u = 0.8$
10^2	0.2664593280	0.4791972770		10^2	0.3213014950	0.3908989020
10^3	0.0734156990	0.1102386910		10^3	0.2323953950	0.1795250140
10^4	0.0141308830	0.0193811630		10^4	0.1261454720	0.0830747330
10^5	0.0023015310	0.0029834410		10^5	0.0624358990	0.0385237410
10^6	0.0003020124	0.0004245390		10^6	0.0000000003	0.0000000001

Table 2: for Figure 2

Table 3: for Figure 3

s	$u = -1.5$	$u = 1.7$		s	$u = -1.7$	$u = 0$
10^2	0.1394054426	0.2344686058		10^2	0.1220869738	0.3145502026
10^3	0.0371720596	0.0532680208		10^3	0.0744788268	0.1694120114
10^4	0.0070782906	0.0091848868		10^4	0.0387853638	0.0854047176
10^5	0.0012405962	0.0010812630		10^5	0.0001428571	0.0000851234
10^6	0.0003972610	0.0003427381		10^6	0.0000307142	0.0000183205

Table 4 : for Figure 4

VII. Conclusion and Remarks.

In this study, the Szász–Chlodovsky–Kantorovich type operators were first discussed in the master’s thesis of Yılmaz [XXVI] on a symmetric interval covering the entire real axis. Some parts of that work were presented at a conference [XXV]. Korovkin-type approximation was considered on a symmetric compact interval, and the rate of approximation was calculated using the modulus of continuity on the same interval. The second modulus of continuity and Peetre’s K-functional were also employed in the investigation. A Voronovskaya-type approximation theorem was obtained. The approximation properties were examined separately for functions satisfying the classical Lipschitz condition and a modified Lipschitz condition. Graphs of two different functions were plotted for different values of b_s on a compact interval, and the corresponding numerical tables were prepared. Finally, the approximation and the rate of approximation were investigated in a weighted space.

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Conflict of Interest

There is no conflict of interest among the authors.

Reference

- I. Akhieser, N.I. Akhieser, Lectures on the theory of approximation. OGIZ, Moscow-Leningrad, 1947 (in Russian), Theory of approximation (in English), Translated by C.J. Hyman, Frederick Ungar Publishing Co., New York, 1956.
- II. Aslan, R., Izgi, A. q -Szász-Kantorovich-Chlodowsky Operatörlerinin Yaklaşımlar, Erciyes Üniversitesi Fen Bilimleri Enstitüsü Dergisi 36(1), (2020), 138-150.
- III. Aslan, R. (2014). Ağırlıklı Uzaylarda Kantorovich-Chlodowsky-Szász Tipi Operatörlerin Yaklaşımı ve Yaklaşım Hızı. Harran Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi, 29s, Şanlıurfa
- IV. Altomare, F. and Campiti, M. Korovkin-type Approximation Theory and its Applications. Berlin: New York: de Gruyter, (1994).

- V. Bawa, P., BhardwaN. j & Bhatia, S. K. (2023). Different Variants of Bernstein Kantorovich Operators and Their Applications in Sciences and Engineering Field. *Chaos Theory and Applications*, 5(4), 293-299, 10.51537/chaos.1320442
- VI. Bernstein, S.N. Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités, *Commun. Kharkov Math. Soc.* 13 (1912/1913) 1–2.
- VII. Bajpeyi, S., Baxhaku, B. & Agrawal, P. N. Exponential Sampling Type Kantorovich Max-Product Neural Network Operators. *Numerical Functional Analysis and Optimization*, 45(13–14), (2024) 631–650..
- VIII. Bohman, H. On approximation of continuous and of analytic functions, *Ark. Mat.* 2 (1) (1952) 43–56.
- IX. Chlodowsky, I. Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de M. S. Bernstein. *Compositio Mathematica*, 4, (1937).380–393.
- X. Çilo, A.Approximation properties and speed of approximation of Bernstein polynomials in the interval $[-1,1]$, Harran University, Institute of Science,(2012), (In Turkish)
- XI. Çilo A.,A.Ural , İzgi, A Properties of Approximation and Rate of Approximation Bernstein Polynomials in the range $[-1, 1]$, Conference: XXV. Ulusal Matematik Sempozyumu - Niğde Üniversitesi, (5-8 Eylül(September),2012).
- XII. Ditzian, Z.; Totik, V. *Moduli of Smoothness*; Springer Series in Computational Mathematics; Springer: New York, NY, USA, 1987; Volume 9.
- XIII. Gadzhiev, A.D. The convergence problem for a sequence of linear operators on unbounded sets and theorem analogous to that of P.P. Korovkin. *Soviet Math. Dokl.* 15, no. 5 (1974), 1433-14.
- XIV. Gadzhiev, A.D. Theorems of the type P.P. Korovkin theorems.*Math. Zametki* 20, no. 5 (1976) , 781-786. English translation in *Math. Notes* 20, no. 5-6 (1976), 996-998.
- XV. Ispir, N. On modi ed Baskakov operators on weighted spaces. *Turk. J.Math.* 26, no. 3 (2001), 355-365.
- XVI. Kantorovich, L.V. Sur certains développements suivant les polynômes de la forme de s. Bernstein, I, II, *CR Acad. URSS* 563(568): (1930)595–600.
- XVII. Karaisa, A. ,Aral, A. 2016. Some Approximation Properties of Kantorovich Variant of Chlodowsky Operators Based on q -integer, 65,97-119 Ankara

- XXVIII. Korovkin, P.P. On convergence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR 90 (1953) 961–964.
- XIX. Mursaleen, M., Alotaibi, A., Ansari, K.J.: On a Kantorovich variant of Szász-Mirakjan operators. J. Funct. Spaces **2016**(1035253), 9 (2016). 10.1155/2016/1035253
- XX. Ozarslan ,M. A. And Aktuglu, H. Local approximation for certain King type operators, Filo mat. 27 (2013) 173–181.
- XXI. Peetre, J. A Theory of Interpolation of Normed Spaces; Notas de Matemática; Instituto de Matemática Pura e Aplicada (IMPA): Rio de Janeiro, Brazil, 1963; Volume 39, pp. 1–86.
- XXII. Szász, O. Generalization of S. Bernstein’s polynomials to the infinite interval, J. Res. Nat. Bur. of Standards 45 (1950), 239-245.
- XXIII. Taşdelen ,F.F. ; Aktaş, R, Altın, A.: Kantorovich type of Szász operators including Brenke-type polynomials. In: Abstract and Applied Analysis, vol. 2012. Hindawi (2012)
- XXIV. Totik, V. Approximation by Szász-Mirakjan-Kantorovich operators in $L_p(p \geq 1)$. Anal. Math. **9**(2), (1983)147–67.
- XXV. Yılmaz, R., İzgi, A. Simetrik Aralıkta Szász Kantorovich Operatörlerinin Yaklaşım Özellikleri (Approximation Properties of Szász -Kantorovich Operators in Symmetric Intervals) V. INTERNATIONAL FIRAT SCIENTIFIC RESEARCH CONGRESS December 5-7 ,2025 ELAZIG,TURKIYE, (CONFERENCE BOOK).
- XXVI. Yılmaz. R, Simetrik Aaralıkta Szász -Kantorovich Operatörlerinin Yaklaşım Özellikleri (Approximation Properties Of Szász -Kantorovich Operators In A Symmetrical Interval), Harran Üniversitesi Lisansüstü Eğitim Enstitüsü , 2026.(56 page)