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A TWO-WAREHOUSE EOQ MODEL FOR NON-INSTANTANEOUS DECAYING PRODUCTS WITH CONSTANT DEMAND UNDER PRESERVATION TECHNOLOGY AND SHORTAGES

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Abstract

Inventory management is an integral part of a country's economic development. Many goods are perishable and gradually worsen over time. Therefore, degradation has a negative effect on perishable goods. If deterioration is not managed properly, it leads to significant inventory loss. Therefore, we propose an EOQ model that incorporates two warehouses and preservation technology to reduce total inventory loss. This proposed model has a constant demand rate with a linearly time-dependent deterioration rate and shortages that are completely backlogged. Goods are first placed in the owned warehouse, and when it is full, the remaining goods are placed in a rented warehouse. For both warehouses, the holding cost is taken as constant. A numerical case and sensitivity analysis table are used to validate the model and examine the behavior of the model under varying parameters.

Keywords: Two warehouses, Preservation technology, deterioration, shortages with complete backlogging.

I. Introduction

No country can achieve economic growth without businesses. Every country wants to strengthen its economy by having good relations with other countries. Without business, the growth of the economy of any nation is unimaginable. In the present high-competition environment, every business firm wants to improve its growth and sustainability. So, every business organization maintains its inventory and fosters a healthy relationship with customers. An analytical approach that is primarily used in inventory management to identify the ideal order quantity and to minimise inventory cost is known as the economic order quantity (EOQ) model. A two-warehouse system is where the organization keeps its inventory in two separate warehouses. Generally, one warehouse is called the owned warehouse, and the other warehouse is known as the rented warehouse.

Harris [V] was the first researcher who developed an inventory model for a single warehouse. After Harris, Hartley [VII] first introduced a two-warehouse inventory model. He considered constant demand without shortages. Goswami and Chaudhuri [IV] proposed a model in which deterioration was not considered, and shortages were allowed. A dual-warehouse EOQ model for decaying products was introduced by Bansal et al. [I]. The proposed model considered the deterioration behavior using a Weibull distribution, and shortages were permitted. Garg et al. [III] presented a two-warehouse EOQ model. According to this model, the deterioration rate was considered constant in both warehouses. The demand was considered ramp-type demand with partial backlogging in the model. Nath and Sen [IX] designed a model based on two storages for non-immediate spoilage items. In their inventory model, demand varying with time and selling price was considered, while the deterioration rate was assumed to be constant. Nath and Sen [X] further developed a partially backlogged two-warehouse EOQ model for non-instantaneously deteriorating items. Their model considered demand dependent on both selling price and time, along with preservation technology and interval-number inventory parameters. Rathore et al. [XII] considered a model for decaying products under a sustainability preservation technique. In their proposed model, demand depended on advertisement and credit terms. Kumar et al. [VIII] proposed a two-warehouse EOQ model for deteriorating items incorporating preservation technology. The model assumed time-dependent demand and shortages were not allowed. Pathak et al. [XI] presented a model based on a dual-warehouse model for deteriorating products. The deteriorated items were considered shelf-life stock. In this model, demand was considered a time-varying bi-quadratic demand. This inventory model consisted of shortages and inflation.

Researchers have developed many inventory models considering different factors such as demand rate, deterioration rate, shortages, and inventory-related costs. These models have been formulated to help decision-making in business organisations. In a recent review, we have observed that very few inventory models have been developed for two-warehouse inventory systems incorporating stochastic demand, composite demand, decaying rate, and preservation techniques. Therefore, this study attempts to address this research gap.

This model is formulated as follows: Section 1 provides a brief introduction to the current study and reviews previous studies. Section 2 explains the assumptions and notation used in the mathematical model and analysis table. Section 3 describes the mathematical model formulation, including the differential equation formulation and its solution. Section 4 explains the methodology for obtaining the different types of costs included in this model. Section 5 presents the method for determining the optimal cost. Section 6 discusses the solution procedure. Section 7 presents a numerical illustration of the model. Section 8 presents an analysis table. ection 9 presents the observations from the sensitivity analysis. Section 10 includes the conclusion of this model and future work. Future work describes that this type of paper can be extended in many dimensions using different types of deterioration rate, demand rate, holding cost, statistical function, and so on. Section 11 contains the list of all the research papers on two-warehouse systems that have been used as references in preparing this research paper.

II. Assumptions and Notation:

II.i. Assumption:

- This model considers a single type of deteriorating item.
- No lead time.
- Restocking occurs without delay.
- Demand rate does not vary in both warehouses.
- The capacity of both warehouses is limited.
- Due to economic reasons, initially items are to be consumed from the rented warehouse (RW), and when the rented warehouse becomes empty, then items are to be consumed from the owned warehouse (OW).
- Deterioration rate begins after a specific duration. Both warehouses have the same rate of deterioration.
- A rented warehouse (RW) incurs more holding cost than an owned warehouse owing to advanced preservation facilities.
- Holding cost is constant in both warehouses.
- The items are not repaired or replaced.
- Shortages are completely backlogged.
- The Preservation technology is utilized to minimize losses due to deterioration.
- The preservation technology reduces the deterioration rate but cannot cause inventory regeneration; the effective deterioration rate must remain non-negative throughout the preservation interval.
- To ensure the physical admissibility of the preservation technology, the effective deterioration rate is required to remain non-negative throughout the preservation interval. Since $\mathrm{f(t)} = a + bt$ is increasing for $b > 0$, it is sufficient to impose the condition $0 \leq m \leq \ a + bt_1$.

II.ii. Notation:

The proposed model uses the notations given below:

Decision-Making Variables

t_1: The onset time of deterioration of goods

t_2 : level of RW is exhausted.

t_3 : level of OW is exhausted.

T: Cycle length

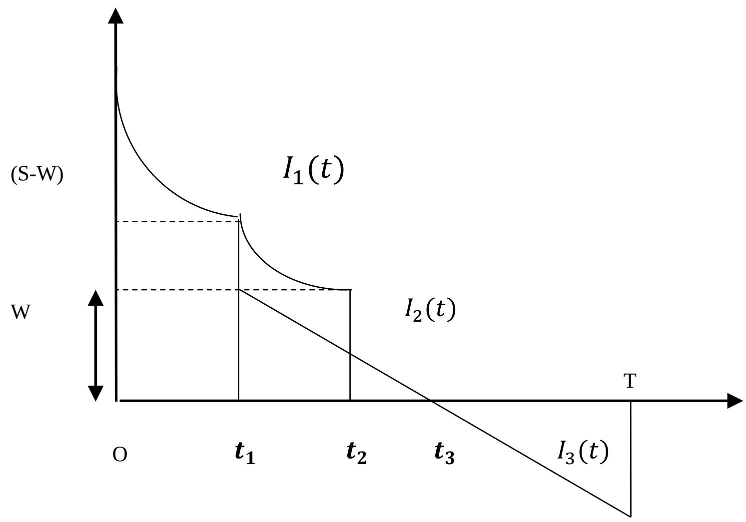
Variables/Parameters: The model uses the following notations:

- Purchasing cost, ordering cost, and holding cost are denoted by PC, OC, and HC, respectively.
- The owned and rented warehouses are denoted by OW and RW, respectively.
- The deterioration rate in the rented warehouse (RW) and owned warehouse (OW) is a function of a linearly time-dependent variable, where f(t) = a + bt, where a, b > 0.
- The parameter q indicates the per-unit preservation technology cost used to reduce deterioration $(q > 0)$.
- m denotes the reduction in the deterioration rate due to preservation technology in RW with $0 \leq m \leq a + bt_1$.
- The unit purchase cost and ordering cost per order are given by p and A, respectively.
- Holding costs per unit in RW and OW are represented by h_r and h_o.
- d and s correspond to deterioration cost and shortage cost per unit.
- The resultant deterioration rate in RW is denoted by $\eta = \text{f(t)} - m \geq 0$, t$\in [t_1, t_2]$
- The total cycle time is expressed by T.
- W represents the stock level in OW.
- D indicates to Demand and is considered as constant.
- Q_B denotes the backorder quantity.
- HC_o and HC_R indicate holding costs in OW and RW.
- DC_o and DC_R represent deterioration costs in OW and RW, respectively.
- The maximum inventory level before the shortage period is denoted by S.
- PTC indicates to preservation technology cost.
- SC indicates to Shortage cost.
- TAC indicates to the total average cost per cycle.
- The inventory levels in RW, OW, and during shortage are expressed by $I_1(t)$, $I_2(t)$, and $I_3(t)$, respectively.
- The imprecise holding costs per item in RW and OW are given by interval numbers $\hat{h}_r = [h_1L, h_1R]$ and $\hat{h}_o = [h_2L, h_2R]$ respectively.
- $\hat{d} = [d_1L, d_2R]$ indicates to the imprecise deterioration cost per unit.
- $\hat{s} = [S_cL, S_cR]$ indicates to the imprecise shortage cost per unit.

III. Mathematical model formulation

This model is based on a two-warehouse inventory system with preservation technology. The model considered as total Quantity Q = S+ Q_B entering this model. Let S be the total quantity of inventory before the shortage occurs. Let W be the quantity of OW, and the remaining quantity $S - W$ is stored in the RW. Q_B is the backorder quantity during the shortage time period. As in the diagram, at time t = 0, the total units S enter the model. First, the RW is used, and after the inventory in the RW is exhausted, then OW is used. According to the diagram, from the RW duration, the time [0, t_1], the amount of RW ($I_1(t)$) falls merely because of the requirement. Whereas the amount of $I_2(t)$ remains constant over the duration [0,$\boldsymbol{t_1}$]. During the time span [0, t_1], no deterioration occurs in OW. Over the time [t_1, t_2], the preservation technique is adopted to decrease the deterioration effect, and the amount of ($I_1(t)$) reduces because of the requirement and resultant degradation in RW. In the time span

$[t_1, t_2]$, deterioration causes a decline in the inventory level $I_2(t)$. The inventory ($I_1(t)$) is exhausted at time t_2. During the time interval [t_2, t_3] the inventory ($\boldsymbol{I_2(t)}$) reduces due to deterioration and demand from OW. At time t_3, the inventory is over, and a shortage occurs during the time [t_3, T].



The amount of rented warehouse (RW) reduces due to demand only during [0, t_1]. The differential equations show the stock amount given by

$$\frac{dI_1(t)}{dt} = -D, 0 \le t \le t_1 \tag{1}$$

Using boundary condition at t = 0, I(0) = S – W

$$\frac{dI_1(t)}{dt} + \eta \boldsymbol{I_1(t)} + D = 0, \quad t_1 \le t \le t_2 \tag{2}$$

Apply the boundary condition, $\boldsymbol{I_1(t_2)}$ = 0 at at t = $\boldsymbol{t_2}$

$$\frac{dI_2(t)}{dt} = 0, 0 \le t \le t1 \tag{3}$$

Using boundary condition I (0) = W at t = 0

$$\frac{dI_2(t)}{dt} + f(t)\boldsymbol{I_2(t)} = 0, \quad t_1 \le t \le t_2 \tag{4}$$

Using boundary condition $\boldsymbol{I_2}(t_1)$ = W at t = t_1

$$\frac{dI_2(t)}{dt} + f(t)\, \boldsymbol{I_2(t)} + D = 0, \quad t_2 \le t \le t_3 \tag{5}$$

Apply boundary condition at t = t_1, $\boldsymbol{I_2}(t_1)$ = W

$$\frac{dI_3(t)}{dt} = -D, \quad t_3 \le t \le T \tag{6}$$

boundary condition at t = $\boldsymbol{t_3}$, hence $\boldsymbol{I_3(t_3)}$ = 0

The solutions of the above equation (1), (2), (3), (4), (5), and (6) respectively are (i), (ii), (iii), (iv), (v), and (vi)

(i) $I_1(t) - (S - W) + Dt = 0$, $0 \le t \le t_1$

(ii) $I_1(t) = De^{-(at + \frac{bt^2}{2} - mt)} \left\{ t_2 - t - \frac{at_2^2}{2} + \frac{at^2}{2} - \frac{mt_2^2}{2} + \frac{mt^2}{2} \right\}$, $t_1 \le t \le t_2$

(iii) $I_2(t) - W = 0$, $0 \le t \le t_1$

(iv) $I_2(t) = We^{(at_1-at)}$, $t_1 \le t \le t_2$ [Neglecting higher power two]

(v) $I_2(t) = D.e^{-(at+\frac{bt^2}{2})}\left(t_3 + \frac{at_3^2}{2} - t - \frac{at^2}{2}\right)$, $t_2 \le t \le t_3$

(vi) $I_3(t) = D(t_3 - t)$, $t_3 \le t \le T$

$I_1(t)$ continuous at $t = t_1$, so putting in the solution (i) and (ii), the maximum level of inventory before shortage (S) is given below as

Maximum Inventory level before shortage (S) = W + D$t_1 + De^{-\left(at_1+\frac{bt_1^2}{2}-mt_1\right)}\left\{t_2 - t_1 - \frac{at_2^2}{2} + \frac{at_1^2}{2} - \frac{mt_2^2}{2} + \frac{mt_1^2}{2}\right\}$

Backordered quantity $(Q_B) = -\left(I_3(t)\right) = D(T - t_3)$, $t_3 \le t \le T$

Total order quantity (Q) = S + Q_B

Optimal order quantity (Q) = $Dt_1 + W + De^{-\left(at_1+\frac{bt_1^2}{2}-mt_1\right)}\left\{t_2 - t_1 - \frac{at_2^2}{2} + \frac{at_2^2}{2} - \frac{mt_2^2}{2} + \frac{mt_1^2}{2}\right\} + D(T - t_3)$

IV. The calculation of TAC per cycle involves the following cost components:

(i) OC = A, order per cycle

(ii) $\boldsymbol{HC_R} = h_r\left[\int_0^{t_2} I_1(t)dt\right]$

$$HC_R = h_r\Bigg[t_1(S - W) + D\left(t_2^2 + t_1^2 - 2t_1t_2 - \frac{at_2^3}{2} + \frac{at_1^2t_2}{2} + \frac{mt_2t_1^2}{2} - \frac{mt_1^3}{3} + \frac{mt_2^3}{6} - \frac{t_1^2}{2}\right)\Bigg]$$

Neglecting higher power three, we get

$$HC_R = h_r\left[(S - W)t_1 - \frac{Dt_1^2}{2} + D(t_2^2 + t_1^2 - 2t_1t_2)\right]$$

(iii) $\boldsymbol{HC_O} = h_O\left[\int_0^{t_3} I_2(t)dt\right]$

$$HC_O = h_O\Bigg[Wt_1 + D\left(\frac{t_3^2}{2} - \frac{at_3^3}{6} + \frac{at_3t_2^2}{2} + \frac{t_2^2}{2} - \frac{at_2^3}{6} - t_3t_2\right) + W\left(t_2 + at_1t_2 - \frac{at_2^2}{2} - \frac{at_1^2}{2} - t_1\right)\Bigg]$$

Neglecting higher power three we get

$$HC_O = h_O\left[Wt_1 + W\left(t_2 - \frac{at_2^2}{2} - t_1 - \frac{at_1^2}{2} + at_1t_2\right) + D\left(\frac{t_3^2}{2} + \frac{t_2^2}{2} - t_2t_3\right)\right] rectio$$

(iv) $\boldsymbol{DC_r} = d\left[\int_{t_1}^{t_2} \eta\, I_1(t)dt\right]$

$$DC_r = Dd\left[(a-m)\frac{t_2^2}{2} - \frac{bt_2^3}{2} + t_2^3\left(\frac{m^2}{2} - \frac{a^2}{2}\right) + \frac{t_1^2}{2}\{(a-m) + t_2(b + a^2 - m^2)\} + (m-a)t_1t_2\right]$$

Neglecting higher power three, we get

$$\boldsymbol{DC_r} = Dd\left[\frac{t_2^2}{2}(a-m) + \frac{t_1^2}{2}(a-m) + t_1t_2(m-a)\right]$$

(v) $\boldsymbol{DC_O} = d\left[\int_{t_1}^{t_3} \mathrm{f(t)}\, I_2(t)dt\right]$

Neglecting higher power three, we get

$$DC_O = d\left[D\left(\frac{(at_3^2)}{2} - a_2t_2t_3 + \frac{at_2^2}{2}\right) + W\left(a(t_2 - t_1) - a^2t_1t_2 - \frac{at_2^2}{2} - \frac{t_1^2}{2}(a^2 + b)\right)\right]$$

(vi) **SC** = $s\int_{t_3}^{T} -I_3(t)dt$

SC = $\frac{Ds}{2}(T^2 - 2t_3T + t_3^2)$

(vii) PTC = $q\int_{t_1}^{t_2} I_1(t)dt$

PTC = $q\left[D\left(t_2^2 + t_1^2 - 2t_1t_2 - \frac{at_2^3}{2} + \frac{at_1^2t_2}{2} + \frac{mt_2t_1^2}{2} - \frac{mt_1^3}{3} + \frac{mt_2^3}{6}\right)\right]$

Neglecting higher power three we get

PTC = $q[D(t_1^2 + t_2^2 - 2t_2t_1)]$

(viii) **PC** : p($Total\ Inventory\ before\ shortage + Total\ Inventory\ after\ shoratge$)

$$p\left[W + t_1D + De^{-\left(at_1 + \frac{bt^2}{2} - mt_1\right)}\left\{\frac{at_2^2}{2} - t_1 - \frac{at_2^2}{2} - \frac{mt_2^2}{2} + \frac{mt_1^2}{2} + t_2\right\} + D(T - t_3)\right]$$

TAC = $\frac{1}{T}[OC + HC_O + HC_R + DC_O + DC_R + SC + PTC + PC]$

$$\text{TAC} = \frac{1}{T}\left[A + h_r\left[(S-W)t_1 - \frac{Dt_1^2}{2} + D(t_2^2 + t_1^2 - 2t_1t_2)\right] + h_O\left[Wt_1 + W\left(t_2 + at_1t_2 - \frac{at_2^2}{2} - t_1 - \frac{at_1^2}{2}\right) + D\left(\frac{t_3^2}{2} - t_2t_3 + \frac{t_2^2}{2}\right)\right] + Dd\left[(a-m)\frac{t_2^2}{2} + \right.\right.$$

$(m-a)t_1t_2+(a-m)\frac{t_1^2}{2}\Big]+\ d\Bigg[W\Bigg(a(t_2-t_1)-\ a^2t_1t_2-\frac{at_2^2}{2}-\ \frac{t_1^2}{2}(a^2+b)\Bigg)+$
$D\left(\frac{at_3^2}{2}-\ a_2t_2t_3+\ \frac{at_2^2}{2}\right)\Bigg]+\ \frac{Ds}{2}(T^2+\ t_3^2-\ 2t_3T)+\ \mathrm{q}[D(t_2^2+\ t_1^2-\ 2t_1t_2)]+$
$\mathrm{p}\left[W+Dt_1+De^{-\left(at_1+\frac{bt_1^2}{2}-mt_1\right)}\left\{t_2-t_1-\frac{at_2^2}{2}+\frac{at_1^2}{2}-\frac{mt_2^2}{2}+\frac{mt_1^2}{2}\right\}+\ D(T-t_3)\right]\Bigg]$

V. Determination of optimal cost:

TAC is a function of four variables $(T, t_3, t_2,\ t_1)$. The required condition for optimal solution is

$$\frac{\partial(TAC)}{\partial T}=0, \frac{\partial(TAC)}{\partial t_1}=0, \frac{\partial(TAC)}{\partial t_2}=0, \frac{\partial(TAC)}{\partial t_3}=0 \qquad (7)$$

$\frac{\partial(TAC)}{\partial T}=\frac{1}{T}\left[\frac{Ds}{2}(2T-2t_3)+pD\right]-\frac{1}{T^2}\Bigg[A+\ h_r\Big[(S-W)t_1-\ \frac{Dt_1^2}{2}+\ D(t_2^2+t_1^2\ -$
$2t_1t_2)\Big]+h_O\left[Wt_1+W\left(t_2+at_1t_2-\ \frac{at_2^2}{2}-t_1-\ \frac{at_1^2}{2}\right)\ +\ D\left(\frac{t_3^2}{2}-\ t_2t_3+\ \frac{t_2^2}{2}\right)\right]+$
$Dd\left[(a-m)\frac{t_2^2}{2}+\ (m-a)t_1t_2+(a-m)\frac{t_1^2}{2}\right]+\ \ d\Bigg[W\Bigg(a(t_2-t_1)-\ a^2t_1t_2-$
$\frac{at_2^2}{2}-\ \frac{t_1^2}{2}(a^2+b)\Bigg)\ +D\left(\frac{at_3^2}{2}-\ a_2t_2t_3+\ \frac{at_2^2}{2}\right)\Bigg]+\frac{Ds}{2}(T^2+\ t_3^2-\ 2t_3T)+$
$\mathrm{q}[D(t_2^2+\ t_1^2-\ 2t_1t_2)]+\ \ \mathrm{P}\Bigg[W+Dt_1+De^{-\left(at_1+\frac{bt_1^2}{2}-mt_1\right)}\Big\{t_2-t_1-\frac{at_2^2}{2}+\frac{at_1^2}{2}-$
$\frac{mt_2^2}{2}+\frac{mt_1^2}{2}\Big\}+\ D(T-t_3)\Bigg]\Bigg]\ \frac{\partial(TAC)}{\partial t_1}=\frac{1}{T}\Big[h_r[(S-W)-\ Dt_1+\ D(2t_1\ -2t_2)]+$
$h_O[W+W(at_2-1-\ at_1)]\ +\ Dd[(m-a)t_2+(a-m)t_1]+\ \ dW\big[\big(-a-$
$a^2t_2-\ t_1(a^2+b)\big)\big]+\ \mathrm{q}[D(\ 2t_1-\ 2t_2)]+\ \ \mathrm{p}\Big[D+D\Big\{e^{-\left(at_1+\frac{bt^2}{2}-mt_1\right)}(-1+$
$at_1+mt_1)-e^{-\left(at_1+\frac{bt^2}{2}-mt_1\right)}\Big[\mathrm{at}_2-\mathrm{at}_1-\frac{\mathrm{a}^2\mathrm{t}_2^2}{2}-\frac{\mathrm{mat}_2^2}{2}+\frac{\mathrm{mat}_1^2}{2}+\mathrm{bt}_1\mathrm{t}_2-\mathrm{bt}_1^2-$
$\frac{\mathrm{abt}_1\mathrm{t}_2^2}{2}-\frac{\mathrm{bmt}_2^2\mathrm{t}_1}{2}+\frac{\mathrm{bmt}_1^3}{2}-\mathrm{mt}_2-\mathrm{mt}_1-\frac{\mathrm{amt}_2^2}{2}-\frac{\mathrm{m}^2\mathrm{t}_2^2}{2}+\frac{\mathrm{m}^2\mathrm{t}_1^2}{2}\Big]\Big\}\Big]\Big]$

$$\frac{\partial(TAC)}{\partial t_2} = \frac{1}{T}\left[2Dh_r(t_2-t_1) + h_0\{W(1+at_1-at_2)+D(t_2-t_3)\}\right.$$
$$+ Dd\{(a-m)t_2+(m-a)t_1\} + Wd\{a-a^2t_1-at_2\}$$
$$+ Dd(-at_3+at_2) + p\left\{De^{-\left(at_1+\frac{bt_1^2}{2}-mt_1\right)}(1-at_2-mt_2)\right\}$$
$$\left. + qD\left(-2t_1-\frac{3at_2^2}{2}+\frac{at_1^3}{2}+\frac{mt_1^2}{2}+\frac{mt_2^2}{2}\right)\right]$$

$$\frac{\partial(TAC)}{\partial t_3} = \frac{1}{T}\left[Dh_o(t_3-t_2) + Dd(at_3-at_2) + \frac{Ds}{2}(2t_3-2T)\right.$$
$$\left. - D^2e^{-\left(at_1+\frac{bt_1^2}{2}-mt_1\right)}\right]$$

The optimal solution of t_1, t_2, t_3 *and* T if the corresponding Hessian matrix, presented below, is positive definite. We can find the minimum total average cost with respect to t_1, t_2, t_3 *and* T

$$H_{es} = \begin{bmatrix} \frac{\partial^2 TAC}{\partial {t_1}^2} & \frac{\partial^2 TAC}{\partial t_1 \delta t_2} & \frac{\partial^2 TAC}{\partial t_1 \partial t_3} & \frac{\partial^2 TAC}{\partial t_1 \partial T} \\ \frac{\partial^2 TAC}{\partial t_2 \partial t_1} & \frac{\partial^2 TAC}{\partial {t_2}^2} & \frac{\partial^2 TAC}{\partial t_2 \partial t_3} & \frac{\partial^2 TAC}{\partial t_2 \partial T} \\ \frac{\partial^2 TAC}{\partial t_3 \partial t_1} & \frac{\partial^2 TAC}{\partial t_3 \partial t_2} & \frac{\partial^2 TAC}{\partial {t_3}^2} & \frac{\partial^2 TAC}{\partial t_3 \partial T} \\ \frac{\partial^2 TAC}{\partial T \partial t_1} & \frac{\partial^2 TAC}{\partial T \partial t_2} & \frac{\partial^2 TAC}{\partial T \partial t_3} & \frac{\partial^2 TAC}{\partial T^2} \end{bmatrix}$$

VI. Solution Procedure

Under such a two-warehouse model, the objective function is highly nonlinear and exponential in nature; hence, finding a closed-form analytical solution is not possible. This model uses a parametric functional form of interval numbers to represent imprecise parameters (Appendix 1). Interval parameters are incorporated into TAC and described using a parametric form. The lower and upper bounds in conventional interval-based methods require different objective functions, which are usually solved using heuristic or meta-heuristic techniques. The parametric approach provides a closed-form objective function, which transforms the problem into a single-goal formulation problem. This optimisation problem can be solved using conventional numerical or classical techniques. This conceptual mathematical formulation of using a parametric approach has already been documented by Das & Roy [II] and Nath & Sen [IX]. Furthermore, owing to the highly non-linear characteristic of the objective function formulated from this problem, this research work relies on existing numerical solutions for solving this problem mathematically. The proposed model has been determined by Maple 2024 software, as the inventory framework does not always provide feasible values for the time-related decision variables. This demonstrates that the interior solution does not lie within the admissible planning horizon.

Algorithm:

Step1: Set the parameters: A, p, D, S, W, q, a, b, m.

Step 2: Express the estimated values of $\hat{h}_r$, $\hat{h}_O$, $\hat{d}$, $\hat{s}$ in interval form.

Step 3: Using a parametric formulation, the values of $\hat{h}_r(m)$, $\hat{h}_O(m)$, $\hat{d}(m)$, $\hat{s}(m)$ are evaluated for a given value of m.

Step 4: Determine the optimal values of TAC and t_1, t_2, t_3, T

VII. Numerical Illustration

For demonstration, the following parameter values are selected. A = 50, p =3, q = 1, D = 250, S = 300, W = 150, m = 0.10, a = 0.10, b = 0.10.

Here, some selected parameters expressed as interval numbers are as follows:
$\hat{h}_r$= [2,4], h_O= [1,2], $\hat{d}[1,2]$, $\hat{s}$ = [1,2], The preservation parameter is fixed at $m = 0.10$, satisfying the physical admissibility condition $0 \le m \le a + bt_1$.
By applying the parametric functional form for m = 0.10, we have $\hat{h}_r$ (m) = 2.30, h_O(m) =1.15, $\hat{d}$(m)= 1.15, $\hat{\boldsymbol{s}}$(m) = 1.15; the solution is computed using Maple 2024 software as T = 1.2206 yr, t_1= 0.0055 yr, t_2=0.1378 yr., t_3 = 0.5758 yr. The value of total average cost (TAC) is 6322.2267.

VIII. Sensitivity Analysis

Sensitivity analysis has been carried out to examine the impact of varying values of one parameter at a time, such as A, m,h_r, a, and b, while other parameters are kept unchanged on the decision variables and the total average cost.

Table 1: Variation of results under varying A

A	t_1	t_2	t_3	T	TAC
100	0.0547	0.1123	0.3359	0.6676	1394.6829
200	0.0699	0.1925	0.2241	0.3009	3058.5960
300	0.0125	0.01324	0.1446	0.1659	4732.9339
400	0.0898	0.1249	0.2390	0.6677	4901.3379
500	0.0125	0.0132	0.1445	0.1657	5944.9529

Table 2: Variation of results under varying a

a	t_1	t_2	t_3	T	TAC
0.20	0.3199	0.3524	0.3640	0.4581	2122.7093
0.30	0.1858	0.3411	0.7584	1.2484	1049.2350
0.40	0.1798	0.4021	0.8535	1.2820	1032.6337
0.50	0.1900	0.4066	0.7963	1.2768	1074.4890
0.60	0.0946	0.2361	1.0766	2.3689	905.0517

Table 3: Variation of results under varying b

b	t_1	t_2	t_3	T	TAC
0.20	0.0183	0.0325	0.1366	0.3277	1733.7458
0.30	0.3516	0.6127	1.5797	0.6317	1623.5526
0.40	1.1136	1.7000	1.7925	1.8796	1156.8146
0.50	0.3953	0.5791	1.3063	1.5000	903.8872
0.60	0.1193	0.3223	1.3114	1.7500	768.6411

IX. Observation

From the above sensitivity analysis tables, it is observed that changing the values of different parameters, such as A, m,h_r, a, b significantly influences the time variables and the total average cost (TAC).

- It has been noted from Table 1, value of parameter A rises, the TAC increases correspondingly, while the decision variables t_1, t_2, t_3 , and Texhibit a non-monotonic variation.
- From Table 2, it is observed that an increase in the parameter (a) results in non-monotonic variations in the decision variables t_1, t_2, t_3, and T. The TAC also exhibits a fluctuating pattern.
- From Table 3, it is observed that as the parameter (b) increases, the decision variables t_1, t_2, t_3, and T exhibit a fluctuating pattern, whereas the TAC decreases correspondingly.

X. Conclusions and Scope for future work

The present work considers the concept of two warehouses with preservation technology for non-instantaneous products. Deterioration is a linearly time-dependent variable, demand is considered constant, and shortages are completely backlogged. Preservation technology has been applied only in rented warehouses for particular time periods to minimise the deterioration. The numerical illustration and sensitivity analysis show how the TAC fluctuates with varying parameter values for parameters A, a, and b. When there is an increase in parameter A, the total average cost also increases. When there is an increase in parameter a, the TAC also exhibits a fluctuating pattern. Now When an increase in parameter b, the TAC decreases correspondingly.

In the future, this model can be extended in several directions. This model is dedicated to non-instantaneous products. Preservation technology has been applied for a particular time period, and the deterioration rate is linearly time-dependent. Future research may consider instantaneously deteriorating items. Preservation technology can be applied for the full time period, and the deterioration rate can be considered in many dimensions, such as a constant rate, a quadratic time-dependent rate, a Weibull distribution deterioration rate, etc.

Appendix 1:

In this appendix, the concept of interval numbers and function-based parametric form is described.

Definition 1: A Closed interval **Î**= [XL, XU] is called an interval number, and it includes all the real numbers X ∈ R such that [XL, XU] here {X ∈ R: XL ≤ x ≤ XU}. In this case, XL and XU are the minimum and maximum boundaries of the interval, respectively, and R represents the set of real numbers. The interval may also be written in terms of its functional form as shown below.

Definition 2: Let an interval number**Î** = [XL, XU], where XU ≥ XL > 0. With respect to the interval number Î, a function f maps the interval [0,1] into R, where $f(m)$ is defined as f(m) = X L+ (XU−XL).m, for all m ∈ [0, 1]. Here f(m) is a monotonic non-decreasing function. Clearly f (0) = XL, f (1) = XU hence f(m) ∈ [XL, XU]. For simplicity, the interval number may also be defined using the interval function associated with it, which is written as**Î** = [XL, XU], as I (m), that is, I(m) = f(m) = XL + m·(XU−XL). The parametric form definition represents an interval number by means of a real-valued function.

The interval numbers shown below have been chosen for illustration purposes only.

$\hat{h}_r = [h_1L, h_1R]$, $\hat{h}_O = [h_2L, h_2R]$, $\hat{d} = [d_1L, d_2R]$, $\hat{s} = [S_cL, S_cR]$

The related parametric forms are given below:

$h_r = (h_1L) +$ m. $(h_1U - h_1L)$, $h_O = (h_2L) +$ m. $(h_2U - h_2L)$, d $=(d_1L) +$ m. $(d_1U - d_1L)$,

s $= (s_1L) +$ m. $(s_1U - s_1L)$

Conflict of Interest:

There is no conflict of interest regarding this article.

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