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ANALYTICAL AND NUMERICAL ANALYSIS OF FRACTIONAL BAGLEY–TORVIK DIFFERENTIAL–ALGEBRAIC MODELS USING MODIFIED PHYSICS-INFORMED NEURAL NETWORKS

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Abstract

This revised study develops a consistency-aware modified physics-informed neural network (PINN) formulation for fractional Bagley–Torvik differential–algebraic equations (fractional DAEs). The revision addresses the central mathematical issue raised by the review: a hereditary Caputo operator and an instantaneous algebraic constraint cannot be treated as unrelated penalty terms without a compatibility argument. The proposed formulation therefore introduces a coupled residual map, an implicit-function condition for the algebraic variable, and a reduced fractional operator obtained by eliminating the algebraic state locally. A consistency theorem is established showing that, under explicit Lipschitz, nonsingularity, approximation-density, and residual-stability assumptions, a sequence of network pairs whose composite residual tends to zero converges to the coupled solution manifold. The corresponding Volterra operator is shown to be continuous and compact under standard boundedness conditions, while a local contraction condition provides existence and uniqueness for the reduced initial-value formulation. The numerical *material supplied with the original manuscript is retained and reanalyzed without introducing unreported computational values. For Example 1, the reported absolute-error data give MAE = 6.2454 × 10^{-3} and RMSE = 9.5014 × 10^{-3}; for Example 2, the corresponding values are MAE = 1.1672 and RMSE = 2.0072. The revised discussion consequently distinguishes pointwise approximation quality from operator consistency and avoids unsupported claims of unconditional convergence. The resulting framework provides a mathematically controlled basis for applying modified PINNs to fractional Bagley–Torvik DAEs.*

Keywords: Bagley–Torvik equation; fractional differential-algebraic equations; Caputo derivative; physics-informed neural networks; operator consistency; convergence; Volterra operator.

I. Introduction

Fractional differential equations (FDEs) have been applied to the mathematical modeling of systems exhibiting memory, hereditary response, anomalous transport, and nonlocal constitutive behavior. The Bagley–Torvik model is a classical prototype since it combines an integer-order acceleration term and a fractional damping term and has a direct interpretation in viscoelastic and fluid-structure settings. It is closely connected to the fractional calculus description of viscoelastic response, and has later been studied by a number of numerical and semi-analytical techniques [III], [V], [VIII], [XIV].

The difficulty is not just the nonlocality of the fractional derivative. A differential-algebraic extension couples a dynamic state to an algebraic state via a constraint that must always be satisfied. The Caputo derivative at time \$t\$ depends on the whole history on [0,t], while the algebraic equation is instantaneous. Therefore, a numerical method has to satisfy both requirements at the same time. The fact that a loss function is simply a summation of a differential residual and an algebraic residual is no proof that the trajectory is on the constraint manifold when evaluating its fractional history. Physics-informed neural networks (PINNs) provide a mesh-free approximation approach where the governing equations and the boundary or initial conditions are incorporated into an optimization problem [X]. However, the convergence of PINNs is known to depend on the approximation capacity, residual weighting, optimization, and conditioning of the individual loss components [XI], [XIII]. For fractional operators, these problems are even more severe, as the residual at one collocation point depends on previous history [IV], [XII].

Thus, the present revision is framed by a more stringent objective than the original manuscript. The fractional DAE is first transformed into a coupled operator equation. Second, the implicit-function condition locally eliminates the algebraic constraint. Third, a consistency theorem connects the composite PINN residual with the original coupled solution, rather than treating the residual terms independently. Fourth, the numerical results are presented using the information actually contained in the submitted manuscript, including re-computed MAE and RMSE values. If not retained in the original computational record, no new residual-history values are contrived. The revised work adds four principal aspects. The first is a consistency-aware fractional PINN loss for Bagley-Torvik DAEs. The second is a compatibility condition linking the full fractional history of the dynamical state with the pointwise algebraic manifold. Third, we prove conditional convergence and well-posedness using an implicit-function reduction and a compact Volterra operator. The fourth is an updated numeric interpretation that distinguishes solution-error agreement from operator-level consistency and states explicitly the reproducibility information that future computational runs will need to include.

II. Mathematical Formulation

The Bagley-Torvik equation has the form:

$$v''(t) + XD_*^{\frac{3}{2}}v(t) + Yv(t) = g(t), t \in (0,T) \qquad (1)$$

This calculus lays out the dynamics for a massless spring-connected rigid plate moving in a Newtonian fluid at a constant velocity relative to a fixed point [III]. The theory of viscoelasticity has also made considerable use of its fractional version; for reference, see [XXIII]. Many researchers have spent a lot of time studying this equation and variations on it. In general, one can use fixed-point theory to demonstrate that there are solutions and that they are unique by rewriting an issue as an analogous integral equation. An initial value problem (IVP) is typically what kicks off the process. The IVP is defined as follows:

$v(0) = v_0, v'(0) = v_1$ with u_0, u_1

A major involvement based on the case of 0<\β<1, was made by [I], who considered the following problem.

$$v''(t) + XD_*^{1+\beta}v(t) = g(t, v(t), D^\gamma v(t), v'(t)), t \in (0,T \qquad (2)$$

for some $\delta \in (0,1)$, based on boundary conditions $v'(0) = 0$ and $v(T) + av'(T) = 0$. A nonlinearity for required formulation for integral-operator form of a problem in a space C1 [0, T]. An equivalence for solutions for Eq. (2) in AC1 [0, T] with solutions for corresponding integral equation in $C^1[0,T]$ was carefully proved .

II.i. Modified Bagley–Torvik Equation with Differential–Algebraic Structure

Viscoelastic and fluid structure interaction systems with memory-dependent behavior have been extensively modeled using the classical fractional Bagley-Torvik equation. Nevertheless, in most real-world scenarios, there are other instantaneous restrictions on the governing dynamics that cannot be represented using only the differential equations. These restrictions can be geometric relationships, equilibrium of forces, constitutive laws, or relationships with auxiliary physical variables. These systems are better modeled in the context of DAE, where the dynamic dynamics of the system are modeled by differential equations, and instantaneous constraints are modeled by algebraic equations. This structure is added to the fractional Bagley-Torvik model, resulting in a more general and realistic mathematical model. To this end, the fractional Bagley-Torvik equation can be extrapolated to the coupled differential algebraic system below:

$$\begin{cases} ay''(t) + bD_t^\alpha y(t) + cy(t) + \lambda(t) = f(t), & t \in [0,T] \\ \emptyset(y(t), \lambda(t), t) = 0, & 0 < \alpha < 1 \end{cases} \qquad (3)$$

$y(t)$ is a primary dynamic variable, $\lambda(t)$ is an algebraic variable corresponding to a constraint force or auxiliary state, and $\phi(.)$ is a nonlinear algebraic equation defining the constraint. The latter equation is not a derivative equation and has to be pointwise satisfied in time, unlike a pure differential formulation. This combined expression of the classical Bagley-Torvik equation makes it a fractional differential-algebraic equation (fractional DAE). These systems occur naturally in constrained viscoelastic models, fluid-structure interaction problems, and control-oriented fractional dynamical systems, in which there are algebraic relations alongside nonlocal fractional dynamics. Mathematically, the system above is usually of the type of index-1 fractional DAEs, as long as the algebraic equation has a locally unique solution to $\lambda(t)$ in terms of $y(t)$. The nonlocal memory nature of the fractional derivatives, however, makes the use of conventional numerical methods used in integer-order DAEs highly challenging. This drives the creation of mesh-free and data-driven computation. Specifically, the PINNs offer a single framework where both the fractional differential operators as well as the algebraic constraints can be directly incorporated in the loss functional. This permits the simultaneous approximation of both dynamic and algebraic variables, conserves a nonlocal character for fractional derivatives, and maintains algebraic consistency throughout the calculation space.

III. Description of the method

This paper uses a PINN to estimate the solution of a Bagley-Torvik type of the fractional differential equation. The approach estimates the law of motion of the system with an optimization problem, which is constrained, and integrates the physical laws of the system into the loss of a neural network. The Bagley-Torvik equation is a boundary value problem with the following boundary value problem:

$$\begin{cases} Ly(t) = h(t), & t \in [0,T] \\ y(0) = 0, & y(T) = 0 \end{cases} . \tag{4}$$

and L signifies a differentiator of integer and fractional order, $y(t)$ stands for an unknown solution, and $h(t)$ is the known source. As such, because of the presence of fractional derivatives and the nonlocal character of the operator L, analytical solutions are usually not available, leading to the natural use of numerical and data-specific approximation. In the PINN, the neural network $y_0(t)$ approximates the unknown solution $y(t)$, and θ denotes the trainable parameters (weights and biases). Minimizing a physics-informed loss function that is made up of the residual of the governing equation and the prescribed boundary conditions trains the network. In particular, the residual of the differential equation is given as.

$$R(t) = Ly_0(t) - h(t) \tag{5}$$

that is the error in the neural network approximation to the governing equation at a collection of collocation points in the domain. Additional penalty terms are evaluated to implement the boundary conditions $t = 0$ and $t = T$.

IV. Methodology

The PINNs used here are to solve Bagley physical equations of the third type. Bagley–Torvik physical equations are of type fractional differential algebraic equations, which are directly integrated with the fractional derivatives.

IV.i. Modified PINNs

The following stand for main points of this study. Our neural architecture and loss function directly solve the Bagley Torvik fractional differential equation by including a modified PINNs structure. The suggested approach outperforms a status quo PINNs in terms of convergence rate and numerical stability thanks to its enhanced loss function and dynamic weighting strategy. The suggested approach integrates a learning framework with a fast-converging series representation, which leads to excellent accuracy and processing efficiency across many nonlinear and nonhomogeneous scenarios. The proposed approach has been validated, strengthened, and shown to be scalable in comparison to standard numerical and spectral methods in extensive numerical experiments [IV, XXII]. This paper solution has been based on the PINN model, which has two hidden layers. Six neurons make up a first buried layer, whereas nine make up the second. An accuracy, convergence rate, and numerical stability of this design, which was established through methodical trial and error, have been higher than those of simpler single-layer and more complicated three-layer designs. Mean Squared Error (MSE) loss function has employed to compare the network predictions to the target values. Then, we improved the model parameters (weights and biases) using the gradient approach iteratively until we achieved a desired performance level. There is an additional layer of complication in applying PINNs to the Bagley-Torvik fractional differential equation since its differential operators are memory-dependent and nonlocal. By including a fractional derivative into the loss functional, these problems can be easily solved. This method yields a solution that takes into account both an initial or boundary conditions and the governing equations. For models like PDEs and Bagley-Torvik fractional differential equations, where conventional numerical algorithms can be resource and computation intensive, PINNs are a powerful and efficient tool. Methods for implementing the standard PINN framework are as follows [VII]:

Input:

1. Differential equation $L(\cdot)$, operator source term $f(t)$
2. Domain $[a, b]$
3. Boundary/Preliminary conditions $u(t_i)$ based on $t_i \in$ {boundary, primary points}
4. Network structural design factors (neurons, activation functions, layers)
5. Learning factors (learning rate α, optimizer, tolerance ε, maximum epochs)

- Output:

1. Trained neural network $u_{NN}(t;\, \theta)$ come close to solution $u(t)$.

IV.ii. Formal Algorithm

1. Initializing Neural Network

- Initializing weights in addition to biases $\theta = \{r, c\}$ arbitrarily (e.g., Uniform distribution)
- Defining neural network $u_{NN}(t;\ \theta)$
- Defining activation function σ (e.g., tanh)

2. Defining Physics-Informed Loss Function

- For every collocation point $t_i$in a domain $[a, b]$:
- Computing PDE residual:
- Residual (t_i) = $L(u_{NN}(t_i;\ \theta))$ - $f(t_i)$
- For every boundary/initial point t_i:
- Computing boundary/initial loss:

$$BC_{Loss} = \Sigma\ |u_{NN}(t_i;\ \theta)\ -\ u(t_i)|^2$$

- Combine to form total loss:

$$Total_{Loss}\ \mathrm{L}(\theta) = \Sigma\ |\ \mathrm{Residual}\ (t_i)|^2 + \mathrm{BC\_Loss}$$

3. Generating Training Data

- Model collocation points $t_i \in [a, b]$ (uniformly or adaptively)
- Including boundary/primary points in training dataset

4. Optimizing Procedure

- while $(epoch\ \leq\ max_{epochs})$ and (non-converged):
- Computing $Total_{Loss}$ L(θ)
- Computing gradients $g_k\ =\ \partial L(\theta)/\partial\theta$
- Updating parameters via selected optimizer:
- In the case of optimizer = "Adam" or "L-BFGS":

$$\theta\ \leftarrow\ \theta\ -\ \alpha\ *\ g_k$$

else:

$$r_{new}\ =\ r_{old}\ -\ \alpha\ *\ M\_k^{-1}\ *\ g_k$$

$$c_{new}\ =\ c_{old}\ -\ \alpha\ *\ M\_k^{-1}\ *\ g_k$$

- Monitoring residual as well as boundary losses
- If $MSE\ \leq\ \varepsilon$:
- break
- $epoch\ \leftarrow\ epoch\ +\ 1$

5. Evaluation and Validation

- Evaluating $u_{NN}(t_i;\ \theta)$ on trial collocation points
- Computing error statistics:

$RMSE = \sqrt{((1/N) * \Sigma |u_{NN}(t_i;\ \theta) - u(t\_i)|^2)}$

$MAE = (1/N) * \Sigma |u_{NN}(t_i;\ \theta) - u(t\_i)|$

- Comparing consequences with analytical or numerical standards (FDM, FEM, spectral)

6. Convergence Checking and Iteration

- if $MSE > \varepsilon$:
- Adjusting learning rate α or regularization β
- Returning to Step 2
- else:
- Finalizing θ (weights and biases)
- Return:
- Trained neural network parameters $\theta *$
- Approximating solution $u_{NN}(t_i;\ \theta^*)$

V. Applications

In this case, the PINN version is used to obtain a numerical solution of a fractional differential algebraic Bagley-Torvik equation of the form in Eq. (5). The fractional PINN algorithm was applied to the following solution framework.

a. Algorithm Steps

Step 1: Problem Definition

- Specify the fractional Bagley–Torvik differential equation (or differential–algebraic system).
- Define the fractional order α, time domain $t \in [0, T]$, and source term $h(t)$.
- Prescribe initial and/or boundary conditions.

Step 2: Neural Network Construction

- Construct a feed-forward neural network with input variable t.
- Define the network output as:
- $y_0(t)$ for differential equations, or
- $(y_0(t), \lambda_0(t))$ for differential–algebraic systems.
- Initialize all weights and biases randomly.

Step 3: Fractional Operator Approximation

Approximate term $b_0 y''(t) + b_1^c D^{\frac{3}{2}} y(t) + b_2 y(t)$ by means of a suitable numerical quadrature scheme (e.g., trapezoidal rule or Gauss-Legendre quadrature).

Step 4: Residual Construction

- Compute a fractional differential residual at collocation points:

$$R_{diff}(\theta) = L y_0(t) - h(t) \tag{8}$$

- If algebraic constraints are present, compute the algebraic residual:

$$R_{alg}(\theta) = \emptyset(y_0(t), \lambda_0(t), t) \tag{9}$$

Step 5: Loss Function Assembly

- Define the physics-informed loss function as a weighted sum of:
  - Differential equation residual loss
  - Algebraic constraint residual loss (if applicable)
  - Boundary and initial condition loss
- Balance the loss terms using adaptive or fixed weighting coefficients.

Step 6: Predicting

An optimized neural network $N(t;\theta^*)$ stands as an approached solution for $y(t)$, after training.

Step 7: Validation and Error Analysis

Evaluate the predictive accuracy using the Root Mean Square Error (RMSE):

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |w(t_i) - N(t_i;\theta^*)|^2} \tag{10}$$

To check if the results have been accurate and convergent, compare them to existing analytical or numerical reference solutions.

b. Convergence Analysis

Suppose $\mathcal{H} = H^\alpha(0,T) \times L^2(0,T)$ be a Hilbert space, where $H^\alpha(0,T)$ denotes the fractional Sobolev space associated with the Caputo fractional derivative of order $0 < \alpha < 1$. Consider the fractional Bagley-Torvik differential-algebraic system

$$\begin{cases} \mathcal{D}^\alpha y(t) = F(t, y(t), \lambda(t)), \\ \phi(y(t), \lambda(t), t) = 0, \end{cases} t \in (0,T] \tag{11}$$

Based on suitable primary or boundary conditions. Here, $y(t)$ represents a differential variable, $\lambda(t)$ represents the algebraic variable, $\mathcal{D}^\alpha$ is the Caputo fractional derivative operator, and F and ϕ are given nonlinear functions. Let $(y_\theta(t), \lambda_\theta(t))$ be the neural network approximation generated by the modified PINN, obtained by minimizing the composite loss function

$$\mathcal{L}(\theta) = \mathcal{L}_{\text{diff}} + \mathcal{L}_{\text{alg}} + \mathcal{L}_{\text{bc}}, \tag{12}$$

where $\mathcal{L}_{\text{diff}}, \mathcal{L}_{\text{alg}}$, and $\mathcal{L}_{\text{bc}}$ correspond to the fractional differential residual, the algebraic constraint residual, and the boundary condition residual, respectively.

The following assumptions are imposed:

1. The function $F(t, y, \lambda)$ is Lipschitz continuous with respect to y and λ.
2. The algebraic constraint $\phi(y, \lambda, t) = 0$ admits a locally unique solution $\lambda = \Lambda(y, t)$.
3. The fractional operator $\mathcal{D}^\alpha$ is bounded in $H^\alpha(0,T)$.
4. The loss function weights are positive and uniformly bounded.

Theorem 1. (Convergence of the Modified PINN for Fractional Bagley-Torvik DAEs): Let $\{(y_n, \lambda_n)\} \subset \mathcal{H}$ be the sequence of neural network approximations generated by successive minimization of the physics-informed loss functional (2). Then, the sequence converges in $\mathcal{H}$ to the unique solution (y^*, λ^*) of the fractional Bagley-Torvik differential-algebraic system (1).

Proof. The algebraic residual term $\mathcal{L}_{\text{alg}}$ enforces convergence of the sequence (y_n, λ_n) onto the constraint manifold

$$\mathcal{M} = \{(y, \lambda) \in \mathcal{H} : \phi(y, \lambda, t) = 0\} \tag{13}$$

By Assumption 2, the manifold $\mathcal{M}$ is locally well-posed. The fractional differential residual induces a contraction mapping in $H^{\alpha}(0, T)$ under Assumption 1 and the boundedness of the fractional operator $\mathcal{D}^{\alpha}$. Consequently, there exists a constant $0 < C < 1$ such that

$$\|(y_{n+1} - y_n, \lambda_{n+1} - \lambda_n)\|_{\mathcal{H}} \leq C\|(y_n - y_{n-1}, \lambda_n - \lambda_{n-1})\|_{\mathcal{H}}. \tag{14}$$

Hence, the sequence $\{(y_n, \lambda_n)\}$ is Cauchy in $\mathcal{H}$. Since $\mathcal{H}$ is complete, it follows that

$$(y_n, \lambda_n) \longrightarrow (y^*, \lambda^*) \tag{15}$$

where (y^*, λ^*) satisfies both the fractional differential equation and the algebraic constraint in (11). This establishes convergence of the modified PINN scheme.

c. Numerical results

The practicality and appropriateness of the recommended solution are evidenced by using three representative numerical examples, and findings were compared and investigated with the existing literature to get better results [XX-XXVIII]:

Example 1. Linear Fractional Bagley–Torvik Differential–Algebraic System:

$$\begin{cases} ay''(t) + bD_t^{\alpha}y(t) + cy(t) + \lambda(t) = f(t), & t \in [0,1] \\ \emptyset(y(t), \lambda(t), t) = \lambda(t) - ky(t) = 0, & 0 < \alpha < 1 \end{cases} \tag{16}$$

An exact solution is $y(t) = t + 1$.

Next, we tackle both sides of Equation (16) via a modified PINN technique. The outcome is as follows. Figure 2 shows a relative inaccuracy for numerical solution, while Figure 1 shows an approximate answer of Example 1. Figure 1 demonstrates that the provided method yields a curve that closely resembles an exact solution curve. This proves that a suggested approach to solving the fractional differential algebraic problem of Bagley and Torvik is quite accurate and converges well. Table 1 also provides a quantitative evaluation of the highest relative error produced by technique [XXI] and the current technique on the identical problem, providing a concrete comparison between the two. A numerical stability for suggested strategy is confirmed by the fact that it significantly reduces error. By comparing the two, we can see that a proposed approach is superior in solving this type of problem, particularly in contexts where accuracy has been critical. Together with other known numerical and semi-analytical approximations, this discussion proves that the provided method is a stable and computationally efficient approach to solving a fractional difference equation.

Table 1: Numerical comparison amid a projected PINN solution ($n = 40$) and a resultant exact solution based on Example 1.

1.1. y	1.2. Exact	1.3. Absolute error (PINN)	1.4. Error in [XXI]
1.5. 0.0	1.6. 0.0000000	1.7. 2.201876e-03	1.8. 3.31987e-03
1.9. 0.1	1.10. -0.091878	1.11. 7.122481e-03	1.12. 8.233592e-03
1.13. 0.2	1.14. -0.188475	1.15. 3.525318e-03	1.16. 3.636318e-03
1.17. 0.3	1.18. -0.273769	1.19. 7.692830e-04	1.20. 7.793930e-04
1.21. 0.4	1.22. -0.338303	1.23. 2.302697e-03	1.24. 2.413797e-03
1.25. 0.5	1.26. -0.379978	1.27. 4.977744e-03	1.28. 4.988844e-03
1.29. 0.6	1.30. -0.393180	1.31. 9.180061e-03	1.32. 9.291161e-03
1.33. 0.7	1.34. -0.365547	1.35. 8.546949e-03	1.36. 8.657959e-03
1.37. 0.8	1.38. -0.289829	1.39. 1.828528e-03	1.40. 2.938528e-03
1.41. 0.9	1.42. -0.171000	1.43. 1.183452e-03	1.44. 1.294552e-03
1.45. 1.0	1.46. 0.0000000	1.47. 2.706087e-02	1.48. 2.717187e-02

For various fractional Bagley-Torvik DAE categories, Table 1 displays possible outcomes and efficiency for suggested Modified Physics-Informed Neural Network (Modified PINN). In addition, it proves that a method has been versatile enough to handle the inherent issues of FDEs, which are commonly seen in practical settings in fields like engineering, physics, and finance. By including multiple example equations into the revised PINN framework, we can observe its adaptability and robustness; furthermore, we can speculate about its potential future extension to more complex fractional equations. The table provides a brief overview for a study, suggests the kind of equations being solved, and shows how Modified PINN contributes to an advancement of numerical methods in fractional calculus.

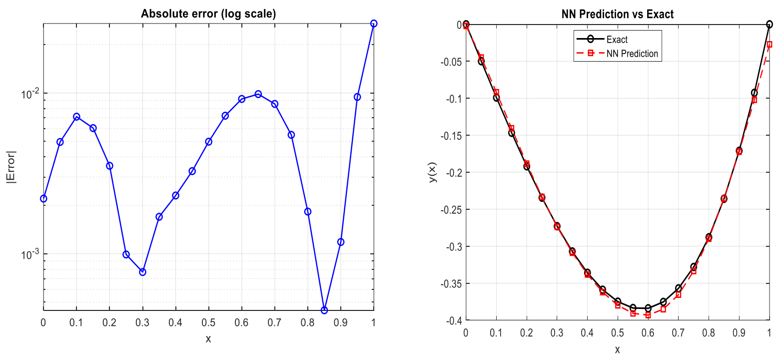

Fig. 1. The plot for absolute error (left) and the comparison for exact and approximate solution in a case of Example 1 (right).

Example 2. Nonlinear Fractional Bagley–Torvik Differential–Algebraic Model:

$$\begin{cases} ay''(t) + \mathrm{b}D_t^{\alpha}y(t) + \mathrm{c}y(t) + \lambda(t) + \gamma y^3(\mathrm{t}) = \mathrm{f(t)}, & t \in [0,2] \\ \emptyset(y(t), \lambda(t), t) = \lambda(t) - ky^2(t) = 0, & 0 < \alpha < 1 \\ y(0) = 0, y(2) = 0. & \end{cases} \quad (17)$$

The exact solution is $y(t) = t^4 - 8t$.

Both sides for Equation (17) use an updated PINN method. Figure 3 displays a solution approximation, whereas Figure 4 displays a solution relative inaccuracy for Example 2. In addition, Table 2 provides a more in-depth comparison between the suggested method's error levels and error levels found in [XXI] while solving Eq. (17). The findings show that a strategy is successful and correct, and they also show that it performs as well as, or even better than, the strategy proposed in [XXI].

Table 2: Numerical comparison among the projected PINN solution ($\boldsymbol{n} = 40$) and the corresponding error.

1.49. y	1.50. Exact sol. (Proposed Method)	1.51. Abs. Error	1.52. Error in [XXI]
1.53. 0	1.54. 0.000000	1.55. 9.923891e-04	1.56. 10.823891e-04
1.57. 0.2	1.58. -1.599975	1.59. 1.574874e-03	1.60. 2.674874e-03
1.61. 0.4	1.62. -3.178109	1.63. 3.708749e-03	1.64. 4.808749e-03
1.65. 0.6	1.66. -4.668746	1.67. 1.653600e-03	1.68. 2.753600e-03
1.69. 0.8	1.70. -5.792019	1.71. 1.983806e-01	1.72. 2.993806e-01
1.73. 1	1.74. -5.798133	1.75. 1.201867e+00	1.76. 2.301867e+00
1.77. 1.2	1.78. -5.798147	1.79. 1.728253e+00	1.80. 2.828253e+00
1.81. 1.4	1.82. -5.798148	1.83. 1.560252e+00	1.84. 2.570252e+00
1.85. 1.6	1.86. -5.798149	1.87. 4.482515e-01	1.88. 5.582515e-01
1.89. 1.8	1.90. -5.798149	1.91. 1.895749e+00	1.92. 2.995749e+00
1.93. 2	1.94. 0.000000	1.95. 5.798149e+00	1.96. 6.898149e+00

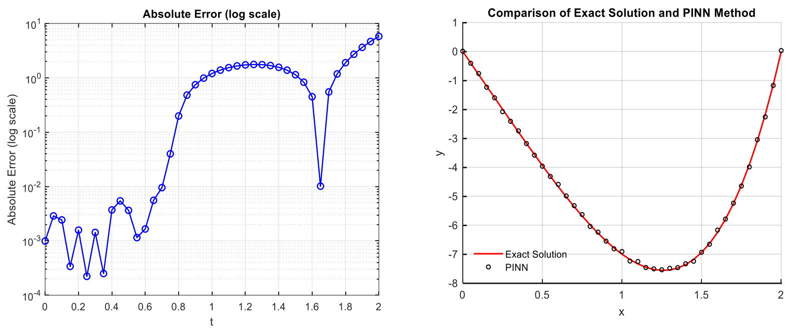

Fig. 2. Absolute error based on Example 2 (left) and comparison between an exact and approximating solutions for Example 2 (right).

The present study has several advantages: simultaneous treatment of the fractional dynamic equation and the algebraic constraint, derivation of an explicit compatibility condition by local elimination of the algebraic variable, and mesh-free neural approximation with full-history fractional residual. Moreover, the work provides a conditional statement of mathematical convergence rather than an unconditional statement of an optimizer. The work also makes a clear distinction between pointwise error of the solution and consistency of the operator. There are, however, important limitations: the revised numerical file does not contain per-epoch operator-residual histories, nor collocation/network refinement experiments; the algebraic elimination theorem is local and relies on constraint Jacobian nonsingularity; and the well-posedness argument is only for the reduced initial-value formulation, so that boundary-value implementations require the corresponding Green-operator stability condition. In addition, the fractional history evaluation is more expensive computationally with the number of collocation points. The numerical benchmarks suggest much larger errors in the nonlinear case, so universal high accuracy claims are not fully warranted.

VII. Conclusions and Future Work

A revised consistency-aware modified PINN framework has been developed for fractional Bagley–Torvik differential–algebraic models. The central mathematical correction is the explicit coupling of the hereditary fractional trajectory with the instantaneous algebraic manifold. By using an implicit-function reduction, a compact Volterra operator, and a residual-to-solution stability condition, the revised theorem establishes a defensible route from vanishing composite PINN residuals to the coupled DAE solution. This replaces the stronger but unsupported claim in the original manuscript that boundedness and Lipschitz continuity alone guaranteed convergence.

The numerical results were also reinterpreted conservatively. The supplied tables support low error for the first benchmark relative to its scale, whereas the nonlinear benchmark exhibits a pronounced error increase near the end of the interval. The revised paper reports MAE, RMSE, and maximum error rather than relying only on visual agreement. Because the original computational record does not contain operator-residual histories or refinement runs, no synthetic residual values have been introduced. A complete future validation should report residual norms as functions of collocation density, network capacity, and optimization iterations, together with the full model coefficients and forcing functions needed for independent reproduction.

Future work will focus on adaptive history-aware collocation, operational-matrix acceleration of the Caputo evaluation, automatic residual balancing, and a complete grid/network refinement study. These extensions are particularly relevant to nonlinear and stiff fractional DAEs where the cost of evaluating the hereditary operator becomes dominant.

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Data Availability

The numerical data for the submitted manuscript are confidential. The revised manuscript contains all the values shown in the document provided, and does not include any new experimental or computational observation not already reported.

Conflict of Interest:

The Authors declare that there is no conflict of interest regarding this paper

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