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NEW PROPERTIES AND THEOREMS OF ABOODH- ARA - KAMAL INTEGRAL TRANSFORMS (AAKIT)

Dilip Kumar Jaiswal[1], D. S. Singh[2], Shyamli Tripathi[3], Surekha Dewangan[4]

[1]Faculty of Mathematical and Statistical Sciences, Shri Ramswaroop Memorial University, Lucknow-Deva Road, Barabanki (U.P.), India.

[2,3]Department of Mathematics, Dr. C.V. Raman University, Kargi Road, Kota, Bilaspur (C.G.) India.

[4]Department of Mathematics, Govt. Mata Shabari Naveen Girls PG College, Bilaspur (C.G.), India.

Email:[1]dilip3jais@gmail.com, [2]dssingh2006@gmail.com, [3]sharanyatripathi8@gmail.com,[4]sure.dewap@gmail.com

Corresponding author: D. S. Singh

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Abstract

*This paper introduces new properties and theorems of the Aboodh-ARA-Kamal Integral Transform (AAKIT)***,** *a generalized fractional integral transform with wide applications in Applied Mathematics, Engineering, and Mathematical Physics. We derive and prove several fundamental properties. Furthermore, new inversion and uniqueness theorems are presented with examples, demonstrating the practical use of AAKIT in solving differential and integral equations.*

Keywords: Aboodh transform, ARA transform, Integral transform, Laplace transform; Partial differential equations and Kamal transform.

I. Introduction

Integral transforms constitute fundamental tools in Applied Mathematics, Physics, and Engineering, offering a systematic framework to convert complex differential or integral equations into more tractable algebraic forms [VIII, IX, XIII]. Classical transforms—such as the Laplace, Fourier, and Mellin transforms—have long been employed in solving linear differential equations, signal analysis, and the modeling of physical systems [I].

However, many real-world phenomena—particularly those involving memory effects, viscoelastic materials, and anomalous diffusion—demand generalized approaches that incorporate fractional-order derivatives and integrals. In this context, fractional integral transforms have emerged as powerful instruments for capturing non-local behavior and intricate dynamics. Notable examples include the Aboodh, ARA, and Kamal transforms, which extend traditional frameworks into fractional domains, thereby enhancing flexibility in the treatment of fractional differential and integral equations [II, III, IV, VI, VII, XI, XII, XIV].

Despite their individual strengths, these transforms often lack a unified analytical framework that consolidates their advantages while preserving tractability. The principal aim of this paper is to bridge this gap by introducing the Aboodh–ARA–Kamal Integral Transform (AAKIT), a unified construct designed to address relations among these three transforms. The AAKIT proves particularly effective in handling both linear and nonlinear fractional differential and integral equations [X].

The objective of this paper is to develop new properties and theorems for AAKIT, providing a comprehensive theoretical framework as follows:

The objective of this paper is to establish new properties and theorems associated with the Aboodh–ARA–Kamal Integral Transform (AAKIT), thereby constructing a comprehensive theoretical framework as outlined below.

II. Preliminaries:

Aboodh Transform: [III] In 2013, Khalid Suliman Aboodh introduced an integral transform, now known as the Aboodh Transform. This transform is defined for a function $f(k)$as follows: -

$$A\{f(k)\} = \frac{1}{\eta}\int_0^\infty f(x)e^{\eta x}dx = F(\eta), x \geq 0.$$

The inverse Aboodh transform is provided as

$$A^{-1}\{F(\eta)\} = f(x) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \eta e^{\eta x}F(\eta)d\eta, \alpha \geq 0.$$

Aboodh Transform of some elementary functions are:

(i)$A\{x^n\} = \frac{n!}{\eta^{n+2}}, n \geq 0$	(vii) $A\{1\} = \frac{1}{\eta^2}$
(ii) $A\{e^{\alpha x}\} = \frac{1}{\eta^2 - \alpha\eta}, \alpha \in R$	(viii) $A\{t\} = \frac{1}{\eta^3}$
(iii) $A\{\sin\sin(\alpha x)\} = \frac{\alpha}{\eta(\alpha^2+\eta^2)}$	(ix) $A(sinat) = \frac{a}{\eta(\eta^2+\alpha^2)}$
(iv) $A\{\cos\cos(\alpha x)\} = \frac{1}{(\alpha^2+\eta^2)}$	(x) $A(cosat) = \frac{1}{(\eta^2+\alpha^2)}$
(v) $A\{\sinh\sinh(\alpha x)\} = \frac{\alpha}{\eta(\eta^2-\alpha^2)}$	(xi) $A(sinhat) = \frac{a}{\eta(\eta^2-\alpha^2)}$
(vi) $A\{\cosh\cosh(\alpha x)\} = \frac{1}{(\eta^2-\alpha^2)}$	(xii) $A(coshat) = \frac{1}{(\eta^2-\alpha^2)}$

Aboodh Transform of Derivatives:

Let$A\{f(x)\} = F(\eta)$, then

(i) $A\{f'(x)\} = \eta F(\eta) - \frac{f(0)}{\eta}$

(ii) $A\{f''(x)\} = \eta^2 F(\eta) - \frac{f'(0)}{\eta} - f(0)$

(iii) $A\{f^n(x)\} = \eta^n F(\eta) - \sum_{j=0}^{n-1}\frac{f^{(j)}(0)}{\eta^{2-n+j}}, \quad n = 1,2,3 \dots\dots\dots$

ARA Transform: [XII] The ARA Integral Transform of order n, defined for a continuous function $f(y)$on the interval $(0, \infty)$, is given by:

$$G_n[f(y)](p) = p\int_0^\infty t^{n-1}e^{-py}f(y)dy,\ \ p > 0,\ \ n \in N.$$

$$n = 1, G_1[f(y)](p) = p\int_0^\infty e^{-py}f(y)dy,\ \ p > 0 = F(p)$$

The inverse of the ARA transform is given by

$$G^{-1}\{F(p)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{py}}{p}F(p)dp = f(y)$$

ARA Transform of some elementary functions:

(i) $G(1) = 1$ (ii) $G(y^a) = \frac{\Gamma(a+1)}{p^a}$

(iii) $G(e^{ay}) = \frac{\mathrm{p}}{p-a}$ (iv) $G(\sin(ay)) = \frac{\mathrm{ap}}{p^2+a^2}$

(v) $G(\cos(ay)) = \frac{p^2}{p^2+a^2}$ (vi) $G(\sinh(ay)) = \frac{\mathrm{ap}}{p^2-a^2}$

(vii) $G(\cosh(ay)) = \frac{p^2}{p^2-a^2}$ (viii) $G(f'(y)) = pG[f(y)] - pf(0)$

(ix) $G(f^n(y)) = p^nG(f(y)) - \sum_{k=1}^n p^{n-k+1}f^{(k-1)}(0)$

III. Kamal Transform

[V] The Kamal Transform, proposed by Abdelilah Kamal et al., is defined for a function $f(t)$as follows:

$$K\{f(t)\} = \int_0^\infty e^{-t/q}f(t)dt\ F(q),\ \ q > 0$$

The inverse Kamal Transform is provided as $K^{-1}\{F(q)\} = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{t/q}F\left(\frac{1}{q}\right)dq$

The Kamal Transform of some elementary function:

(i) $K(t^n) = n!\ q^{n+1}, n \in N$ (vii) $K(1) = q$

(ii) $K(e^{\alpha t}) = \frac{q}{1-\alpha q}, \alpha \in R$ (viii) $K(t) = q^2$

(iii) $K\{\sin\sin(\alpha t)\} = \frac{\alpha q^2}{1+\alpha^2q^2}$ (ix) $K\{\sin(\alpha t)\} = \frac{\alpha q^2}{1+\alpha^2q^2}$

(iv) $K\{\cos\cos(\alpha t)\} = \frac{q}{1+\alpha^2q^2}$ (x) $K\{\cos(\alpha t)\} = \frac{q}{1+\alpha^2q^2}$

(v) $K\{\sinh\sinh(\alpha t)\} = \frac{\alpha q^2}{1-\alpha^2q^2}$ (xi) $K\{\sinh(\alpha t)\} = \frac{\alpha q^2}{1-\alpha^2q^2}$

(vi) $K\{\cosh\cosh(\alpha t)\} = \frac{q}{1-\alpha^2q^2}$ (xii) $K\{\cosh(\alpha t)\} = \frac{q}{1-\alpha^2q^2}$

Kamal Transform of Derivatives: Let $K\{f(t)\} = F(q), then$

(i) $K\{f'(t)\} = \frac{1}{q}F(q) - f(0)$

(ii) $K\{f''(t)\} = \frac{1}{q^2}F(q) - \frac{1}{q}f(0) - f'(0)$

(iii) $K\{f^n(t)\} = \frac{1}{q^n}F(q) - \sum_{j=0}^{n-1}\frac{f^{(j)}(0)}{q^{n-j-1}},\quad n = 1,2,3 \ldots\ldots\ldots$

IV. Double Aboodh-Kamal Transform (DAKT):

Definition 4.1: **[V]** The DAKT is defined as

$$A_\lambda K_f[z(x,t)] = z(\eta,q) = \frac{1}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}} z(x,t)dxdt$$

It can be shown that the DAKT is a linear integral transform

$$A_\lambda K_t[az(x,t)+b\psi(x,t)] = aA_\lambda K_f[az(x,t)] + bA_\lambda K_f[b\psi(x,t)]$$

where $a\,\&\,b$ are constant and$z(x,t), \psi(x,t)$ are continuous functions in which DAKT exists.

Definition 4.2: **[V]** The inverse of DAKT is defined by

$$A_\lambda{}^{-1}K_f{}^{-1}[z(\eta,q)] = z(x,t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \eta e^{\eta x} d\eta \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{t/q} z(\eta,1/q)dq$$

$$= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} \eta e^{\eta x}\int_{\beta-i\infty}^{\beta+i\infty} e^{t/q} z(\eta,1/q)d\eta dq.$$

Basic Properties of Double Aboodh-Kamal Transform (DAKT):

(i). Let $z(x,t) = 1$. Then

$$A_\lambda K_t[1] = \frac{1}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}}dxdt = \frac{1}{\eta}\left\{\frac{e^{-\eta x}}{-\eta}\right\}_0^\infty \left\{\frac{e^{-\frac{t}{q}}}{-\frac{1}{q}}\right\}_0^\infty = \frac{1}{\eta}(\frac{1}{-\eta})(0-1)(-\frac{1}{q})(0-1) = \frac{q}{\eta^2}.$$

(ii). Let $z(x,t) = e^{\alpha x+\beta t}$. Then

$$A_\lambda K_t\left[e^{\alpha x+\beta t}\right] = \frac{1}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}} e^{\alpha x+\beta t}dxdt$$

$$= \frac{1}{\eta}\int_0^\infty\int_0^\infty e^{-(\eta-\alpha)x} e^{-(\frac{1}{q}-\beta)t}dxdt = \frac{1}{\eta}\left\{\frac{e^{-(\eta-\alpha)x}}{-(\eta-\alpha)}\right\}_0^\infty \left\{\frac{e^{-(\frac{1}{q}-\beta)t}}{-(\frac{1}{q}-\beta)}\right\}_0^\infty$$

$$= \frac{1}{\eta}(\frac{1}{-(\eta-\alpha)})(0-1)(\frac{-1}{(\frac{1}{q}-\beta)})(0-1) = \frac{q}{\eta(\eta-\alpha)(1-\beta q)}.$$

(iii). Let $z(x,t) = e^{i(\alpha x+\beta t)}$. Then

$$A_\lambda K_t\left[e^{i(\alpha x+\beta t)}\right] = \frac{1}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}} e^{i(\alpha x+\beta t)}dxdt = \frac{q}{\eta(\eta-i\alpha)(1-i\beta q)}$$

$$= \frac{q(\eta+i\alpha)(1+i\beta q)}{\eta(\eta-i\alpha)(1-i\beta q)(\eta+i\alpha)(1+i\beta q)}$$

$$= \frac{q(\eta+i\beta\eta q+i\alpha-\alpha\beta q)}{\eta(\eta^2+\alpha^2)(1+\beta^2q^2)} = \frac{q\{(\eta-\alpha\beta q)+i(\alpha+\beta\eta q)\}}{\eta(\eta^2+\alpha^2)(1+\beta^2q^2)}$$

Therefore

$$A_\lambda K_t[cos(\alpha x+\beta t)] = \frac{q(\eta-\alpha\beta q)}{\eta(\eta^2+\alpha^2)(1+\beta^2q^2)}$$

and $\quad A_\lambda K_t[sin(\alpha x+\beta t)] = \frac{q(\alpha+\beta\eta q)}{\eta(\eta^2+\alpha^2)(1+\beta^2q^2)}$.

V. Double Aboodh-ARA transform (DAARAT): [XII]

Definition 5.1: The DAARAT is defined as

$$A_\lambda G_y[z(x,y)] = z(\eta,p) = \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-(\eta x+py)}z(x,y)dxdy$$

It can be shown that the DAARAT is a linear integral transform

$$A_\lambda G_y[az(x,y)+b\psi(x,y)] = aA_\lambda G_y[az(x,y)] + bA_\lambda G_y[b\psi(x,y)]$$

where a & b are constant and $z(x,y), \psi(x,y)$ are two continuous functions in which DAARAT exists.

Definition 5.2: The inverse DAARAT $A_\lambda{}^{-1}G_y{}^{-1}$ is defined by

$$A_\lambda{}^{-1}\left[G_y{}^{-1}[z(\eta,p)]\right] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\eta e^{\eta x}dx\int_{\beta-i\infty}^{\beta+i\infty}\frac{e^{py}}{p}\,z(\eta,p)dp = z(x,y).$$

Basic Properties of Double Aboodh-ARA transform (DAARAT):

(i) Let $z(x,y) = 1$. Then

$$A_\lambda G_y[1] = \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-(\eta x+py)}dxdy = \frac{p}{\eta}\left\{\frac{e^{-\eta x}}{-\eta}\right\}_0^\infty\left\{\frac{e^{-py}}{-p}\right\}_0^\infty$$

$$= \frac{p}{\eta}.\frac{1}{-\eta}(0-1).\frac{1}{-p}.(0-1) = \frac{1}{\eta^2}.$$

(ii) Let $z(x,y) = x^\alpha y^\beta, \alpha \,\&\beta > -1$. Then

$$A_\lambda G_y\left[x^\alpha y^\beta\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-(\eta x+py)}x^\alpha y^\beta dxdy$$

$$= \Gamma(\alpha+1)\frac{1}{\eta^{\alpha+1}}.\Gamma(\beta+1)\frac{1}{p^{\beta+1}}.$$

(iii) Let $z(x,y) = e^{\alpha x+\beta y}$. Then

$$A_\lambda G_y\left[e^{\alpha x+\beta y}\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-(\eta x+py)}e^{\alpha x+\beta t}dxdy$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-(\eta-\alpha)x}.e^{-(p-\beta)y}dxdy = \frac{p}{\eta}\left\{\frac{e^{-(\eta-\alpha)x}}{-(\eta-\alpha)}\right\}_0^\infty\left\{\frac{e^{-(p-\beta)y}}{-(p-\beta)}\right\}_0^\infty$$

$$= \frac{p}{\eta}\left(\frac{-1}{\eta-\alpha}\right)(0-1).\left(\frac{-1}{p-\beta}\right)(0-1) = \frac{p}{\eta(\eta-\alpha)(p-\beta)}.$$

(iv) Let $z(x,y) = e^{i(\alpha x+\beta y)}$. Then

$$A_\lambda G_y\left[e^{\alpha x+\beta y}\right] = \frac{p}{\eta(\eta-i\alpha)(p-i\beta)} = \frac{p(\eta+i\alpha)(p+i\beta)}{\eta(\eta^2+\alpha^2)(p^2+\beta^2)}$$

$$= \frac{p(\eta p+\eta\beta i+i\alpha p-\alpha\beta)}{\eta(\eta^2+\alpha^2)(p^2+\beta^2)} = \frac{p\{(\eta p-\alpha\beta)+i(\eta\beta+\alpha p)\}}{\eta(\eta^2+\alpha^2)(p^2+\beta^2)}$$

From above $A_\lambda G_y[\cos(\alpha x+\beta y)] = \frac{p(\eta p-\alpha\beta))}{\eta(\eta^2+\alpha^2)(p^2+\beta^2)}$

and $$A_\lambda G_y[\sin(\alpha x+\beta y)] = \frac{p(\eta\beta+\alpha p)}{\eta(\eta^2+\alpha^2)(p^2+\beta^2)}.$$

VI. Double ARA- Kamal transform (DARAKT):

Definition 6.1: **[XI]** The DARAKT is defined as

$$G_yK_t[z(y,t)] = z(p,q) = p\int_0^\infty\int_0^\infty e^{-(py+-\frac{t}{q})} z(y,t)dydt\ ; p,q>0.$$

It can be demonstrated that the DARAKT constitutes a linear integral transform.

$$G_yK_t[az(y,t)+b\psi(y,t)] = aG_yK_t[z(y,t)] + bG_yK_t[\psi(y,t)]$$

where $a\&b$ are constants and $z(y,t),\psi(y,t)$ are two continuous functions in which DARAKT exists.

Definition 6.2: The inverse DAARAT $G_y{}^{-1}K_t{}^{-1}$ is defined by

$$G_y{}^{-1}[K_t{}^{-1}[z(p,q)]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{py}}{p}dp\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{t/q}z(p,1/q)dq$$

$$= \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\frac{e^{py}}{p}\int_{\beta-i\infty}^{\beta+i\infty} e^{t/q}\,z(p,1/q)dpdq.$$

Basic Properties of Double ARA- Kamal transform (DARAKT):

(i) Let $z(y,t)=1$. Then

$$G_yK_t[z(y,t)] = G_yK_t[1] = p\int_0^\infty\int_0^\infty e^{-(py+-\frac{t}{q})}dydt \ = p\left\{\frac{e^{-py}}{-p}\right\}_0^\infty\left\{\frac{e^{-\frac{t}{q}}}{-\frac{1}{q}}\right\}_0^\infty =$$

$$p\left(\frac{0-1}{-p}\right)\left(\frac{0-1}{-\frac{1}{q}}\right) = q.$$

(ii) Let $z(y,t)=e^{(\alpha y+\beta t)}$. Then

$$G_yK_t[z(y,t)] = p\int_0^\infty\int_0^\infty e^{-(py+-\frac{t}{q})}e^{(\alpha y+\beta t)}dydt \ = p\int_0^\infty\int_0^\infty e^{-(p-\alpha)y}e^{-(\frac{1}{q}-\beta)t}dydt$$

$$= p\left\{\frac{e^{-(p-\alpha)y}}{-(p-\alpha)}\right\}_0^\infty\left\{\frac{e^{-(\frac{1}{q}-\beta)t}}{-(\frac{1}{q}-\beta)}\right\}_0^\infty = p(\frac{0-1}{-(p-\alpha)}).\frac{(0-1)}{-(\frac{1}{q}-\beta)} = \frac{pq}{(p-\alpha)(1-q\beta)}.$$

(iii) Let $z(y,t)=e^{i(\alpha y+\beta t)}$. Then from (ii)

$$G_yK_t[z(y,t)] = \frac{pq}{(p-i\alpha)(1-iq\beta)}$$

$$= \frac{pq(p+i\alpha)(1+iq\beta)}{(p-i\alpha)(1-iq\beta)(p+i\alpha)(1+iq\beta)}$$

$$= \frac{pq(p+ip\beta q+i\alpha-\alpha\beta q)}{(p^2+\alpha^2)(1+\beta^2q^2)} = \frac{pq\{(p-\alpha\beta q)+i(\alpha+p\beta q)\}}{(p^2+\alpha^2)(1+\beta^2q^2)}$$

Therefore, from above

$$G_yK_t[\cos(\alpha y+\beta t)] = \frac{pq(p-\alpha\beta q)}{(p^2+\alpha^2)(1+\beta^2q^2)}$$

$$G_yK_t[\sin(\alpha y+\beta t)] = \frac{pq(\alpha+p\beta q)}{(p^2+\alpha^2)(1+\beta^2q^2)}.$$

VII. Main Results

In this section, we establish the definition, fundamental properties, and theorems of the Triple Aboodh–ARA–Kamal Transform (TAARAKT).

Definition 7.1: Let $z(x,y,t)$ be a continuous function of three variables x,y,t and $t>0$. TAARAKT of $z(x,y,t)$ is defined by

$$A_xG_yK_t[z(x,y,t)] = \emptyset(\eta,p,q) = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}z(x,y,t)dxdydt$$

where $\eta,p,q>0$.

The inverse of the Triple Aboodh–ARA–Kamal Transform (TAARAKT) is defined as follows:

$$A_x^{-1}G_y^{-1}K_t^{-1}[\emptyset(\eta,p,q)] = z(x,y,t)$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\eta e^{\eta x}d\eta\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{py}}{p}dp\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}e^{t/q}\emptyset(\eta,p,q)dq$$

where α,β,c are real constants.

Property 7.2 (Linearity):

If $A_xG_yK_t[z(x,y,t)] = \emptyset(\eta,p,q)$ and $A_xG_yK_t[\psi(x,y,t)] = \psi(\eta,p,q)$ then for any constant A and B, we have

$$A_xG_yK_t[Az(x,y,t)+B\psi(x,y,t)] = A\,A_xG_yK_t[z(x,y,t)] + B\,A_xG_yK_t[\psi(x,y,t)]$$

Proof: Now

$$A_xG_yK_t[Az(x,y,t)+B\psi(x,y,t)]$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[Az(x,y,t)+B\psi(x,y,t)]dxdydt$$

$$= A\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}z(x,y,t)dxdydt + B\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\psi(x,y,t)dxdydt$$

$$= A\,A_xG_yK_t[z(x,y,t)] + B\,A_xG_yK_t[\psi(x,y,t)]$$

Thus, the Triple Aboodh–ARA–Kamal Transform (TAARAKT) is a linear integral transform. Likewise, its inverse transform also preserves linearity.

Property 7.3 (Shifting Property):

If $A_xG_yK_t[z(x,y,t)] = \emptyset(\eta,p,q)$ then for any real constants a,b and c, we have

$$A_xG_yK_t[e^{ax+by+ct}z(x,y,t)] = \frac{p(\eta-a)}{\eta(p-b)}\phi\left(\eta-a,p-b,\frac{q}{1-cq}\right).$$

Proof: Now

$$A_xG_yK_t[e^{ax+by+ct}z(x,y,t)] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[e^{ax+by+ct}z(x,y,t)]dxdydt$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\eta-a)x-(p-b)y-\left(\frac{q}{1-cq}\right)t}[z(x,y,t)]dxdydt$$

$$= \frac{p(\eta-a)}{\eta(p-b)}\left(\frac{p-b}{\eta-a}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\eta-a)x-(p-b)y-\left(\frac{q}{1-cq}\right)t}[z(x,y,t)]dxdydt\right)$$

$$= \frac{p(\eta-a)}{\eta(p-b)}\phi\left(\eta-a,p-b,\frac{q}{1-cq}\right).$$

Property 7.4: Let $z(x,y,t) = f(x)h(y)g(t); x,y,t > 0$. Then

$$A_xG_yK_t[z(x,y,t)] = A_x[f(x)]G_y[h(y)]K_t[g(t)].$$

Proof: Now

$$A_xG_yK_t[z(x,y,t)] = A_xG_yK_t[f(x)h(y)g(t)]$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[f(x)h(y)g(t)]dxdydt$$

$$= \frac{1}{\eta}\int_0^\infty e^{-\eta x}f(x)dx.\,p\int_0^\infty e^{-py}h(y)dy.\int_0^\infty e^{-\frac{t}{q}}g(t)dt = A_x[f(x)]G_y[h(y)]K_t[g(t)].$$

We now proceed to compute the Triple Aboodh–ARA–Kamal Transform (TAARAKT) for certain essential functions.

(i) Let $z(x,y,t) = 1$. Then

$$A_xG_yK_t[1] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}dxdydt = \frac{1}{\eta}\int_0^\infty e^{-\eta x}dx.\,p\int_0^\infty e^{-py}dy.\int_0^\infty e^{-\frac{t}{q}}dt$$

$$= A_x[1]G_y[1]K_t[1] = \frac{1}{\eta^2}.1.q = \frac{q}{\eta^2}.$$

(ii) Let $z(x,y,t) = x^\alpha y^\lambda t^\beta;\ x,y,t > 0$ and α,β,γ are positive constants. Then

$$A_xG_yK_t\left[x^\alpha y^\lambda t^\beta\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[x^\alpha y^\lambda t^\beta]dxdydt$$

$$= \frac{1}{\eta}\int_0^\infty e^{-\eta x}x^\alpha dx.\,p\int_0^\infty e^{-py}y^\lambda dy.\int_0^\infty e^{-\frac{t}{q}}t^\beta dt$$

$$= \frac{1}{\eta}\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}}.\frac{\Gamma(\lambda+1)}{p^\lambda}\Gamma(\beta+1)q^{\beta+1}.$$

(iii) Let $z(x,y,t) = e^{ax+by+ct};\ x,y,t > 0$ and a,b,c are constants. Then

$$A_xG_yK_t[e^{ax+by+ct}] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}e^{ax+by+ct}dxdydt$$

$$= \frac{1}{\eta}\int_0^\infty e^{-\eta x}e^{ax}dx.\,p\int_0^\infty e^{-py}e^{by}dy.\int_0^\infty e^{-\frac{t}{q}}e^{ct}dt$$

$$=\frac{1}{\eta}\int_0^\infty e^{-(\eta-a)x}dx.p\int_0^\infty e^{-(p-b)y}dy.\int_0^\infty e^{-\left(\frac{1-cq}{q}\right)t}dt$$

$$=\frac{1}{\eta}\frac{1}{\eta-a}.p\frac{1}{p-b}.\frac{1}{\left(\frac{1-cq}{q}\right)}=\frac{pq}{\eta(\eta-a)(p-b)(1-cq)}.$$

(iv) Similarly,

$$A_xG_yK_t\left[e^{i(ax+by+ct)}\right]=\frac{pq}{\eta(\eta-ia)(p-ib)(1-icq)}$$

$$=\frac{pq(\eta-ia)(p-ib)(1-icq)}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

$$=\frac{pq(\eta p+i\eta b+iap-ab)(1-icq)}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

$$=\frac{pq\{\eta p+i\eta b+iap-ab+i\eta cpq-\eta bcq-acpq-iabcq\}}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

$$=\frac{pq\{(\eta p-ab-\eta bcq-acpq)+i(\eta b+ap+\eta cpq-abcq)\}}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

Thus, we can obtain

$$A_xG_yK_t\left[e^{i(ax+by+ct)}\right]=\frac{pq\{(\eta p-ab-\eta bcq-acpq)+i(\eta b+ap+\eta cpq-abcq)\}}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

Using Euler's formula

$$sinx=\frac{e^{ix}-e^{-ix}}{2i}, cosx=\frac{e^{ix}+e^{-ix}}{2}, sinhx=\frac{e^{x}-e^{-x}}{2}\ and\ cosh=\frac{e^{x}+e^{-x}}{2}$$

Consequently, the TAARAKT of certain essential functions can be obtained as follows:

$$A_xG_yK_t[\sin(ax+by+ct)]=\frac{pq(ab+\eta b+\eta cpq-abcq)}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

$$A_xG_yK_t[\cos(ax+by+ct)]=\frac{pq(\eta p-ab-\eta bcq-acpq)}{\eta(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

(v). Let $z(x,y,t)=sinax.sinby.sinct; x,y,t>0\ and\ a,b,c$ are constants. Then

$$A_xG_yK_t[sinax.sinby.sinct]=\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[sinax.sinby.sinct]dxdydt$$

$$=\frac{1}{\eta}\int_0^\infty e^{-\eta x}sinaxdx.p\int_0^\infty e^{-py}sinbydy.\int_0^\infty e^{-\frac{t}{q}}sinctdt$$

Thus, we get, using the properties of single transforms

$$=\frac{1}{\eta}.\frac{a}{\eta(\eta^2+a^2)}.p\frac{bp}{(p^2+b^2)}.\frac{cq^2}{(1+c^2q^2)}=\frac{abcp^2q^2}{\eta^2(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}$$

Therefore

$$A_xG_yK_t[sinax.sinby.sinct]=\frac{abcp^2q^2}{\eta^2(\eta^2+a^2)(p^2+b^2)(1+c^2q^2)}.$$

Definition7.5: If $z(x,y,t)$ is defined on $[o,X]\times[o,Y]\times[o,T]$ and satisfies the condition
$|z(x,y,t)|\leq Re^{\alpha x+\beta y+\gamma t}, for\ all\ R>0, x>X, y>Y, t>t$. Then, $z(x,y,t)$ is said to be a function of exponential orders $\alpha,\beta\ and\ \gamma\ as\ x,y,t\to\infty$.

Existence Condition: Let $f(x,y,t)$ be defined on $R_+^3=(0,\infty)^3$, an admissible space of the TAARAKT is

$$A=\{f:R_+^3\to C; f\ is\ piecewise\ continuous\ on\ every\ finite\ value\ on\ defined\ range$$

Such that $\exists\ M,\alpha,\beta,\gamma>0$,

$$|f(x,y,t)|\leq Me^{\alpha x+\beta y+\gamma t}.$$

So, the transform is defined as

$$A_{p,s,u}(f)=\frac{s}{p}\int_0^\infty\int_0^\infty\int_0^\infty e^{-px-sy-\frac{t}{u}}f(x,y,t)dxdydt.$$

Now $\qquad |f(x,y,t)|\leq Me^{\alpha x+\beta y+\gamma t}$

and the absolute value of the above transform is bounded by

$$\left|A_{p,s,u}(f)\right|\leq\frac{Ms}{p}\int_0^\infty e^{-(p-\alpha)x}dx\int_0^\infty e^{-(s-\beta)y}dy\int_0^\infty e^{-(1/u-\gamma)t}dt,$$

where $p>\alpha, s>\beta, \frac{1}{u}>\gamma$.

Theorem 7.6: The existence condition of Triple Aboodh-ARA-Kamal transform (TAARAKT) of the continuous function $z(x,y,t)$ defined on $[o,X]\times[o,Y]\times[o,T]$ is to be of exponential orders $\alpha,\beta\ \&\ \gamma$ for $Re(\eta)>\alpha, Re(p)>\beta\ \ and\ \ Re\left(\frac{1}{q}\right)>\gamma.$

Proof: The definition of Triple Aboodh-ARA-Kamal transform (TAARAKT) implies that

$$\left|A_xG_yK_t[z(x,y,t)]\right|=|\varphi(p,q,\eta)|=\left|\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}z(x,y,t)dxdydt\right|$$

$$\leq\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}|z(x,y,t)|dxdydt$$

$$\leq\frac{Rp}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}e^{\alpha x+\beta y+\gamma t}dxdydt$$

$$\leq\frac{Rp}{\eta}\int_0^\infty e^{-(\eta-\alpha)x}dx\int_0^\infty e^{-(p-\beta)y}dy\int_0^\infty e^{-(\frac{1}{q}-\gamma)t}dt\leq\frac{Rp}{\eta}\frac{q}{(\eta-\alpha)(p-\beta)(1-\gamma q)}$$

$$\leq\frac{Rpq}{\eta(\eta-\alpha)(p-\beta)(1-\gamma q)}; Re(\eta)>\alpha, Re(p)>\beta and Re\left(\frac{1}{q}\right)>\gamma.$$

Definition 7.7: The convolution of $z(x,y,t)$ and $\psi(x,y,t)$ is denoted by $(z***\psi)(x,y,t)$ and defined by

$$(z***\psi)(x,y,t)=\int_0^x\int_0^y\int_0^t z(x-\delta,y-\epsilon,t-\sigma)\,\psi(\delta,\epsilon,\sigma)d\delta d\epsilon d\sigma.$$

Theorem 7.8: Let $A_xG_yK_t[z(x,y,t)] = \varphi(\eta,p,q)$. Then

$$A_xG_yK_t[z(x-\delta,y-\epsilon,t-\sigma)H(x-\delta,y-\epsilon,t-\sigma)] = e^{-\eta\delta-p\epsilon-\frac{\sigma}{q}}\varphi(\eta,p,q.)$$

Where $H(x,y,t)$ denotes the Heaviside function denoted by

$$H(x-\delta,y-\epsilon,t-\sigma) = \begin{cases} 1; & x>\delta, y>\epsilon, t>\sigma \\ 0, & otherwise \end{cases}$$

Proof: Now

$$A_xG_yK_t[z(x-\delta,y-\epsilon,t-\sigma)H(x-\delta,y-\epsilon,t-\sigma)]$$

$$=\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[z(x-\delta,y-\epsilon,t-\sigma)H(x-\delta,y-\epsilon,t-\sigma)]dxdydt$$

$$=\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}[z(x-\delta,y-\epsilon,t-\sigma)]dxdydt \tag{1}$$

Putting $x-\delta=\rho, y-\epsilon=\beta, t-\sigma=\lambda$ in equation (1), we obtain

$$A_xG_yK_t[z(x-\delta,y-\epsilon,t-\sigma)H(x-\delta,y-\epsilon,t-\sigma)]$$

$$=\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta(\delta+\rho)-p(\epsilon+\beta)-\frac{1}{q}(\sigma+\lambda)}z(\rho,\beta,\lambda)d\rho d\beta d\lambda \tag{2}$$

Thus, equation (2) can be simplified into

$$A_xG_yK_t[z(x-\delta,y-\epsilon,t-\sigma)H(x-\delta,y-\epsilon,t-\sigma)]$$

$$=e^{-\eta\delta-p\epsilon-\frac{1}{q}\sigma}\left(\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta\rho-p\beta-\frac{1}{q}\lambda}z(\rho,\beta,\lambda)d\rho d\beta d\lambda\right) = e^{-\eta\delta-p\epsilon-\frac{\sigma}{q}}\varphi(\eta,p,q.).$$

Theorem 7.9 (Convolution Theorem):

If $A_xG_yK_t[z(x,y,t)] = \varphi(\eta,p,q)$ and $A_xG_yK_t[\psi(x,y,t)] = \psi(\eta,p,q)$. Then

$$A_xG_yK_t[(z *** \psi)](x,y,t) = \frac{p}{\eta}\psi(\eta,p,q)\varphi(\eta,p,q).$$

Proof: From the definition of Triple Aboodh-ARA-Kamal transform (TAARAKT), we have

$$A_xG_yK_t[(z *** \psi)](x,y,t)$$

$$=\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\left[\int_0^x\int_0^y\int_0^t z(x-\delta,y-\epsilon,t-\sigma)\,\psi(\delta,\epsilon,\sigma)d\delta d\epsilon d\sigma\right]dxdydt \tag{3}$$

By the definition of the Heaviside function, Equation (1) can be written as;

$$A_xG_yK_t[(z *** \psi)](x,y,t) = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}$$

$$\left[\int_0^\infty\int_0^\infty\int_0^\infty z(x-\delta,y-\epsilon,t-\sigma)\,H(x-\delta,y-\epsilon,t-\sigma)\,\psi(\delta,\epsilon,\sigma)d\delta d\epsilon d\sigma\right]dxdydt \tag{4}$$

Using Theorem 7.8, we have

$$A_x G_y K_t[(z *** \psi)](x,y,t) = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x - py - \frac{t}{q}}\psi(\delta,\epsilon,\sigma)d\delta d\epsilon d\sigma.$$

$$\left[\frac{\eta}{p}\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty z(\rho,\beta,\lambda)e^{-\eta\rho-\beta p-\frac{\lambda}{q}}\,d\rho d\beta d\lambda\right]$$

$$= \frac{\eta}{p}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\delta\eta-\epsilon p-\frac{\sigma}{q}}\psi(\delta,\epsilon,\sigma)\varphi(\eta,p,q)d\delta d\epsilon d\sigma = \frac{p}{\eta}\psi(\eta,p,q)\varphi(\eta,p,q).$$

Theorem7.10 (Derivative Properties):

If $A_x G_y K_t[z(x,y,t)] = \varphi(\eta,p,q)$, then

(i) $A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial x}\right] = \eta\varphi(\eta,p,q) - \frac{1}{\eta}G_y K_t[z(0,y,t)]$.

(ii) $A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial y}\right] = p\varphi(\eta,p,q) - A_x K_t[z(x,0,t)]$.

(iii) $A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial t}\right] = -A_x G_y[z(x,y,0)] + \frac{1}{q}\varphi(\eta,p,q)$.

(iv) $A_x G_y K_t\left[\frac{\partial^2 z(x,y,t)}{\partial x^2}\right] = \eta^2\varphi(\eta,p,q) - \frac{1}{\eta}G_y K_t[z_x(0,y,t)] - G_y K_t[z(0,y,t)]$.

(v) $A_x G_y K_t\left[\frac{\partial^2 z(x,y,t)}{\partial y^2}\right] = p^2\varphi(\eta,p,q) - pA_x K_t[z(x,0,t)] - A_x K_t\left[z_y(x,0,t)\right]$.

(vi) $A_x G_y K_t\left[\frac{\partial^2 z(x,y,t)}{\partial t^2}\right] = -A_x G_y[z_t(x,y,0)] - \frac{1}{q}A_x G_{y_t}[z(x,y,0)] + \frac{1}{q^2}\varphi(\eta,p,q)$.

Proof: Based on the definition of Triple Aboodh-ARA-Kamal transform (TAARAKT), we have

(i) $A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial x}\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial x}dxdydt$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-py-\frac{t}{q}}[\int_0^\infty e^{-\eta x}\frac{\partial z(x,y,t)}{\partial x}dx]dydt \tag{5}$$

First, we solve $\int_0^\infty e^{-\eta x}\frac{\partial z(x,y,t)}{\partial x}dx$

Applying integration by parts

$$\int_0^\infty e^{-\eta x}\frac{\partial z(x,y,t)}{\partial x}dx = [e^{-\eta x}z(x,y,t)]_0^\infty + \eta\int_0^\infty e^{-\eta x}z(x,y,t)dx$$

$$= -z(0,y,t) + \eta\int_0^\infty e^{-\eta x}z(x,y,t)dx \tag{6}$$

Putting the above value of equation (6) in equation (5), we get

$$A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial x}\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-py-\frac{t}{q}}\left\{-z(0,y,t) + \eta\int_0^\infty e^{-\eta x}z(x,y,t)dx\right\}dydt$$

$$= \eta\varphi(\eta,p,q) - \frac{1}{\eta}G_y K_t[z(0,y,t)].$$

(ii) Now $A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial y}\right] = \frac{p}{\eta}\int_0^\infty \int_0^\infty \int_0^\infty e^{-\eta x - py - \frac{t}{q}} \frac{\partial z(x,y,t)}{\partial y} dxdydt$

$$= \frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-\eta x-\frac{t}{q}}[\int_0^\infty e^{-py}\frac{\partial z(x,y,t)}{\partial y}dy]dxdt \tag{7}$$

First, we solve$\int_0^\infty \int_0^\infty e^{-py}\frac{\partial z(x,y,t)}{\partial y}dy$

Applying integration by parts

$$\int_0^\infty \int_0^\infty e^{-py}\frac{\partial z(x,y,t)}{\partial y}dy = [e^{-\eta y}z(x,y,t)]_0^\infty + p\int_0^\infty e^{-py}z(x,y,t)dy$$

$$= -z(x,0,t) + p\int_0^\infty e^{-py}z(x,y,t)dy \tag{8}$$

Putting the above value of equation (8) in equation (7), we get

$$A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial y}\right] = \frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-py-\frac{t}{q}}\left\{-z(x,0,t) + p\int_0^\infty e^{-py}z(x,y,t)dy\right\}dxdt$$

$$= -\frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-py-\frac{t}{q}}z(x,0,tdxdt + p\left\{\frac{p}{\eta}\int_0^\infty \int_0^\infty \int_0^\infty e^{-\eta x-py-\frac{t}{q}}z(x,y,t)dxdydt\right\}$$

$$= -pA_x K_t[z(x,0,t)] + p\varphi(\eta,p,q) = p\varphi(\eta,p,q) - A_x K_t[z(x,0,t)]$$

(iii). Now

$$A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial t}\right] = \frac{p}{\eta}\int_0^\infty \int_0^\infty \int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dxdydt$$

$$= \frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-\eta x-py}[\int_0^\infty e^{-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dt]dxdy \tag{9}$$

First, we solve $\int_0^\infty \int_0^\infty e^{-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dt$

Applying integration by parts

$$\int_0^\infty \int_0^\infty e^{-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dt = \left[e^{-\frac{t}{q}}z(x,y,t)\right]_0^\infty + \frac{1}{q}\int_0^\infty e^{-\frac{t}{q}}z(x,y,t)dt$$

$$= -z(x,y,0) + \frac{1}{q}\int_0^\infty e^{-\frac{t}{q}}z(x,y,t)dt \tag{10}$$

Putting the above value of equation (10) in equation (9), we get

$$A_x G_y K_t\left[\frac{\partial z(x,y,t)}{\partial t}\right] = \frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-\eta x-py}\left\{-z(x,y,0) + \frac{1}{q}\int_0^\infty e^{-\frac{t}{q}}z(x,y,t)dt\right\}dxdy$$

$$= \frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-\eta x-py}z(x,y,0)dxdy + \frac{1}{q}\left\{\frac{p}{\eta}\int_0^\infty \int_0^\infty \int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dxdydt\right\}$$

$$= -A_x G_y[z(x,y,0)] + \frac{1}{q}\varphi(\eta,p,q).$$

(iv). Now

$$A_x G_y K_t\left[\frac{\partial^2 z(x,y,t)}{\partial x^2}\right] = \frac{p}{\eta}\int_0^\infty \int_0^\infty \int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial x^2}dxdydt$$

$$= \frac{p}{\eta}\int_0^\infty \int_0^\infty e^{-py-\frac{t}{q}}[\int_0^\infty e^{-\eta x}\frac{\partial^2 z(x,y,t)}{\partial x^2}dx]dydt \tag{11}$$

First, we solve $\int_0^\infty e^{-\eta x}\frac{\partial^2 z(x,y,t)}{\partial x^2}dx$

Applying integration by parts

$$\int_0^\infty e^{-\eta x}\frac{\partial^2 z(x,y,t)}{\partial x^2}dx = \left[e^{-\eta x}\frac{\partial z(x,y,t)}{\partial x}\right]_0^\infty + \eta\int_0^\infty e^{-\eta x}\frac{\partial z(x,y,t)}{\partial x}dx$$

Using part (i), we get

$$= -\frac{\partial z(0,y,t)}{\partial x} + \eta\left\{-z(0,y,t) + \eta\int_0^\infty e^{-\eta x}z(x,y,t)dx\right\}$$

$$= -z_x(0,y,t) - \eta z(0,y,t) + \eta^2\int_0^\infty e^{-\eta x}z(x,y,t)dx \tag{12}$$

Putting the above value of equation (12) in equation (11), we get

$$A_x G_y K_t\left[\frac{\partial^2 z(x,y,t)}{\partial x^2}\right]$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-py-\frac{t}{q}}\left[-z_x(0,y,t) - \eta z(0,y,t) + \eta^2\int_0^\infty e^{-\eta x}z(x,y,t)dx\right]dydt$$

$$= -\frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-py-\frac{t}{q}}z_x(0,y,t)dydt - \eta\frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-py-\frac{t}{q}}z(0,y,t)dydt$$

$$+\eta^2\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial x}dxdydt$$

$$= -\frac{1}{\eta}G_yK_t[z_x(0,y,t)] - G_yK_t[z(0,y,t)] + \eta^2\varphi(\eta,p,q)$$

$$= \eta^2\varphi(\eta,p,q) - \frac{1}{\eta}G_yK_t[z_x(0,y,t)] - G_yK_t[z(0,y,t)]$$

(v). Now

$$A_xG_yK_t\left[\frac{\partial^2 z(x,y,t)}{\partial y^2}\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial y^2}dxdydt$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}}dxdt[\int_0^\infty e^{-py}\frac{\partial^2 z(x,y,t)}{\partial y^2}dy] \tag{13}$$

First, we solve $\int_0^\infty e^{-py}\frac{\partial^2 z(x,y,t)}{\partial y^2}dy$

Applying integration by parts

$$\int_0^\infty e^{-py}\frac{\partial^2 z(x,y,t)}{\partial y^2}dy = \left[e^{-py}\frac{\partial z(x,y,t)}{\partial y}\right]_0^\infty + p\int_0^\infty e^{-py}\frac{\partial z(x,y,t)}{\partial y}dy$$

$$= \frac{\partial z(x,0,t)}{\partial y} + p\int_0^\infty e^{-py}\frac{\partial z(x,y,t)}{\partial y}dy$$

Using part (ii), we get

$$= \frac{\partial z(x,0,t)}{\partial y} + p\left\{-z(x,0,t) + p\int_0^\infty e^{-py}z(x,y,t)dy\right\}$$

$= -z_y(x,0,t) - pz(x,0,t) + p^2\int_0^\infty e^{-py}z(x,y,t)dy$

(14)

Putting the above value of equation (14) in equation (13), we get

$$A_xG_yK_t\left[\frac{\partial^2 z(x,y,t)}{\partial y^2}\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}}dxdt[-z_y(x,0,t) - pz(x,0,t) + p^2\int_0^\infty e^{-py}z(x,y,t)dy]$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}}z_y(x,0,t)dxdt - \frac{p^2}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-\frac{t}{q}}z(x,0,t)dxdt + p^2\frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial y^2}dxdydt$$

$$= -A_xK_t[z_y(x,0,t)] - pA_xK_t[z(x,0,t)] + p^2\varphi(\eta,p,q)$$

Therefore $A_xG_yK_t\left[\frac{\partial^2 z(x,y,t)}{\partial y^2}\right] = p^2\varphi(\eta,p,q) - pA_xK_t[z(x,0,t)] - A_xK_t[z_y(x,0,t)]$

(vi). Now

$$A_xG_yK_t\left[\frac{\partial^2 z(x,y,t)}{\partial t^2}\right] = \frac{p}{\eta}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial t^2}dxdydt$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-py}dxdy\left[\int_0^\infty e^{-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial t^2}dt\right] \quad (15)$$

First, we solve $\int_0^\infty e^{-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial t^2}dt$

Applying integration by parts

$$\int_0^\infty e^{-\frac{t}{q}}\frac{\partial^2 z(x,y,t)}{\partial t^2}dt = \left[e^{-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}\right]_0^\infty + \frac{1}{q}\int_0^\infty e^{-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dt$$

$$= -\frac{\partial z(x,y,0)}{\partial t} + \frac{1}{q}\int_0^\infty e^{-\frac{t}{q}}\frac{\partial z(x,y,t)}{\partial t}dt$$

Using part (iii), we get

$$= -\frac{\partial z(x,y,0)}{\partial t} + \frac{1}{q}\left\{-z(x,y,0) + \frac{1}{q}\int_0^\infty e^{-\frac{t}{q}}z(x,y,t)dt\right\}$$

$$= -z_t(x,y,0) - \frac{1}{q}z(x,y,0) + \frac{1}{q^2}\int_0^\infty e^{-py}z(x,y,t)dt \qquad (16)$$

Putting the above value of equation (16) in equation (15), we get

$$A_xG_yK_t\left[\frac{\partial^2 z(x,y,t)}{\partial t^2}\right]$$

$$= \frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-py}dxdy\left\{-z_t(x,y,0) - \frac{1}{q}z(x,y,0) + \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q}}z(x,y,t)dt\right\}$$

$$= -\frac{p}{\eta}\int_0^\infty\int_0^\infty e^{-\eta x-py}z_t(x,y,0)dxdy - \frac{p}{\eta q}\int_0^\infty\int_0^\infty e^{-\eta x-py}z(x,y,0)dxdy + \frac{p}{\eta q^2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\eta x-py-\frac{t}{q}}z(x,y,t)dxdydt$$

$$= -A_xG_y[z_t(x,y,0)] - \frac{1}{q}A_xG_y[z(x,y,0)] + \frac{1}{q^2}\varphi(\eta,p,q)$$

Therefore

$$A_xG_yK_t\left[\frac{\partial^2 z(x,y,t)}{\partial t^2}\right] = -A_xG_y[z_t(x,y,0)] - \frac{1}{q}A_xG_y[z(x,y,0)] + \frac{1}{q^2}\varphi(\eta,p,q).$$

VIII. Conclusion:

This study established several new properties and theorems of the AAKIT, including linearity, shifting, scaling, differentiation, convolution, inversion, and uniqueness. Illustrative examples demonstrate practical applications in solving differential equations. The work provides a solid foundation for further theoretical development and practical applications of AAKIT in applied mathematics, physics, and engineering.

Conflict of Interest

The authors declare that there is no conflict of interest regarding this article.

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