JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-21, No.-07, July (2026) pp 146-160 ISSN (Print) 0973-8975

FIXED POINT THEOREMS FOR SIMULATION FUNCTIONS IN INTERPOLATIVE $\mathfrak{S}$-METRIC SPACE: A NOVEL APPROACH

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https://doi.org/10.26782/jmcms.2026.07.00009

(Received: April 08, 2026; Revised: June 30, 2026; Accepted: July 12, 2026)

Abstract

In the present manuscript, first of all, we shall introduce a new concept of simulation function in interpolative $\mathfrak{S}$-metric space. Further, we shall prove a fixed point theorem for the same. An example is also provided to prove the validity of our result. Some results from the literature are also deduced, and an integral equation and boundary value problem are also solved with the help of our main result.

Keywords: Fixed point, simulation function, interpolative $\mathfrak{S}$-metric space.

I. Introduction

The idea of a metric serves as the essential foundation of fixed point theory, since many of its key results depend on the existence of an appropriate metric framework. One of the most celebrated results in this direction is the Banach Contraction Principle [II], which states that if $(\aleph, \Upsilon)$ is a complete metric space, $T: \aleph \to \aleph$ is a self-mapping satisfying:

$\Upsilon(T\vartheta, T\mho) \le q\Upsilon(\vartheta, \mho)$, for all $\vartheta, \mho \in \aleph$, for some $q \in [0,1)$, then T possesses a unique fixed point.

Throughout the years, multiple authors [III, IV, V, IX, X] have presented their unique generalizations of metric spaces and established relevant theoretical foundations. These developments have significantly broadened the scope of fixed point theory, enabling its application in different fields such as nonlinear analysis, optimization theory, and the study of differential and integral equations.

In 2004, Mustafa and Sims [VIII, IX] introduced the following concept of $\mathfrak{S}$-metric space as a natural generalization of the standard metric space, which further expanded the scope of fixed point theory:

Definition I.i. [VIII] Let $\aleph$ be a non-empty set, and let $\mathfrak{S} : \aleph \times \aleph \times \aleph \to \mathbb{R}^+$ be a function satisfying the following properties:

(1) $\mathfrak{S}(\vartheta, \mho, \tilde{\imath}) = 0$ if $\vartheta = \mho = \tilde{\imath}$;
(2) $\mathfrak{S}(\vartheta, \vartheta, \mho) > 0$; for all $\vartheta, \mho \in \aleph$, with $\vartheta \neq \mho$;
(3) $\mathfrak{S}(\vartheta, \vartheta, \mho) \leq \mathfrak{S}(\vartheta, \mho, \tilde{\imath})$, for all $\vartheta, \mho, \tilde{\imath} \in \aleph$, with $\tilde{\imath} \neq \mho$;
(4) $\mathfrak{S}(\vartheta, \mho, \tilde{\imath}) = \mathfrak{S}(\vartheta, \tilde{\imath}, \mho) = \mathfrak{S}(\mho, \tilde{\imath}, \vartheta) = \ldots$, (symmetry in all three variables);
(5) $\mathfrak{S}(\vartheta, \mho, \tilde{\imath}) \leq \widetilde{\mathfrak{S}}(\vartheta, \not{f}, \not{f}) + \mathfrak{S}(\not{f}, \mho, \tilde{\imath})$, for all $\vartheta, \mho, \tilde{\imath}, \not{f} \in \aleph$, (rectangle inequality).

The pair $(\aleph, \mathfrak{S})$ is called $\mathfrak{S}$-metric space.

Karapinar [V] presented the concept of generalized metric spaces known as interpolative metric spaces by substituting the traditional triangle inequality with a new condition called the interpolative inequality as follows:

Definition I.ii. [V] Let $\aleph$ be a non-empty set. We say that $\Upsilon : \aleph \times \aleph \to [0, +\infty)$ is $(\tilde{\gamma}, \tilde{\varkappa})$-interpolative metric if

$(k1)$ $\Upsilon(\vartheta, \mho) = 0$, if and only if, $\vartheta = \mho$ for all $\vartheta, \mho \in \aleph$;

$(k2)$ $\Upsilon(\vartheta, \mho) = \Upsilon(\mho, \vartheta)$, for all $\vartheta, \mho \in \aleph$;

$(k3)$ There exists an $\tilde{\gamma} \in (0,1)$ and $\tilde{\varkappa} \geq 0$ such that

$$\Upsilon(\vartheta, \mho) \leq \tilde{\Upsilon}(\vartheta, \tilde{\imath}) + \Upsilon(\tilde{\imath}, \mho) + \tilde{\varkappa}\left[\left(\Upsilon(\vartheta, \tilde{\imath})\right)^{\tilde{\gamma}} \left(\tilde{\Upsilon}(\tilde{\imath}, \mho)\right)^{1-\tilde{\gamma}}\right],$$

for all $\vartheta, \mho, \tilde{\imath} \in \aleph$.

Then, we call $(\aleph, \Upsilon)$ an $(\tilde{\gamma}, \tilde{\varkappa})$-interpolative metric space.

Example I.iii. Let $\aleph = [0,1]$ and define a function $\Upsilon : \aleph \times \aleph \to [0, \infty]$ as follows:

$$\Upsilon(\vartheta, \mho) = |\vartheta - \mho|^5, \text{ for all } \vartheta, \mho \in \aleph.$$

Conditions $(k1)$ and $(k2)$ hold trivially. Pertaining to condition $(k3)$, we have

$$\begin{aligned}\Upsilon(\vartheta, \mho) &= |\vartheta - \mho|^5 = |\vartheta - \tilde{\imath} + \tilde{\imath} - \mho|^5 \\ &\leq |\vartheta - \tilde{\imath}|^5 + 5|\vartheta - \tilde{\imath}|^4 |\tilde{\imath} - \mho| + 10|\vartheta - \tilde{\imath}|^3 |\tilde{\imath} - \mho|^2 + 10|\vartheta - \tilde{\imath}|^2 |\tilde{\imath} - \mho|^3 \\ &\quad + 5|\vartheta - \tilde{\imath}| |\tilde{\imath} - \mho|^4 + |\tilde{\imath} - \mho|^5 \\ &\leq |\vartheta - \tilde{\imath}|^5 + 5|\vartheta - \tilde{\imath}|^4 |\tilde{\imath} - \mho| + 5|\vartheta - \tilde{\imath}||\tilde{\imath} - \mho|^4 + 10|\vartheta - \tilde{\imath}|^3 |\tilde{\imath} - \mho|^2 + \\ &\quad 10|\vartheta - \tilde{\imath}|^2 |\tilde{\imath} - \mho|^3 + |\tilde{\imath} - \mho|^5 \\ &\leq \Upsilon(\vartheta, \tilde{\imath}) + 5[\Upsilon(\vartheta, \tilde{\imath})^{\frac{4}{5}} \Upsilon(\tilde{\imath}, \mho)^{\frac{1}{5}}] + 5[\Upsilon(\vartheta, \tilde{\imath})^{\frac{1}{5}} \Upsilon(\tilde{\imath}, \mho)^{\frac{4}{5}}] + 10\left[\Upsilon(\vartheta, \tilde{\imath})^{\frac{3}{5}} \Upsilon(\tilde{\imath}, \mho)^{\frac{2}{5}}\right] + \\ &\quad 10\left[\Upsilon(\vartheta, \tilde{\imath})^{\frac{2}{5}} \Upsilon(\tilde{\imath}, \mho)^{\frac{3}{5}}\right] + \Upsilon(\tilde{\imath}, \mho)\end{aligned}$$

Without loss of generality, we assume that $\Upsilon(\vartheta, \tilde{\imath}) \geq \Upsilon(\tilde{\imath}, \mho)$

$$\leq \Upsilon(\vartheta, \tilde{\imath}) + \Upsilon(\tilde{\imath}, \mho) + 30\,[\Upsilon(\vartheta, \tilde{\imath})^{\frac{4}{5}} \Upsilon(\tilde{\imath}, \mho)^{\frac{1}{5}}].$$

Thus, $(k3)$ condition is satisfied for $\tilde{\varkappa} \geq 30$ for $\tilde{\gamma} = \frac{1}{5} \in (0, 1)$.

Hence, $(\aleph, \Upsilon)$ is $(\frac{1}{5}, 30)$-interpolative metric space.
In 2025, Karapinar [VI] further contributed to this framework by establishing some fixed point results on interpolative metric spaces.

In 2026, Kajal *et al.* [III] introduced a new approach to interpolative $\mathfrak{S}$-metric spaces, which is defined as follows.

Definition I.iv. Let $\aleph$ be a non-empty set. A function $\mathfrak{S}:\aleph^3 \to [0,+\infty)$ is said to be $(\tilde{\gamma},\tilde{\varkappa})$ −interpolative $\mathfrak{S}$-metric if

($\mathfrak{S}$1) $\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) = 0$ if $\vartheta = \mho = \tilde{\iota}$;

($\mathfrak{S}$2) $\mathfrak{S}(\vartheta,\vartheta,\mho) > 0$; for all $\vartheta,\mho \in \aleph$, with $\vartheta \neq \mho$;

($\mathfrak{S}$3) $\mathfrak{S}(\vartheta,\vartheta,\mho) \leq \mathfrak{S}(\vartheta,\mho,\tilde{\iota})$, for all $\vartheta,\mho,\tilde{\iota} \in \aleph$, with $\tilde{\iota} \neq \mho$;

($\mathfrak{S}$4) $\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) = \mathfrak{S}(\vartheta,\tilde{\iota},\mho) = \mathfrak{S}(\mho,\tilde{\iota},\vartheta) = \ldots$, (symmetry in all three variables);

($\mathfrak{S}$5) $\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) \leq \mathfrak{S}(\vartheta,\not{f},\not{f}) + \mathfrak{S}(\not{f},\mho,\tilde{\iota}) + \tilde{\varkappa}\left[\left(\mathfrak{S}(\vartheta,\not{f},\not{f})\right)^{\tilde{\gamma}}\left(\mathfrak{S}(\not{f},\mho,\tilde{\iota})\right)^{1-\tilde{\gamma}}\right]$,

for all $\vartheta,\mho,\tilde{\iota},\not{f} \in \aleph$.

The pair $(\aleph,\mathfrak{S})$ is called an $(\tilde{\gamma},\tilde{\varkappa})$-interpolative $\mathfrak{S}$-metric space.

Example I.v. Let $(\aleph,\mathcal{H})$ be a usual metric space. Define a function $\mathfrak{S}:\aleph^3 \to [0,\infty)$ as follow:

$\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) := [\delta(\vartheta,\mho,\tilde{\iota})]^2 + \mathcal{H}\delta(\vartheta,\mho,\tilde{\iota})$, for all $\vartheta,\mho,\tilde{\iota} \in \aleph$.

Since δ is a metric on $\aleph$, the conditions ($\mathfrak{S}$1)-($\mathfrak{S}$4) are satisfied. For ($\mathfrak{S}$5), it is adequate to assume $\tilde{\varkappa} \geq 2$ for any $\tilde{\gamma} \in (0,1)$. Thus, $(\aleph,\mathfrak{S})$ is $(\frac{1}{2},\ 2)$-interpolative $\mathfrak{S}$-metric space.

We have

$$\begin{aligned}
\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) &= [\delta(\vartheta,\mho,\tilde{\iota})]^2 + \mathcal{H}\delta(\vartheta,\mho,\tilde{\iota}) \\
&\leq (\delta(\vartheta,\not{f},\not{f}) + \delta(\not{f},\mho,\tilde{\iota}))(\delta(\vartheta,\not{f},\not{f}) + \delta(\not{f},\mho,\tilde{\iota}) + \mathcal{H}) \\
&\leq [\delta(\vartheta,\not{f},\not{f})(\delta(\vartheta,\not{f},\not{f}) + \mathcal{H}) + \delta(\vartheta,\not{f},\not{f})\delta(\not{f},\mho,\tilde{\iota})] \\
&\quad + [\delta(\not{f},\mho,\tilde{\iota})(\delta(\not{f},\mho,\tilde{\iota}) + \mathcal{H}) + \delta(\not{f},\mho,\tilde{\iota})\delta(\vartheta,\not{f},\not{f})] \\
&\leq [\delta(\vartheta,\not{f},\not{f})(\delta(\vartheta,\not{f},\not{f}) + \mathcal{H})] + [\delta(\not{f},\mho,\tilde{\iota})(\delta(\not{f},\mho,\tilde{\iota}) + \mathcal{H})] + \\
&\quad 2\delta(\vartheta,\not{f},\not{f})\delta(\not{f},\mho,\tilde{\iota}) \\
&\leq \tilde{\mathfrak{S}}(\tilde{\vartheta},\not{f},\not{f}) + \tilde{\mathfrak{S}}(\not{f},\tilde{\mho},\tilde{\iota}) + 2\left(\delta(\tilde{\vartheta},\not{f},\not{f})\right)^{\frac{1}{2}}\left(\delta(\tilde{\vartheta},\not{f},\not{f})\right)^{\frac{1}{2}} \\
&\quad (\delta(\not{f},\mho,\tilde{\iota}))^{\frac{1}{2}}(\delta(\not{f},\mho,\tilde{\iota}))^{\frac{1}{2}} \\
&\leq \mathfrak{S}(\vartheta,\not{f},\not{f}) + \mathfrak{S}(\not{f},\mho,\tilde{\iota}) + 2(\delta(\vartheta,\not{f},\not{f}))^{\frac{1}{2}}[\delta(\vartheta,\not{f},\not{f}) + \\
&\quad \mathcal{H}]^{\frac{1}{2}}(\delta(\not{f},\mho,\tilde{\iota}))^{\frac{1}{2}}[\delta(\not{f},\mho,\tilde{\iota}) + \mathcal{H}]^{\frac{1}{2}} \\
&\leq \mathfrak{S}(\vartheta,\not{f},\not{f}) + \mathfrak{S}(\not{f},\mho,\tilde{\iota}) + 2(\mathfrak{S}(\vartheta,\not{f},\not{f}))^{\frac{1}{2}}(\mathfrak{S}(\not{f},\mho,\tilde{\iota}))^{\frac{1}{2}}
\end{aligned}$$

Therefore, the function $\mathfrak{S}(\vartheta,\mho,\tilde{\iota})$ does not form a $\mathfrak{S}$-metric.

Example I.vi. Let $\aleph$ be a non-empty set and define a function $\mathfrak{S}:\aleph^3 \to [0,\infty)$ as follows:

$$\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) = |\vartheta - \mho| + |\mho - \tilde{\iota}| + |\tilde{\iota} - \vartheta|$$

Clearly, the Conditions ($\mathfrak{S}$1)-($\mathfrak{S}$4) are satisfied trivially.As for the condition ($\mathfrak{S}$5), we have

$\mathfrak{S}(\vartheta,\mho,\tilde{\iota}) \leq (|\vartheta - a| + |\not{f} - \mho|) + (|\mho - \not{f}| + |\tilde{\iota} - a|) + (|\tilde{\iota} - \not{f}| + |\vartheta - \not{f}|)$

$= 2|\vartheta - a| + 2(|\sigma - \not{f} + |\tilde{\iota} - \not{f}|)$

Now,
$\mathfrak{S}(\vartheta, \not{f}, \not{f}) = |\vartheta - \not{f}| + |\not{f} - \not{f}| + |\not{f} - \vartheta|$
$\mathfrak{S}(\vartheta, \not{f}, \not{f}) = 2|\vartheta - \not{f}|$
$\mathfrak{S}(\not{f}, \sigma, \tilde{\iota}) = |\not{f} - \sigma| + |\sigma - \tilde{\iota}| + |\tilde{\iota} - \not{f}|$
$\mathfrak{S}(\not{f}, \sigma, \tilde{\iota}) \geq |\sigma - \not{f}| + |\tilde{\iota} - \not{f}|$
So, we have
$\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) \leq \mathfrak{S}(\vartheta, \not{f}, \not{f}) + 2\mathfrak{S}(\not{f}, \sigma, \tilde{\iota})$

Hence, condition ($\mathfrak{S}$3) holds with $\tilde{\gamma} = \frac{1}{2}, \tilde{\gamma} \geq 2$.

Hence $(\aleph, \mathfrak{S})$ is $(\frac{1}{2}, 2)$-interpolative $\mathfrak{G}$-metric space.

Definition I.vii. Let $(\aleph, \mathfrak{S})$be a $(\tilde{\gamma}, \varkappa)$-interpolative $\mathfrak{S}$-metric space and let $\{\vartheta_n\}$ be a sequence in $\aleph$. We say that $\{\vartheta_n\}$ is $\mathfrak{S}$-converges to ϑ in $\aleph$, if $\mathfrak{S}(\vartheta, \vartheta_n, \vartheta_{\mathcal{J}}) \to 0$ as $n, \mathcal{J} \to \infty$.

Definition I.viii. Let $(\aleph, \mathfrak{S})$ be a $(\tilde{\gamma}, \varkappa)$-interpolative $\mathfrak{S}$-metric space and let $\{\vartheta_n\}$ be a sequence in $\aleph$. We say that $\{\vartheta_n\}$ is $\mathfrak{S}$-Cauchy sequence in $\aleph$, if and only if, $\mathfrak{S}(\vartheta_n, \vartheta_{\mathcal{J}}, \vartheta_l) \to 0$ as $n, \mathcal{J}, l \to \infty$.

Definition I.ix. Let $(\aleph, \mathfrak{S})$ be a $(\tilde{\gamma}, \varkappa)$-interpolative $\mathfrak{S}$-metric space. We say that $(\aleph, \mathfrak{S})$ is a complete $(\tilde{\gamma}, \varkappa)$-interpolative $\mathfrak{S}$ -metric space if every $\mathfrak{S}$-Cauchy sequence converges in $\aleph$.

Recently, Khojasteh *et al.* [VII] proposed a novel class of mappings known as simulation functions, and later, Argoubi *et al.* [I] refined this condition by omitting one condition.

Definition I.x. [VII] A simulation function is a mapping $\varsigma : [0, \infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(1) $\varsigma(0,0) = 0$;

(2) $\varsigma(m, \mathcal{J}) < \mathcal{J} - m$ for all $m, \mathcal{J} > 0$;

(3) If $\{m_n\}$and $\{\mathcal{J}_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty}\{m_n\} = \lim_{n\to\infty}\{\mathcal{J}_n\} = t \in (0, \infty),$$

then

$$\lim_{n\to\infty} sup \varsigma(m, \mathcal{J}_n) < 0.$$

In 2025, Alsahli and Ozyurt [X] introduced the notion of simulation functions in interpolative metric space.

"Let $T : S \to S$ be an interpolative Z_α-contraction, where (S, d) is a complete (ξ, c)-interpolative metric space.

Then T possesses a unique fixed point m^* in M and for every $m_0 \in S$ the Picard sequence $\{m_n\}$, where $m_n = Tm_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T."

By getting motivation from the above results, we shall introduce a new concept of simulation function in an interpolative $\mathfrak{S}$-metric space.

II. Main Result

In this section, we shall prove a fixed point theorem for a simulation function in an interpolative $\mathfrak{S}$-metric space. We shall also provide an example to prove the validity of the result.

Theorem II.i. Let $(\aleph, \mathfrak{S})$ be an interpolative $\mathfrak{S}$-metric space and let T be a self mapping on $\aleph$ with respect to $\zeta \in \mathbb{Z}$, such that

$$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}), \mathfrak{S}(\vartheta, \mho, \tilde{\imath})) \geq 0, \text{ for all } \vartheta, \mho, \tilde{\imath} \in \aleph, \tag{2.1}$$

Then T has a unique fixed point.

Proof: Let $\vartheta \in \aleph$ be arbitrary and denote it by $\vartheta_0 = \vartheta$. Using that initial point, we construct the sequence ϑ_n defined by $\vartheta_n = T_{\vartheta_{n-1}}$, $n \in \mathbb{N}$. If there exists some $k \in \mathbb{N}$ with $\vartheta_k = \vartheta_{k+1}$, then it follows that $T^k\vartheta = T^{k-1}\vartheta$. Therefore, without loss of generality, we may assume that $T^n\vartheta \neq T^{n-1}\vartheta$, for all $n \in \mathbb{N}$.

$$\begin{aligned} 0 &\leq \zeta(\mathfrak{S}(T^{n+1}\vartheta, T^n\vartheta, T^n\vartheta), \mathfrak{S}(T^n\vartheta\, T^{n-1}\vartheta, T^{n-1}\vartheta)) \\ &= \zeta(\mathfrak{S}(TT^n\vartheta, TT^{n-1}\vartheta, TT^{n-1}\vartheta), \mathfrak{S}(T^n\vartheta, T^{n-1}\vartheta, T^{n-1}\vartheta)) \\ &\leq \mathfrak{S}(T^n\vartheta, T^{n-1}\vartheta, T^{n-1}\vartheta) - \mathfrak{S}(T^{n+1}\vartheta, T^n\vartheta, T^n\vartheta). \end{aligned}$$

The inequality presented indicates that $\{\mathfrak{S}(T^n\vartheta, T^{n-1}\vartheta, T^{n-1}\vartheta)\}$ is a monotonically decreasing sequence of non-negative real numbers; thus it must be convergent.

Let us denote $\underset{n\to\infty}{Lim}\, \mathfrak{S}(T^n\vartheta,\ T^{n+1}\vartheta, T^{n+1}\vartheta) = r) \geq 0$.

We have

$$0 \leq \underset{n\to\infty}{\operatorname{Lim}} sup\zeta(\,\mathfrak{S}(T^{n+1}\vartheta,\ T^n\vartheta, T^n\vartheta), \mathfrak{S}(T^n\vartheta, T^{n-1}\vartheta, T^{n-1}\vartheta)) < 0,$$

a contradiction.

Thus, the limit $r = 0$, that is,

$$\underset{n\to\infty}{Lim}\, \mathfrak{S}(T^n\vartheta,\ T^{n+1}\vartheta, T^{n+1}\vartheta) = 0. \tag{2.2}$$

Therefore, we deduce that the self-mapping T is an asymptotically regular mapping in $\aleph$.

To achieve this, we assert that

$$\underset{n\to\infty}{Lim}\, \mathfrak{S}(T^{n-1}\vartheta,\ T^{n+r}\vartheta, T^{n+r}\vartheta) = 0. \tag{2.3}$$

To establish this result, we proceed by mathematical induction, starting with the following limit.

$$\begin{aligned} \mathfrak{S}(T^{n-1}\vartheta, T^{n+1}\vartheta, T^{n+1}\vartheta) &\leq \mathfrak{S}(\vartheta_n, T^n\vartheta, T^n\vartheta) + \mathfrak{S}(T^n\vartheta, T^{n+1}\vartheta, T^{n+1}\vartheta) + \\ &\tilde{\varkappa}[(\mathfrak{S}(\vartheta_n, T^n\vartheta, T^n\vartheta))^{\zeta}(\mathfrak{S}(T^n\jmath, T^{n+1}\vartheta, T^{n+1}\vartheta))^{1-\zeta}] \end{aligned} \tag{2.4}$$

Letting n→ ∞ in the above inequality and applying (2.4), we conclude that

$$\lim_{n\to\infty} \mathfrak{S}(T^{n-1}\vartheta,\ T^{n+1}\vartheta, T^{n+1}\vartheta) = 0. \qquad (2.5)$$

Moreover, we have

$$\mathfrak{S}(T^{n-1}\vartheta, T^{n+2}\vartheta, T^{n+2}\vartheta) \le \mathfrak{S}(\vartheta_n, T^{n+1}\vartheta, T^{n+1}\vartheta) + \mathfrak{S}(T^{n+1}\vartheta, T^{n+2}\vartheta, T^{n+2}\vartheta) + [(\mathfrak{S}(\vartheta_n, T^{n+1}\vartheta, T^{n+1}\vartheta))^{\zeta}(\mathfrak{S}(T^{n+1}\tilde{\vartheta}, T^{n+2}\vartheta, T^{n+2}\vartheta))^{1-\zeta}] \qquad (2.6)$$

On account of limits (2.2) and (2.6), by taking the limit of the above inequality as n→ ∞, we get that

$$\lim_{n\to\infty} \mathfrak{S}(T^{n-1}\vartheta,\ T^{n+2}\vartheta, T^{n+2}\vartheta) = 0. \qquad (2.7)$$

Let us now assume the general case of our assertion, that is, we have

$$\lim_{n\to\infty} \mathfrak{S}(T^{n-1}\vartheta,\ T^{n+r-1}\vartheta, T^{n+r-1}\vartheta) = 0, \text{ for some } r \in \mathbb{N}. \qquad (2.8)$$

Consequently, according to the Theorem, we obtain

$$\mathfrak{S}(T^{n-1}\vartheta, T^{n+r}\vartheta, T^{n+r}\vartheta) \le \mathfrak{S}(T^{n-1}\vartheta, T^{n+r-1}\vartheta, T^{n+r-1}\vartheta) + \mathfrak{S}(T^{n+r-1}\vartheta, T^{n+r}\vartheta, T^{n+r}\vartheta) + \tilde{\varkappa}[(\mathfrak{S}(T^{n-1}\vartheta, T^{n+r-1}\vartheta, T^{n+r-1}\vartheta))^{\zeta} (\mathfrak{S}(T^{n+r-1}\vartheta, T^{n+r}\vartheta, T^{n+r}\vartheta)^{1-\zeta}] \qquad (2.9)$$

By taking the limits (2.2) and (2.8) into account, by letting $n \to \infty$, we observe that

$$\lim_{n\to\infty} \mathfrak{S}(T^{n-1}\vartheta,\ T^{n+r}\vartheta, T^{n+r}\vartheta) = 0. \qquad (2.10)$$

Consequently, we deduce that

$$\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) < 1, \qquad (2.11)$$

For $\vartheta > n > M_2$for some $M_2 \in \mathbb{N}$.

We now prove that the iterative sequence $\{T^{n-1}\vartheta\}$ is a Cauchy.

For $\vartheta > n > M_2$, for some $M_2 \in \mathbb{N}$, consider

$$\begin{aligned}\mathfrak{S}(T^{n-1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) &\le \mathfrak{S}(T^{n-1}\vartheta, T^{n}\vartheta, T^{n}\vartheta) + \mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) + \tilde{\varkappa}[(\mathfrak{S}(T^{n-1}\vartheta, T^{n}\vartheta, T^{n}\vartheta))^{\zeta}(\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta))^{1-\zeta}]\\ &\le \lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta) + \mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) + \tilde{\varkappa}[\lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))^{\zeta}(\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta))^{1-\zeta}]. \end{aligned} \qquad (2.12)$$

Using fact (2.11), it follows that $\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) < 1$ and consequently

$$(\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta))^{1-\zeta}] < 1. \qquad (2.13)$$

Considering this fact, we can evaluate the right side of the inequality(2.12) as shown:

$$\mathfrak{S}(T^{n-1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) \le \lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta) + \mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) + \tilde{\varkappa}[(\lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\mathfrak{I}, T^{k}\vartheta)))^{\zeta}\left(\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta)\right)^{\zeta} \mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta)] \qquad (2.14)$$

In conclusion, we have

$$\mathfrak{S}(T^{n-1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) \leq \lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta) + [1+\tilde{\varkappa}(\lambda^{n-k})^{\zeta}] \quad \mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta). \tag{2.15}$$

Notice also that

$$\mathfrak{S}(T^{n}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) \leq \mathfrak{S}(T^{n}\mathcal{J}, T^{n+1}\vartheta, T^{n+1}\vartheta) + \mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) +$$

$$\tilde{\varkappa}[(\mathfrak{S}(T^{n}\vartheta, T^{n+1}\vartheta, T^{n+1}\vartheta))^{\zeta}(\mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta))^{1-\zeta}]$$

$$\leq \lambda^{n-k+1}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta) + \mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta)$$

$$+ \tilde{\varkappa}[\lambda^{n-k+1}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))^{\zeta}\left(\mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta)\right)^{-\zeta}$$

$$\mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta)]$$

$$\leq \lambda^{n-k+1}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta) + [1+\tilde{\varkappa}(\lambda^{n-k+1})^{\zeta}]$$

$$\mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta). \tag{2.16}$$

Merging the inequalities (2.15) and (2.16), we find that

$$\mathfrak{S}(T^{n-1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) \leq \lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta)) + \lambda^{n-k+1}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))$$

$$[1+\tilde{\varkappa}(\lambda^{n-k})^{\zeta}] + \left[1+\tilde{\varkappa}(\lambda^{n-k})^{\zeta}\right][1+\tilde{\varkappa}(\lambda^{n-k+1})^{\zeta}]$$

$$\mathfrak{S}(T^{n+1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) \tag{2.17}$$

Keeping all these observations, we obtained that

$$\mathfrak{S}(T^{n-1}\vartheta, T^{s-1}\vartheta, T^{s-1}\vartheta) \leq \lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta)) \sum_{i=0}^{\vartheta-n-1}\lambda^{i}\prod_{j=0}^{i-1}(1 + \tilde{\varkappa}\lambda^{n-k+\vartheta})$$

$$(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))^{\zeta}$$

$$\leq \lambda^{n-k}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta)) \sum_{i=0}^{\vartheta-n-1}\lambda^{i}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))$$

$$\prod_{j=0}^{i-1}(1 + \tilde{\varkappa}\lambda^{i}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))^{\zeta}. \tag{2.18}$$

The sequence $\sum_{i=0}^{\infty} S_i$, governs the right-hand side of the inequality above, which is convergent as $n, \vartheta \to \infty$ where,

$$S_i = \lambda^{i}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta)) \prod_{j=0}^{i-1}(1 + \tilde{\varkappa}\lambda^{i}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))^{\zeta}$$

$$< \lambda^{i}(\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))\prod_{j=0}^{i-1}(1 + \tilde{\varkappa}\mathfrak{S}(T^{k-1}\vartheta, T^{k}\vartheta, T^{k}\vartheta))^{\zeta}. \tag{2.19}$$

It follows that the iterative sequence $\{T^{n-1}\vartheta\}$ is a Cauchy sequence.

Since $(\aleph, \Upsilon)$ is a complete $(\tilde{\gamma}, \tilde{\varkappa})$-interpolative $\mathfrak{S}$-metric space, the sequence $\{T^{n-1}\vartheta\}$ converges to $\vartheta^* \in \aleph$.

We claim that ϑ^* is the fixed point of T.

Suppose, for contradiction, that $\mathfrak{S}(\vartheta^*, T\vartheta^*, T\vartheta^*) > 0$. Note that,

$$\mathfrak{S}(T^{n}\vartheta, T\vartheta^*, T\vartheta^*) = \mathfrak{S}(TT^{n-1}\vartheta, T\vartheta^*, T\vartheta^*) \tag{2.20}$$

By taking the lim sup of both sides of the inequalities, we get

$\mathfrak{S}(T\vartheta^*, \vartheta^*, \vartheta^*) < \mathfrak{S}(T\vartheta^*, \vartheta^*, \vartheta^*)$ since we assume $\mathfrak{S}(\vartheta^*, T\vartheta^*, T\vartheta^*) > 0$

Which is a contradiction.

Therefore $T\vartheta^* = \vartheta^*$ is the fixed point of T in $\aleph$.

Example II.ii: Let $\aleph$ be a non-empty set and define a function $\mathfrak{S} : \aleph^3 \to [0,\infty]$ as follows:

$$\mathfrak{S}(\vartheta, \mathcal{O}, \tilde{\imath}) = |\vartheta - \mathcal{O}|^2 + |\mathcal{O} - \tilde{\imath}|^2 + |\tilde{\imath} - \vartheta|^2, \text{ for all } \vartheta, \mathcal{O}, \tilde{\imath} \in \aleph.$$

We verify that $(\aleph, \mathfrak{S})$ is $(\frac{1}{2}, 2)$-interpolative $\mathfrak{S}$-metric space.

($\mathfrak{S}$1) Suppose $\vartheta = \mho = \tilde{\iota}$. Then

$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = |\vartheta - \mho|^2 + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \vartheta|^2 = 0.$

Conversely, if $\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = 0$, then

$|\vartheta - \mho|^2 = |\mho - \tilde{\iota}|^2 = |\tilde{\iota} - \vartheta|^2 = 0,$

Which implies $\vartheta = \mho = \tilde{\iota}$.

Hence, condition ($\mathfrak{S}$1) is satisfied.

($\mathfrak{S}$2) *Let* $\vartheta, \mho \in \aleph$ with $\vartheta \neq \mho$. Then

$\mathfrak{S}(\vartheta, \vartheta, \tilde{\iota}) = 2|\vartheta - \mho|^2 > 0.$

Therefore, condition ($\mathfrak{S}$2) holds.

($\mathfrak{S}$3) For every $\tilde{\iota} \neq \mho$,

$\mathfrak{S}(\vartheta, \vartheta, \mho) = 2|\vartheta - \mho|^2 \leq |\vartheta - \mho|^2 + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \vartheta|^2 = \mathfrak{S}(\vartheta, \mho, \tilde{\iota})$, for all $\vartheta, \mho, \tilde{\iota} \in \aleph$.

($\mathfrak{S}$4) $\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = |\vartheta - \mho|^2 + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \vartheta|^2$,

the function is symmetric in all three variables.

Therefore,

$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = \mathfrak{S}(\vartheta, \tilde{\iota}, \mho) = \mathfrak{S}(\mho, \tilde{\iota}, \vartheta) = \mathfrak{S}(\mho, \vartheta, \tilde{\iota}) = \mathfrak{S}(\tilde{\iota}, \vartheta, \mho) = \mathfrak{S}(\tilde{\iota}, \mho, \vartheta).$

Hence, condition ($\mathfrak{S}$4) is satisfied.

($\mathfrak{S}$5) we have

$$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = |(\vartheta - \rlap{/}f) - (\mho - \rlap{/}f)|^2 + |\mho - \tilde{\iota}|^2 + |(\tilde{\iota} - \rlap{/}f) - (\vartheta - \rlap{/}f)|^2$$

$$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) \leq |\vartheta - \rlap{/}f|^2 + |\mho - \rlap{/}f|^2 + 2|\vartheta - \rlap{/}f|\,|\mho - \rlap{/}f| + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \rlap{/}f|^2 + |\vartheta - \rlap{/}f|^2 + 2|\tilde{\iota} - \rlap{/}f|\,|\vartheta - \rlap{/}f|$$

$$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) \leq 2|\vartheta - \rlap{/}f|^2 + |\mho - \rlap{/}f|^2 + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \rlap{/}f|^2 + 2|\vartheta - \rlap{/}f|\,|\mho - \rlap{/}f| + 2|\tilde{\iota} - \rlap{/}f|\,|\vartheta - \rlap{/}f|$$

$$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) \leq 2|\vartheta - \rlap{/}f|^2 + |\mho - \rlap{/}f|^2 + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \rlap{/}f|^2 + 2|\vartheta - \rlap{/}f|\,(|\mho - \rlap{/}f| + |\tilde{\iota} - \rlap{/}f|)$$

$$\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) \leq \mathfrak{S}(\vartheta, \rlap{/}f, \rlap{/}f) + \mathfrak{S}(\rlap{/}f, \mho, \tilde{\iota}) + 2[\mathfrak{S}(\vartheta, \rlap{/}f, \rlap{/}f)^{\frac{1}{2}}\, \mathfrak{S}(\rlap{/}f, \mho, \tilde{\iota})^{\frac{1}{2}}].$$

Therefore, condition ($\mathfrak{S}$5) is satisfied for all $\vartheta, \mho, \tilde{\iota}, \rlap{/}f \in \aleph$ with admissible interpolation parameters $\tilde{\gamma} = \frac{1}{2}, \tilde{\varkappa} = 2$.

Thus, the function $\mathfrak{S}(\vartheta, \mho, \tilde{\iota})$ does not form a $\mathfrak{S}$-metric space.

Hence, $(\aleph, \mathfrak{S})$ is a $(\frac{1}{2}, 2)$-interpolative $\mathfrak{S}$-metric space.

Now, define the mapping $T: \aleph \to \aleph$ by $T\vartheta = \frac{\vartheta}{2}$ and $\zeta(m, \mathcal{J}) = \mathcal{J} - m$

$$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\iota}) = \left|\frac{\vartheta}{2} - \frac{\mho}{2}\right|^2 + \left|\frac{\mho}{2} - \frac{\tilde{\iota}}{2}\right|^2 + \left|\frac{\tilde{\iota}}{2} - \frac{\vartheta}{2}\right|^2$$

$$= \frac{1}{4}[|\vartheta - \mho|^2 + |\mho - \tilde{\iota}|^2 + |\tilde{\iota} - \vartheta|^2]$$

$$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\iota}) = \frac{1}{4}\mathfrak{S}(\vartheta, \mho, \tilde{\iota})$$

$$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\iota}), \mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = \mathfrak{S}(\vartheta, \mho, \tilde{\iota}) - \mathfrak{S}(T\vartheta, T\mho, T\tilde{\iota})$$

$$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\iota}), \mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = \mathfrak{S}(\vartheta, \mho, \tilde{\iota}) - \frac{1}{4}\mathfrak{S}(\vartheta, \mho, \tilde{\iota})$$

$$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\iota}), \mathfrak{S}(\vartheta, \mho, \tilde{\iota}) = \frac{3}{4}\mathfrak{S}(\vartheta, \mho, \tilde{\iota}) \geq 0.$$

Hence, all the hypotheses of Theorem II.i. are satisfied. Therefore, the mapping T has a unique fixed point 0.

Example II.iii. Let $\aleph = [0,\infty)$ and define a function $\mathfrak{S} : \aleph^3 \to [0,\infty]$ as follows:

$$\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) = \max\{|\vartheta - \sigma|^2, |\sigma - \tilde{\iota}|^2, |\tilde{\iota} - \vartheta|^2\}, \text{ for all } \vartheta, \sigma, \tilde{\iota} \in \aleph.$$

We verify that $(\aleph, \mathfrak{S})$ is $(\frac{1}{2}, 2)$-interpolative $\mathfrak{S}$-metric space.

(**$\mathfrak{S}$1**) If $\vartheta = \sigma = \tilde{\iota}$, then

$\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) = \max\{0,0,0\} = 0.$

Conversely, if $\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) = 0$, then

$|\vartheta - \sigma|^2 = |\sigma - \tilde{\iota}|^2 = |\tilde{\iota} - \vartheta|^2 = 0,$

Which implies $\vartheta = \sigma = \tilde{\iota}$.

Hence, condition ($\mathfrak{S}$1) is satisfied.

(**$\mathfrak{S}$2**) *Let* $\vartheta \neq \sigma$. Then

$\mathfrak{S}(\vartheta, \vartheta, \sigma) = \max\{0, |\vartheta - \sigma|^2, |\sigma - \vartheta|^2\} = |\vartheta - \sigma|^2 > 0.$

Therefore, condition ($\mathfrak{S}$2) holds.

(**$\mathfrak{S}$3**)For every $\tilde{\iota} \neq \sigma$,

$\mathfrak{S}(\vartheta, \vartheta, \sigma) = |\vartheta - \sigma|^2 \leq \max\{|\vartheta - \sigma|^2 + |\sigma - \tilde{\iota}|^2 + |\tilde{\iota} - \vartheta|^2 = \mathfrak{S}(\vartheta, \sigma, \tilde{\iota}).$

Therefore, condition ($\mathfrak{S}$3) is satisfied.

(**$\mathfrak{S}$4**)Since $\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) = \mathfrak{S}(\vartheta, \tilde{\iota}, \sigma) = \mathfrak{S}(\sigma, \tilde{\iota}, \vartheta) = \mathfrak{S}(\sigma, \vartheta, \tilde{\iota}) = \mathfrak{S}(\tilde{\iota}, \vartheta, \sigma) = \mathfrak{S}(\tilde{\iota}, \sigma, \vartheta)$.

Hence, condition ($\mathfrak{S}$4) is satisfied.

(**$\mathfrak{S}$5**) Let $\vartheta, \sigma, \tilde{\iota}, \not{f} \in \aleph$.

Since

$\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) = \max\{|\vartheta - \sigma|^2, |\sigma - \tilde{\iota}|^2, |\tilde{\iota} - \vartheta|^2\},$

$|\vartheta - \sigma|^2 = |(\vartheta - \not{f}) + (\not{f} - \sigma)|^2$

$|\vartheta - \sigma|^2 \leq (|\vartheta - \not{f}| + |\not{f} - \sigma|)^2$

$|\vartheta - \sigma|^2 \leq \mathfrak{S}(\vartheta, \not{f}, \not{f}) + \mathfrak{S}(\not{f}, \sigma, \tilde{\iota}) + 2\sqrt{\mathfrak{S}(\vartheta, \not{f}, \not{f})}\ \sqrt{\mathfrak{S}(\not{f}, \sigma, \tilde{\iota})}$

Now,

$|\sigma - \tilde{\iota}|^2 \leq \mathfrak{S}(\not{f}, \sigma, \tilde{\iota})$

Similarly,

$|\tilde{\iota} - \vartheta|^2 = |(\tilde{\iota} - \not{f}) + (\not{f} - \vartheta)|^2$

$|\tilde{\iota} - \vartheta|^2 \leq (|\tilde{\iota} - \not{f}| + |\not{f} - \sigma|)^2$

$|\vartheta - \sigma|^2 \leq \mathfrak{S}(\not{f}, \sigma, \tilde{\iota}) + \mathfrak{S}(\vartheta, \not{f}, \not{f}) + 2\sqrt{\mathfrak{S}(\not{f}, \sigma, \tilde{\iota})}\ \sqrt{\mathfrak{S}(\vartheta, \not{f}, \not{f})}$

Therefore,

$\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) = \max\{|\vartheta - \sigma|^2, |\sigma - \tilde{\iota}|^2, |\tilde{\iota} - \vartheta|^2\},$

$\mathfrak{S}(\vartheta, \sigma, \tilde{\iota}) \leq \mathfrak{S}(\vartheta, \not{f}, \not{f}) + \mathfrak{S}(\not{f}, \sigma, \tilde{\iota}) + 2[\mathfrak{S}(\vartheta, \not{f}, \not{f})^{\frac{1}{2}}\ \mathfrak{S}(\not{f}, \sigma, \tilde{\iota})^{\frac{1}{2}}].$

Hence, condition ($\mathfrak{S}$5) is satisfied with $\tilde{\gamma} = \frac{1}{2}, \tilde{\varkappa} = 2$.

Thus, the function $\mathfrak{S}(\vartheta, \sigma, \tilde{\iota})$ does not form a $\mathfrak{S}$-metric space.

Define the self-map $T: \aleph \to \aleph$ by $T(\vartheta) = \frac{\vartheta}{\vartheta + 1}$

Consider the simulation function $\zeta: [0, \infty) \times [0, \infty) \to \mathbb{R}$ given by

$\zeta(m, \mathcal{J}) = \mathcal{J} - m$, for all $m, \mathcal{J} \in (0, \infty)$

$$|T(\vartheta) - T(\mho)| = |\frac{\vartheta}{\vartheta+1} - \frac{\mho}{\mho+1}|$$
$$|T(\vartheta) - T(\mho)| = |\frac{|\vartheta(\mho+1) - \mho(\vartheta+1)|}{(\vartheta+1)(\mho+1)}$$
$$|T(\vartheta) - T(\mho)| = \frac{|\vartheta\mho + \vartheta - \mho\vartheta - \mho|}{(\vartheta+1)(\mho+1)}$$
$$|T(\vartheta) - T(\mho)| = \frac{|\vartheta - \mho|}{(\vartheta+1)(\mho+1)}$$
$$|T(\vartheta) - T(\mho)| \leq |\vartheta - \mho|$$

Squaring both sides, we get,

$$|T(\vartheta) - T(\mho)|^2 \leq |\vartheta - \mho|^2$$

Similarly,

$$|T(\mho) - T(\tilde{\imath})|^2 \leq |\mho - \tilde{\imath}|^2,$$
$$|T(\tilde{\imath}) - T(\vartheta)|^2 \leq |\tilde{\imath} - \vartheta|^2.$$

Hence for any $\vartheta, \mho, \tilde{\imath} \in \aleph$,

$$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}) = max\{|T(\vartheta) - T(\mho)|^2,\ |T(\mho) - T(\tilde{\imath})|^2, |T(\tilde{\imath}) - T(\vartheta)|^2\}$$
$$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}) \leq \mathfrak{S}(\vartheta, \mho, \tilde{\imath}).$$

Since the simulation function is defined by

$$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}), \mathfrak{S}(\vartheta, \mho, \tilde{\imath}) = \mathfrak{S}(\vartheta, \mho, \tilde{\imath}) - \mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath})$$
$$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}), \mathfrak{S}(\vartheta, \mho, \tilde{\imath})) \geq 0, \text{ for all } \vartheta, \mho, \tilde{\imath} \in \aleph.$$

Now, all the assumptions of Theorem II.i. are satisfied. Consequently, T has a fixed point.

Clearly, the unique fixed point of T is $\vartheta = 0$.

III. Consequences

Corollary III.i. Let $(\aleph, \mathfrak{S})$ be a interpolative complete $\mathfrak{S}$-metric space and $T: \aleph \to \aleph$ be a mapping satisfying the following condition:

$\mathfrak{S}(T\vartheta, T\mho, T\mho) \leq \lambda\mathfrak{S}(\vartheta, \mho, \mho)$, for all $\vartheta, \mho, \mho \in \aleph$, where $\lambda \in [0,1]$.

Then, T has a unique fixed point in $\aleph$.

Proof: Define $\varsigma_B: [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\varsigma_B(m, \mathcal{I}, \mathcal{I}) = \lambda\mathcal{I} - m$ for all $\mathcal{I}, m \in [0,\infty)$.

Note that the mapping T is a Z-contraction with respect to $\varsigma_B \in Z$. Therefore, the result follows by taking $\varsigma = \varsigma_B$ in Theorem II.i.

Corollary III.ii. Let $(\aleph, \mathfrak{S})$ be a interpolative complete $\mathfrak{S}$-metric space and $T: \aleph \to \aleph$ be a mapping satisfying the following condition:

$\mathfrak{S}(T\vartheta, T\mho, T\mho) \leq \mathfrak{S}(\vartheta, \mho, \mho) - \varphi(\mathfrak{S}(\vartheta, \mho, \mho))$, for all $\vartheta, \mho, \mho \in \aleph$, where $\varphi : [0,\infty) \to [0,\infty)$ is lower semi continous function and $\varphi^{-1}(0) = \{0\}$.

Then, T has a unique fixed point in $\aleph$.

Proof: Define $\varsigma_R: [0,\infty) \to [0,\infty) \to \mathbb{R}$ by $\varsigma_R(m, \mathcal{I}, \mathcal{I}) = \mathcal{I} - \varphi(\mathcal{I}) - m$, for all $\mathcal{I}, m \in [0,\infty)$. Note that the mapping is a Z-contraction with respect to $\varsigma_R \in Z$. Therefore, the result follows by taking $\varsigma = \varsigma_R$ in Theorem II.i.

Corollary III.iii. Let $(\aleph, \mathfrak{S})$ be a interpolative complete $\mathfrak{S}$-metric space and $T : \aleph \to \aleph$ be a mapping satisfying the following condition:
$\mathfrak{S}(T\vartheta, T\mho, T\mho) \leq \varphi(\mathfrak{S}(\vartheta, \mho, \mho)) \times G(\vartheta, \mho, \mho)$, for all $\vartheta, \mho, \mho \in \aleph$, where φ : $[0, +\infty) \to [0,1)$ be a mapping such that $\mathrm{Lim}sup_{t\to r^+}\varphi(m) < 1$, for all $r > 0$.
Then, T has a unique fixed point.

Proof: Define $\varsigma_R: [0, \infty) \to [0, \infty) \to \mathbb{R}$ by $\varsigma_R(m, \mathcal{J}, \mathcal{J}) = \mathcal{J}\varphi(\mathcal{J}) - m$ for all $\mathcal{J}, m \in [0, \infty)$. Note that the mapping T is a Z-contraction with respect to $\varsigma_R \in Z$. Therefore, the result follows by taking $\varsigma = \varsigma_R$ in Theorem II.i.

Corollary III.iv. Let $(\aleph, \mathfrak{S})$ be a interpolative complete $\mathfrak{S}$-metric space and $T: \aleph \to \aleph$ be a mapping satisfying the following condition:

$\mathfrak{S}(T\vartheta, T\mho, T\mho) \leq \eta(\mathfrak{S}(\vartheta, \mho, \mho))$, for all $\vartheta, \mho, \mho \in \aleph$, where $\eta : [0, +\infty) \to [0,1)$ be an upper semi continous mapping such that $\eta(m) < m$, for all $m > 0$ and $\eta(0) = 0$.

Then, T has a unique fixed point.

Proof: Define $\varsigma_B W: [0, \infty) \to [0, \infty) \to \mathbb{R}$ by $\varsigma_B W(m, \mathcal{J}, \mathcal{J}) = \mathcal{J}\eta(\mathcal{J}) - m$ for all $\mathcal{J}, m \in [0, \infty)$. Note that the mapping T is a Z-contraction with respect to $\varsigma_B W \in Z$. Therefore, the result follows by taking $\varsigma = \varsigma_B W$ in Theorem II.i.

Corollary III.v. Let $(\aleph, \mathfrak{S})$ be a interpolative complete $\mathfrak{S}$-metric space and $T: \aleph \to \aleph$ be a mapping satisfying the following condition:

$\int \binom{\mathfrak{S}(T\vartheta, T\mho, T\mho)}{0} \phi(m)\mathrm{d}m \leq \mathfrak{S}(\vartheta, \mho, \mho)$, for all $\vartheta, \mho \in \aleph$, $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\int \binom{m}{0} \phi(m)\mathrm{d}m$ exists and $\int \binom{c}{0} \phi(m)\mathrm{d}m > \epsilon$, for each $\epsilon > 0$.

Then, T has a unique fixed point.

Proof: Define $\varsigma_k: [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\varsigma_k(m, \mathcal{J}, \mathcal{J}) = \mathcal{J} - \int \binom{m}{0} \phi(\mathrm{u})\mathrm{du}$, for all $\mathcal{J}, m \in [0, \infty)$.

Then $\varsigma_k \in Z$. Therefore, the result follows by taking $\varsigma = \varsigma_k$ in Theorem II.i.

IV. Application to solve an integral equation

Consider the nonlinear Fredholm integral equation

$\vartheta(t) = \int_0^1 \frac{\vartheta(s)}{1+\vartheta(s)} ds,\ t \in [0,1]$.
Let $\aleph = C([0,1], \mathbb{R})$ with $\mathfrak{S}(\vartheta, \mho, \tilde{\imath}) = \max\{||\vartheta - \mho||_\infty^2, ||\mho - \tilde{\imath}||_\infty^2, ||\tilde{\imath} - \vartheta||_\infty^2\}$.
From Example II.iii., we can say that the function $\mathfrak{S}(\vartheta, \mho, \tilde{\imath})$ does not form a $\mathfrak{S}$-metric space but it is interpolative $\mathfrak{S}$-metric space with $\tilde{\gamma} = \frac{1}{2}, \tilde{\varkappa} = 2$.
Assume that $\vartheta(t), \mho(t), \tilde{\imath}(t) \geq 0$, for all $t \in [0,1]$.
Define the operator
$(T\vartheta)(t) = \int_0^1 \frac{\vartheta(s)}{1+\vartheta(s)} ds,\ t \in [0,1]$.
For any $\vartheta, \mho, \tilde{\imath} \in \aleph$,

$$|(T\vartheta_n)(t) - (T\mho)(t)| \leq \int_0^1 \frac{\vartheta(s)}{1+\vartheta(s)} - \frac{\mho(s)}{1+\mho(s)}|ds$$

$$|(T\vartheta_n)(t) - (T\mho)(t)| \leq \int_0^1 \frac{\vartheta(s) - \mho(s)}{1+|\vartheta(s) - \mho(s)|} ds. \tag{4.1}$$

For $\vartheta, \mho \geq 0$,

$|\frac{\vartheta}{1+\vartheta} - \frac{\mho}{1+\mho}| = \frac{\vartheta-\mho}{(1+\vartheta)(1+\mho)}$

Since

$(1+\vartheta)(1+\mho) \geq 1+|\vartheta-\mho|$

We obtain

$|\frac{\vartheta}{1+\vartheta} - \frac{\mho}{1+\mho}| = \frac{\vartheta-\mho}{1+|\vartheta-\mho|}$

From (4.1), we get

$|(T\vartheta_n)(t) - (T\mho)(t)| \leq \frac{\mathfrak{L}}{1+\mathfrak{L}}$

where $k = \max\{||\vartheta-\mho||_\infty^2, ||\mho-\tilde{\imath}||_\infty^2, ||\tilde{\imath}-\vartheta||_\infty^2\}$

Then,

$\mathfrak{S}(\vartheta, \mho, \tilde{\imath}) = \mathfrak{L}^2$

Hence

$||T\vartheta - T\mho||_\infty \leq \frac{\mathfrak{L}}{1+\mathfrak{L}}.$

Similarly

$$||T\mho - T\tilde{\imath}||_\infty \leq \frac{\mathfrak{L}}{1+\mathfrak{L}}$$

$||T\tilde{\imath} - T\vartheta||_\infty \leq \frac{\mathfrak{L}}{1+\mathfrak{L}}$

Consequently

$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}) = \frac{\mathfrak{L}^2}{(1+\mathfrak{L})^2}$

Using $\mathfrak{L}^2 = \mathfrak{S}(\vartheta, \mho, \tilde{\imath})$

$$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}) \leq \frac{\mathfrak{S}(\vartheta,\mho,\tilde{\imath})}{\left(1+\sqrt{\mathfrak{S}(\vartheta,\mho,\tilde{\imath})}\right)^2} \tag{4.2}$$

Choose the simulation function

$$\text{Define } \zeta(\mathcal{I}, m) = \frac{m}{(1+\sqrt{m})^2} - \mathcal{I}, \quad \mathcal{I}, m \geq 0. \tag{4.3}$$

Put $\mathcal{I} = \mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath})$ and $m = \mathfrak{S}(\vartheta, \mho, \tilde{\imath})$.

By inequality (4.2), we get

$\mathcal{I} \leq \frac{m}{(1+\sqrt{m})^2}.$

Therefore, we have

$\zeta(\mathcal{I}, m) = \frac{m}{(1+\sqrt{m})^2} - \mathcal{I} \geq 0.$

$\zeta(\mathcal{J}, m) \geq 0$.

Hence

$\zeta\big(\mathfrak{S}(T\vartheta, T\mho, T\iota), \mathfrak{S}(\vartheta, \mho, \iota)\big) \geq 0$.

Thus, all the conditions of Theorem II.i. are satisfied. Clearly, T admits a unique fixed point 0.

Next, we show that the operator T does not satisfy the Banach contraction as well as the contraction given by Mustafa and Sims in $\mathfrak{S}$ metric spaces.

For this, let us consider

$\vartheta_n(t) = \frac{1}{n}, \mho(t) = 0$.

Then

$||\vartheta_n - \mho||_\infty = \frac{1}{n}$.

$$(T\vartheta_n)(t) = \int_0^1 \frac{1/n}{1 + 1/n} ds$$

$(T\vartheta_n)(t) = \frac{1}{n+1}$,

while $(T\mho)(t) = 0$.

Hence

$\frac{|(T\vartheta_n)(t) - (T\mho)(t)|}{||\vartheta_n - \mho||_\infty} = \frac{n}{n+1}$.

Since $\frac{n}{n+1} \to 1$, there exists no $k < 1$ satisfying

$||T\vartheta - T\mho||_\infty \leq k||\vartheta - \mho||_\infty$, for all $\vartheta, \mho \in \aleph$.

Hence, the classical Banach contraction principle is not applicable.

The failure also occurs directly in the $\mathfrak{S}$-metric space because

$\frac{\mathfrak{S}(T\vartheta_n, T\mho, T\mho)}{\mathfrak{S}(\vartheta_n, \mho, \mho)} = \left(\frac{n}{n+1}\right)^2 \to 1$.

From here, we can say that no $k < 1$ exists such that $\mathfrak{S}(T\vartheta, T\mho, T\iota) \leq k\mathfrak{S}(\vartheta, \mho, \iota)$.

V. Applications to solve a boundary value problem

Consider the linear BVP

$\vartheta''(t) + \lambda\vartheta(t) = 0, t \in [0,1], \vartheta(0) = 0, \ \vartheta(1) = 0$.

Using the Green's function of the Dirichlet problem, the above BVP can be written in equivalent integral form

$\vartheta(t) = \lambda \int_0^1 \mathfrak{S}(t, \mathfrak{s})\vartheta(\mathfrak{s}) d\mathfrak{s}, t \in [0,1]$

Where, $\mathfrak{S}(t, \mathfrak{s}) = \begin{cases} \mathfrak{s}(1 - t), & \mathfrak{s} \leq t, \\ t(1 - \mathfrak{s}), & \mathfrak{s} \geq t. \end{cases}$

Let $\aleph = C([0,1], \mathbb{R})$ with $\mathfrak{S}(\vartheta, \mho, \iota) = ||\vartheta - \mho||_\infty^2 + ||\mho - \iota||_\infty^2 + ||\iota - \vartheta||_\infty^2$.

By Example II.iii., we can say that the function $\mathfrak{S}(\vartheta, \mho, \tilde{\imath})$ does not form a $\mathfrak{S}$-metric space but it is interpolative $\mathfrak{S}$-metric space with $\tilde{\gamma} = \frac{1}{2}, \tilde{\varkappa} = 2$.
Assume that $\vartheta(t), \mho(t), \tilde{\imath}(t) \geq 0$, for all $t \in [0,1]$.
Define the operator

$$(T\vartheta)(t) = \lambda \int_0^1 \mathfrak{S}(t, \mathfrak{s})\vartheta(\mathfrak{s})d\mathfrak{s}$$

$$||T\vartheta||_\infty \leq (sup_{t\in[0,1]} \frac{t(1-t)}{2}) \, ||\vartheta||_\infty$$

$$||T\vartheta||_\infty = \frac{1}{8}||\vartheta||_\infty$$

Similarly $||T\mho||_\infty = \frac{1}{8}||\mho||_\infty$

We have

$||T\vartheta - T\mho||_\infty \leq \mathfrak{L}||\vartheta - \mho||_\infty$, with $\mathfrak{L} = \frac{|\lambda|}{8}$ <1.
Define $\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}) = ||T\vartheta - T\mho||^2_\infty + ||T\mho - T\tilde{\imath}||^2_\infty + ||T\tilde{\imath} - T\vartheta||^2_\infty$
$\mathfrak{S}(\vartheta, \mho, \tilde{\imath}) = ||\vartheta - \mho||^2_\infty + ||\mho - \tilde{\imath}||^2_\infty + ||\tilde{\imath} - \vartheta||^2_\infty\}$
$\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}) \leq \mathfrak{L}^2\mathfrak{S}(\vartheta, \mho, \tilde{\imath})$, where $0 < \mathfrak{L} < 1$
Choose $\zeta(m, \mathcal{J}) = \mathcal{J} - m$
$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}), \mathfrak{S}(\vartheta, \mho, \tilde{\imath})) = \mathfrak{S}(\vartheta, \mho, \tilde{\imath})) - \mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}))$
$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}), \mathfrak{S}(\vartheta, \mho, \tilde{\imath})) \geq (1-\mathfrak{L}^2)\, \mathfrak{S}(\vartheta, \mho, \tilde{\imath})$, where $\mathfrak{L}^2 = \frac{\lambda^2}{64}$
$\zeta(\mathfrak{S}(T\vartheta, T\mho, T\tilde{\imath}), \mathfrak{S}(\vartheta, \mho, \tilde{\imath})) \geq 0$
Thus, the condition of Theorem II.i. is satisfied.

VI. Conclusion:

In this paper, we established a unique fixed point theorem in interpolative metric spaces using simulation functions. The validity of the result is supported through suitable examples, and its applicability has been demonstrated by proving the existence and uniqueness of solutions for Volterra integral equations and boundary value problems.

Conflict of Interest:

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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