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MATHEMATICAL MODELING AND STABILITY ANALYSIS OF FOREST RESOURCE DYNAMICS UNDER POPULATION PRESSURE AND AFFORESTATION EFFORTS

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Abstract

In this article, a nonlinear dynamical model is developed to study the combined effect of population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts on forest resources dynamics. The governing dynamics of the system were represented by six nonlinear differential equations describing the interactions among forest resources, population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts. The positivity and boundedness of the model solutions are established to ensure the biological validity of the system. Equilibrium points are identified, and sufficient conditions for feasibility and uniqueness of the interior equilibrium are discussed. Local stability of the equilibrium points is analysed through the Jacobian matrix, characteristic equation, and Routh-Hurwitz Stability criteria, while global stability is examined using a suitable Lyapunov function under appropriate parameter restrictions. Numerical simulations are carried out using the MATLAB ODE45 solver to validate the theoretical findings and to illustrate the long-term behaviour of the system. The results show that increased population pressure, urbanization, industrialization, and residential expansion significantly reduce forest resources, while afforestation efforts contribute positively to forest regeneration and restoring ecological balance. Global sensitivity analysis is also performed using Latin Hypercube Sampling and PRCC indices. The findings suggest that sustainable forest management requires controlling anthropogenic pressures along with strengthening afforestation programs. The findings provide important insights for environmental planners and policymakers to design effective forest conservation and sustainable development strategies.

Keywords: Afforestation, Forest resource, Mathematical model, population pressure, Stability analysis, Industrialization.

I. Introduction

Forest resources are being depleted at an alarming rate in different parts of the world due to increasing human activities. One of the main reasons for deforestation is population pressure, since a growing population needs more land for agriculture, infrastructure, and settlements. Urbanization further accelerates forest loss by converting forest areas into roads and urban settlements. Similarly, industrialization increases the exploitation of forest resources for mining, raw materials, timber extraction, and industry growth. Due to clearing forest areas for the construction of homes and residential colonies, residential land expansion also plays a crucial role in the depletion of forest resources. Population growth, urbanization, industrialization, and residential development all work together to reduce forest cover and disrupt ecosystem stability. On the other hand, conservation and afforestation initiatives can lessen these negative impacts on forests. Therefore, sustainable forest management and environmental planning depend on how these factors interact with forest resources.

Mathematical modeling has become a powerful tool for understanding the interaction between forest ecosystems and anthropogenic pressures. Over the past decades, various researchers have developed mathematical systems to analyse and study the depletion and conservation of forest resources. One of the earliest mathematical approaches to model forest resource depletion was presented by Shukla and Dubey [XXV, XXIV]. These studies demonstrated that increasing population pressure and environmental pollution are the major causes of forest depletion, while conservation measures can stabilize the forest ecosystem. Shukla and Mishra [XXVI] extended this mathematical framework by incorporating industrialization and technological progress and demonstrated that industry expansion intensifies forest depletion unless sustainable technologies and environmental policies are adopted. Institutional aspects of forest resources conservation were emphasized by Agrawal and Gibson [I, XII], who argued that local institutions and community-based governance are essential for the prevention of overexploitation of forest resources. Ramdhani and Sebastian [XXI, XXII, XXIII] modelled the combined effect of population growth and industrialization and concluded that industrial crowding and population growth significantly accelerate forest depletion, although appropriate management policies can control deforestation. Economic and social interventions introduced by Misra and Lata [XVI, XVIII] conclude that economic incentives can reduce population pressure and improve forest conservation. Subsequently, Eswari, Hussen and Pratama [VII, XIV, XX] incorporated environmental awareness, industrialization and spatial interactions in a forest conservation mathematical framework. Their studies revealed that environmental awareness programs and regional management policies can contribute positively to forest sustainability. The role of socio-economic changes and human development in forest degradation was studied by Aquino and Bologna [IV]. Their finding indicated that reducing population growth with optimum socio-economic changes can significantly improve forest sustainability and help maintain ecological balance. Similarly, Didiharyono and Ezeorah [V, VIII] emphasized broader anthropogenic impacts. They confirmed that population density and industrialization remain dominant drivers for forest depletion. Meanwhile, Goshu and Endalew [XIII] highlighted the importance of forest conservation, economic incentives, and reforestation programs. In recent advances, Matia [XVII] developed an imprecise forest model to study the optimal management of forest resources influenced by population growth. In addition, Fanuel and co-authors [X] analysed conservation of forest resources and wildlife populations by quantifying uncertainty of parameters. Further, they [XI] worked on fuzzy mathematical modeling to investigate forest depletion and forest-dependent wildlife populations. They suggested that human activities significantly affect forest dynamics and that uncertainty modeling can improve the accuracy of ecological prediction. Other recent studies by [II, III, VI, XV] developed mathematical modeling frameworks to analyse the effect of industrialization, population growth, and associated socio-economic factors. Their studies examined the optimum industrial growth and existence of a solution for the optimal control problem for forest management.

In the above studies and literature, the combined effect of population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts has not been studied; therefore, this research work proposed an integrated non-linear differential equation-based mathematical unified framework incorporating effects of population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts on forest resources dynamics.

II. Formulation of Mathematical Model

This section presents a non-linear differential equation-based mathematical framework to analyze the effects of population pressure, urbanization, industrialization, residential land expansion, and afforestation strategies. The proposed model assumes that forest resources show logistic growth in the absence of human interference. Population growth pressure, industrialization, urbanization, and residential land expansion are the major contributors to forest depletion, whereas afforestation enhances forest resource recovery. Population pressure increases with urban and industrial growth but decreases with forest resource availability. Urbanization depends upon population demands and is limited by environmental & infrastructural capacity. Industrial growth is promoted by urbanization but controlled by conservation efforts. Residential land expansion is driven by urban development and population growth, whereas afforestation is aided by declining forest resources and hampered by competing urban and industrial activities.

In the model formation process, $F(t)$ denotes forest resource density with intrinsic growth rate r and carrying capacity K. $P(t)$ denotes population pressure at time t with intrinsic growth rate α_1 and maximum sustainable population pressure M. Consider $U(t)$ denotes urbanization level at time t with maximum urbanization level $U_{\max}$. $I(t)$ and $I_{\max}$ represent industrialization level at time t and Maximum industrialization capacity, respectively. $R(t)$ and $R_{\max}$ denotes residential land expansion level at time t and its maximum capacity, respectively. Similarly, $A(t)$ and $A_{\max}$ denotes afforestation level at time t and maximum afforestation capacity. The system dynamics is governed by the following nonlinear differential equations

$$\left.\begin{aligned}
\frac{dF}{dt} &= rF\left(1-\frac{F}{K}\right)-\beta_1 PF-\beta_2 UF-\beta_3 IF-\beta_4 RF+\beta_5 AF\\
\frac{dP}{dt} &= \alpha_1 P\left(1-\frac{P}{M}\right)+\alpha_2 UP+\alpha_3 IP-\alpha_4 FP-\alpha_5 P\\
\frac{dU}{dt} &= \gamma_1 P\left(1-\frac{U}{U_{\max}}\right)-\gamma_2 FU-\gamma_3 AU-\gamma_4 U\\
\frac{dI}{dt} &= \delta_1 U\left(1-\frac{I}{I_{\max}}\right)+\delta_2\frac{P}{1+P}I-\delta_3 FI-\delta_4 AI-\delta_5 I\\
\frac{dR}{dt} &= \eta_1 U\left(1-\frac{R}{R_{\max}}\right)+\eta_2\frac{P}{1+P}R-\eta_3 AR-\eta_4 R\\
\frac{dA}{dt} &= \theta_1 A\left(1-\frac{A}{A_{\max}}\right)+\theta_2\left(1-\frac{F}{K}\right)A-\theta_3 UA-\theta_4 IA-\theta_5 A
\end{aligned}\right\} \qquad (1)$$

with initial conditions
$F(0)=F_0>0, P(0)=P_0>0, U(0)=U_0>0, I(0)=I_0>0, R(0)=R_0>0, A(0)=A_0>0$
and $F(t),P(t),U(t),I(t),R(t),A(t)$ are non-negative state variables for $t\geq 0$. The positivity and boundedness of the solutions are established in the subsequent analysis. In the nonlinear system (1), $\beta_1,\beta_2,\beta_3,\beta_4$ and β_5 be the measures of depletion rate of forest due to population pressure, urbanization, industrialization, residential expansion, and recovery rate of forest due to afforestation, respectively. α_2 is the urbanization effect, α_3 is the industrialization effect, α_4 is the reduction effect of forest on population pressure, and α_5 be the Natural decay or regulation rate of population pressure. γ_1 is the urbanization growth rate due to population pressure, γ_2 and γ_3 are the inhibitory effects of forest and afforestation on urbanization, respectively, and γ_4 is the natural decay rate of urbanization. δ_1 is the Industrialization growth rate due to urbanization, δ_2 is the effect of population pressure on industrialization, δ_3 is the inhibitory effect of forest on industrialization, δ_4 is the inhibitory effect of afforestation and δ_5 natural decay rate of industrialization. η_1,η_2,η_3 and η_4 are the measures of residential growth rate due to urbanization, effect of population pressure on residential expansion, inhibitory effect of afforestation on residential expansion, and natural decay rate or regulation rate of residential expansion, respectively. $\theta_1,\theta_2,\theta_3,\theta_4$ and θ_5 represent afforestation intrinsic growth rate, stimulation of afforestation due to forest decline, suppression of afforestation due to urbanization, suppression of afforestation due to industrialization, and Natural decay of afforestation efforts.

III. Model Analysis

Positivity of the Solution

Theorem 1: If $F(0)>0, P(0)>0, U(0)>0, I(0)>0, R(0)>0, A(0)>0$ then the solution set $\{F(t),P(t),U(t),I(t),R(t),A(t)\}$ of the system (1) remains positive for all $t\geq 0$.

Proof: We write each equation from Model (1) in inequality form and then obtain a positive lower bound. Now, for the first equation in Model (1)

$$\frac{dF}{dt} = F\left[r\left(1-\frac{F}{K}\right) - \beta_1 P - \beta_2 U - \beta_3 I - \beta_4 R + \beta_5 A\right] \quad \frac{dF}{dt} \geq -\left(\beta_1 P + \beta_2 U + \beta_3 I + \beta_4 R\right) F$$

After integration, we get

$$F(t) \geq F(0)\exp\left\{-\int_0^t \left(\beta_1 P(s) + \beta_2 U(s) + \beta_3 I(s) + \beta_4 R(s)\right) ds\right\} > 0 \tag{2}$$

From the second equation,

$$\frac{dP}{dt} = P\left[\alpha_1\left(1-\frac{P}{M}\right) + \alpha_2 U + \alpha_3 I - \alpha_4 F - \alpha_5\right] \Rightarrow \frac{dP}{dt} \geq -\left(\alpha_4 F + \alpha_5\right) P$$

which gives, $P(t) \geq P(0)\exp\left\{-\int_0^t \left(\alpha_4 F(s) + \alpha_5\right) ds\right\} > 0$ (3)

From the third equation,

$$\frac{dU}{dt} = \gamma_1 P - \left[\frac{\gamma_1 P}{U_{\max}} + \gamma_2 F + \gamma_3 A + \gamma_4\right] U \Rightarrow \frac{dU}{dt} \geq -\left[\frac{\gamma_1 P}{U_{\max}} + \gamma_2 F + \gamma_3 A + \gamma_4\right] U$$

And therefore, $U(t) \geq U(0)\exp\left\{-\int_0^t \left(\frac{\gamma_1 P(s)}{U_{\max}} + \gamma_2 F(s) + \gamma_3 A(s) + \gamma_4\right) ds\right\} > 0$ (4)

From the fourth equation,

$$\frac{dI}{dt} = \delta_1 U\left[\delta_2 \frac{P}{1+P} - \frac{\delta_1 U}{I_{\max}} - \delta_3 F - \delta_4 A - \delta_5\right] I \Rightarrow \frac{dI}{dt} \geq -\left[\frac{\delta_1 U}{I_{\max}} + \delta_3 F + \delta_4 A + \delta_5\right] I$$

So, $$I(t) \geq I(0)\exp\left\{-\int_0^t \left(\frac{\delta_1 U(s)}{I_{\max}} + \delta_3 F(s) + \delta_4 A(s) + \delta_5\right) ds\right\} > 0 \tag{5}$$

From the fifth equation,

$$\frac{dR}{dt} = \eta_1 U + \left[\eta_2 \frac{P}{1+P} - \frac{\eta_1 U}{R_{\max}} - \eta_3 A - \eta_4\right] R \Rightarrow \frac{dR}{dt} \geq -\left[\frac{\eta_1 U}{R_{\max}} + \eta_3 A + \eta_4\right] R$$

After integration, we get $R(t) \geq R(0)\exp\left\{-\int_0^t \left(\frac{\eta_1 U(s)}{R_{\max}} + \eta_3 A(s) + \eta_4\right) ds\right\} > 0$ (6)

Finally, for the sixth equation,

$$\frac{dA}{dt} = \left[\theta_1\left(1-\frac{A}{A_{\max}}\right) + \theta_2\left(1-\frac{F}{K}\right) - \theta_3 U - \theta_4 I - \theta_5\right] A \Rightarrow \frac{dA}{dt} \geq -\left[\theta_3 U + \theta_4 I + \theta_5\right] A$$

Which gives, $$A(t) \geq A(0)\exp\left\{-\int_0^t \left(\theta_3 U(s) + \theta_4 I(s) + \theta_5\right) ds\right\} > 0 \tag{7}$$

Therefore, each state variable admits a positive exponential lower bound. Hence, all components of the model solution remain positive for $t \geq 0$, and the positive orthant is positively invariant for system (1). This completes the proof.

Boundedness of the Solution

Theorem 2: If $\delta_5 > \delta_2$ and $\eta_4 > \eta_2$, then the set

$\Delta = \{(F,P,U,I,R,A): 0 \le F \le M_F, 0 \le P \le M_P, 0 \le U \le M_U, 0 \le I \le M_I, 0 \le R \le M_R, 0 \le A \le M_A\}$

is a region of attraction for all solutions initiating in the interior of the positive orthant,

$M_U = \max\{U_0, U_{\max}\}, M_A = \max\left\{A_0, \frac{A_{\max}}{\theta_1}(\theta_1 + \theta_2 - \theta_5)_+\right\}, M_F = \max\left\{F_0, \frac{K}{r}(r + \beta_5 M_A)\right\},$

$M_I = \max\left\{I_0, \frac{\delta_1 M_U}{\frac{\delta_1 M_U}{\mathrm{I}_{\max}} + \delta_5 - \delta_2}\right\}, M_P = \max\left\{P_0, \frac{M}{\alpha_1}(\alpha_1 + \alpha_2 M_U + \alpha_3 M_I - \alpha_5)_+\right\},$

and $M_R = \max\left\{R_0, \frac{\eta_1 M_U}{\frac{\eta_1 M_U}{R_{\max}} + \eta_4 - \eta_2}\right\}$

Proof: From the model (1) third equation, $\frac{dU}{dt} = \gamma_1 P\left(1 - \frac{U}{U_{\max}}\right) - \gamma_2 FU - \gamma_3 AU - \gamma_4 U$

If $U > U_{\max}$, then $1 - \frac{U}{U_{\max}} < 0$, and all the other terms in the equation are negative.

Hence $\frac{dU}{dt} < 0$ for $U > U_{\max}$. Therefore, $0 \le U(t) \le M_U$.

From the sixth equation in Model (1),

$$\frac{dA}{dt} = \theta_1 A\left(1 - \frac{A}{A_{\max}}\right) + \theta_2\left(1 - \frac{F}{K}\right)A - \theta_3 UA - \theta_4 IA - \theta_5 A$$

Since $F(t) \ge 0, U(t) \ge 0, I(t) \ge 0$ and $(\theta_1 + \theta_2) > \theta_5$,

$\frac{dA}{dt} \le \theta_1 A\left(1 - \frac{A}{A_{\max}}\right) + \theta_2 A - \theta_5 A$. Thus, $\frac{dA}{dt} \le A\left\{(\theta_1 + \theta_2 - \theta_5) - \frac{\theta_1 A}{A_{\max}}\right\}$.

Then by comparison theorem, we get $0 \le A(t) \le M_A$.

For the first equation in system (1),

$$\frac{dF}{dt} = rF\left(1 - \frac{F}{K}\right) - \beta_1 PF - \beta_2 UF - \beta_3 IF - \beta_4 RF + \beta_5 AF$$

Since all the depletion terms are negative and $A(t) \le M_A$,

$\frac{dF}{dt} \le rF\left(1 - \frac{F}{K}\right) + \beta_5 AF$ or $\frac{dF}{dt} \le F\left\{(r + \beta_5 M_A) - \frac{rF}{K}\right\}$

Therefore, $0 \le F(t) \le M_F$.

For the fourth equation in system (1),

$$\frac{dI}{dt} = \delta_1 U\left(1 - \frac{I}{I_{\max}}\right) + \delta_2 \frac{P}{1+P} I - \delta_3 FI - \delta_4 AI - \delta_5 I$$

Using $U(t) \le M_U$ and $0 \le \frac{P}{1+P} \le 1$, we obtain $\frac{dI}{dt} \le \delta_1 M_U\left(1 - \frac{I}{I_{\max}}\right) + \delta_2 I - \delta_5 I$, thus

$\frac{dI}{dt} \le \delta_1 M_U - I\left\{\frac{\delta_1 M_U}{I_{\max}} + \delta_5 - \delta_2\right\}$. Since the coefficient of I is positive and, $\delta_5 > \delta_2$ then $0 \le I(t) \le M_I$.

Now, for the population pressure system equation in (1)

$\frac{dP}{dt} = \alpha_1 P\left(1 - \frac{P}{M}\right) + \alpha_2 UP + \alpha_3 IP - \alpha_4 FP - \alpha_5 P$, using $U(t) \le M_U$ and $I(t) \le M_I$, we obtain $\frac{dP}{dt} \le P\left\{\alpha_1 + \alpha_2 M_U + \alpha_3 M_I - \alpha_5 - \frac{\alpha_1 P}{M}\right\}$. Then, by the comparison theorem, we get $0 \le P(t) \le M_P$.

Similarly for the fifth equation in (1) $\frac{dR}{dt} = \eta_1 U\left(1 - \frac{R}{R_{\max}}\right) + \eta_2 \frac{P}{1+P} R - \eta_3 AR - \eta_4 R$,

using $U(t) \le M_U$ and $0 \le \frac{P}{1+P} \le 1$, we obtain $\frac{dR}{dt} \le \eta_1 M_U\left(1 - \frac{R}{R_{\max}}\right) + \eta_2 R - \eta_4 R$.

Hence $\frac{dR}{dt} \le \eta_1 M_U - R\left\{\frac{\eta_1 M_U}{R_{\max}} + \eta_4 - \eta_2\right\}$. Since $\eta_4 > \eta_2$, it follows that $0 \le R(t) \le M_R$. Therefore, every solution of the system (1) starting in the positive orthant remains bounded and ultimately enters the compact positively invariant set Δ. Hence, the system is dissipative in the biologically feasible region.

Equilibrium Analysis

Existence, Feasibility and Uniqueness of the Interior Equilibrium

For the interior coexistence equilibrium $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$, biological feasibility requires $F^* > 0, P^* > 0, U^* > 0, I^* > 0, R^* > 0, A^* > 0$. At equilibrium, system (1) gives

$$r\left(1 - \frac{F^*}{K}\right) - \beta_1 P^* - \beta_2 U^* - \beta_3 I^* - \beta_4 R^* + \beta_5 A^* = 0, \alpha_1\left(1 - \frac{P^*}{M}\right) + \alpha_2 U^* + \alpha_3 I^* - \alpha_4 F^* - \alpha_5 = 0,$$
$$\gamma_1 P^*\left(1 - \frac{U^*}{U_{\max}}\right) - \gamma_2 F^* - \gamma_3 A^* - \gamma_4 = 0, \frac{\delta_1 U^*}{1+P^*} + \delta_2 P^*\left(1 - \frac{I^*}{I_{\max}}\right) - \delta_3 F^* - \delta_4 A^* - \delta_5 = 0,$$
$$\frac{\eta_1 U^*}{1+P^*} + \eta_2 P^*\left(1 - \frac{R^*}{R_{\max}}\right) - \eta_3 A^* - \eta_4 = 0, \theta_1\left(1 - \frac{A^*}{A_{\max}}\right) + \theta_2\left(1 - \frac{F^*}{K}\right) - \theta_3 U^* - \theta_4 I^* - \theta_5 = 0.$$

Thus, the following sufficient conditions ensure positivity and feasibility of the interior equilibrium:

$$r + \beta_5 A^* > \beta_1 P^* + \beta_2 U^* + \beta_3 I^* + \beta_4 R^*,\ \alpha_1 + \alpha_2 U^* + \alpha_3 I^* > \alpha_4 F^* + \alpha_5,$$
$$\gamma_1 P^* > \gamma_2 F^* + \gamma_3 A^* + \gamma_4,\ \frac{\delta_1 U^*}{1+P^*} + \delta_2 P^* > \delta_3 F^* + \delta_4 A^* + \delta_5,$$
$$\frac{\eta_1 U^*}{1+P^*} + \eta_2 P^* > +\eta_3 A^* + \eta_4,\ \theta_1 + \theta_2\left(1 - \frac{F^*}{K}\right) > \theta_3 U^* + \theta_4 I^* + \theta_5.$$

Theorem 3: Assume that all parameters of system (1) are non-negative and that the feasibility conditions for $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$ are satisfied in the positively invariant region Δ. Then system (1) admits at least one biologically feasible interior equilibrium E^*. Moreover, if the associated equilibrium mapping with is a contraction on Δ, then E^* is unique.

Proof: From Theorem 2, all the solutions of the system (1) are ultimately bounded in the compact invariant region Δ. The steady state equations define a continuous equilibrium mapping $\Phi : \Delta \to \Delta$. Under the stated feasibility conditions, Φ maps the compact convex set Δ into itself. Therefore, by Brouwer's fixed point theorem, there exists at least one fixed point of Φ in Δ, and this fixed point corresponds to a positive coexistence equilibrium of system (1).

Furthermore, if there exists a constant $0 < L < 1$ such that

$$\|\Phi(X) - \Phi(Y)\| \le L\|X - Y\| \text{ where } X, Y \in \Delta,$$

Then Φ is a contraction mapping on Δ. Hence, by the Banach fixed point theorem, the fixed point is unique. Thus, under the stated conditions, the interior equilibrium exists, is biologically feasible, and unique within the region Δ.

Equilibrium Points

All the equilibrium points of the system (1) are obtained by putting

$$\frac{dF}{dt} = \frac{dP}{dt} = \frac{dU}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = \frac{dA}{dt} = 0 \tag{8}$$

Let the equilibrium point be $E = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$, then the system admits the following equilibrium points-

(i) Trivial Equilibrium Point: $E_0 = (0,0,0,0,0,0)$

This equilibrium always exists, which represents the complete absence of forest resources density as well as population pressure, urbanization, industrialization, residential land expansion, and afforestation.

(ii) Only Forest Equilibrium Point: $E_1 = (K,0,0,0,0,0)$

This equilibrium is also always existing and represents the situation where forest resources reach their carrying capacity in the absence of all human-induced pressure and afforestation.

(iii) Forest- Afforestation Equilibrium Point: $E_2 = (F_2,0,0,0,0,A_2)$

Where $A_2 = \dfrac{\theta_1 - \theta_5}{\dfrac{\theta_1}{A_{\max}} + \dfrac{\theta_2\beta_5}{r}}$ and $F_2 = K\left(1 + \dfrac{\beta_5 A_2}{r}\right)$. This equilibrium exists if $\theta_1 > \theta_5$ so that $A_2 > 0$ and $F_2 > 0$. This equilibrium depicts a state dominated by conservation and shows coexistence of forest resources and afforestation and nonexistence of population pressure, urbanization, industrialization, and residential expansion of land.

(iv) Interior Equilibrium Point: $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$,

where $F^* > 0, P^* > 0, U^* > 0, I^* > 0, R^* > 0, A^* > 0$ and satisfy the following equations-

$$P^* = M\left\{1+\frac{\alpha_2 U^* + \alpha_3 I^* - \alpha_4 F^* - \alpha_5}{\alpha_1}\right\}, F^* = K\left\{1-\frac{\beta_1 P^* + \beta_2 U^* + \beta_3 I^* + \beta_4 R^* - \beta_5 A^*}{r}\right\},$$

$$A^* = A_{\max}\left\{1-\frac{\theta_3 U^* + \theta_4 I^* + \theta_5 - \theta_2\left(1-\frac{F^*}{K}\right)}{\theta_1}\right\}, I^* = \frac{\delta_1 U^*}{\frac{\delta_1 U^*}{I_{\max}} - \delta_2 \frac{P^*}{1+P^*} + \delta_3 F^* + \delta_4 A^* + \delta_5},$$

$$R^* = \frac{\eta_1 U^*}{\frac{\eta_1 U^*}{R_{\max}} - \eta_2 \frac{P^*}{1+P^*} + \eta_3 A^* + \eta_4}, U^* = \frac{\gamma_1 P^*}{\frac{\gamma_1 P^*}{U_{\max}} + \gamma_2 F^* + \gamma_3 A^* + \gamma_4}$$

This equilibrium depicts the long-term coexistence of forest resources with population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts. The existence, feasibility, and uniqueness of this equilibrium follows from Theorem 3 under the stated parameter restrictions.

Local and Global Stability Analysis

For the system (1), let $f_1 = \frac{dF}{dt}, f_2 = \frac{dP}{dt}, f_3 = \frac{dU}{dt}, f_4 = \frac{dI}{dt}, f_5 = \frac{dR}{dt}, f_6 = \frac{dA}{dt}$, then the Jacobian is defined as

$J(F,P,U,I,R,A) = \left(\frac{\partial f_i}{\partial x_i}\right)_{6\times 6}$, $(x_1,x_2,x_3,x_4,x_5,x_6) = (F,P,U,I,R,A)$ and written as

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix}$$

Where

$$a_{11} = r - \frac{2rF}{K} - \beta_1 P - \beta_2 U - \beta_3 I - \beta_4 R + \beta_5 A,\ a_{12} = -\beta_1 F, a_{13} = -\beta_2 F, a_{14} = -\beta_3 F,$$

$$a_{15} = -\beta_4 F, a_{16} = \beta_5 F, a_{21} = -\alpha_4 P, a_{22} = \alpha_1 - \frac{2\alpha_1 P}{M} + \alpha_2 U + \alpha_3 I - \alpha_4 F - \alpha_5,$$

$$a_{23} = \alpha_2 P, a_{24} = \alpha_3 P, a_{25} = 0, a_{26} = 0, a_{31} = -\gamma_2 U, a_{32} = \gamma_1\left(1-\frac{U}{U_{\max}}\right), a_{33} = -\frac{\gamma_1 P}{U_{\max}} - \gamma_2 F - \gamma_3 A - \gamma_4,$$

$$a_{34} = 0, a_{35} = 0, a_{36} = -\gamma_3 U, a_{41} = -\delta_3 I, a_{42} = \delta_2 \frac{I}{(1+P)^2}, a_{43} = \delta_1\left(1-\frac{I}{I_{\max}}\right),$$

$$a_{44} = -\frac{\delta_1 U}{I_{\max}} + \delta_2 \frac{P}{1+P} - \delta_3 F - \delta_4 A - \delta_5, a_{45} = 0, a_{46} = -\delta_4 I, a_{51} = 0, a_{52} = \eta_2 \frac{R}{(1+P)^2},$$

$$a_{53}=\eta_1\left(1-\frac{R}{R_{\max}}\right),a_{54}=0,a_{55}=-\frac{\eta_1 U}{R_{\max}}+\eta_2\frac{P}{1+P}-\eta_3 A-\eta_4,a_{56}=-\eta_3 R,a_{61}=-\frac{\theta_2 A}{K},$$
$$a_{62}=0,a_{63}=-\theta_3 A,a_{64}=-\theta_4 A,a_{65}=0,a_{66}=\theta_1+\theta_2\left(1-\frac{F}{K}\right)-\theta_3 U-\theta_4 I-\theta_5-\frac{2\theta_1 A}{A_{\max}}$$

For the local stability of the trivial equilibrium $E_0=(0,0,0,0,0,0)$, the Jacobian matrix becomes

$$J(E_0)=\begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1-\alpha_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & -\gamma_4 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & -\delta_5 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & -\eta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_1+\theta_2-\theta_5 \end{pmatrix},$$

hence the eigenvalues are

$$\lambda_1=r,\lambda_2=\alpha_1-\alpha_5,\lambda_3=-\gamma_4,\lambda_4=-\delta_5,\lambda_5=-\eta_4,\lambda_6=\theta_1+\theta_2-\theta_5$$

Since $r>0$, the first eigenvalue $\lambda_1=r$ is positive; therefore, the trivial equilibrium point E_0 is unstable.

For the only Forest Equilibrium $E_1=(K,0,0,0,0,0)$, the Jacobian matrix is

$$J(E_1)=\begin{pmatrix} -r & -\beta_1 K & -\beta_2 K & -\beta_3 K & -\beta_4 K & -\beta_5 K \\ 0 & \alpha_1-\alpha_4 K-\alpha_5 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & -(\gamma_2 K+\gamma_4) & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & -(\delta_3 K+\delta_5) & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & -\eta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_1-\theta_5 \end{pmatrix}$$

and the eigenvalues are $\upsilon_1=-r,\upsilon_2=\alpha_1-\alpha_4 K-\alpha_5,\upsilon_3=-(\gamma_2 K+\gamma_4),\upsilon_4=-(\delta_3 K+\delta_5),\upsilon_5=-\eta_4,\upsilon_6=\theta_1-\theta_5$

Since for the positive parameters, eigenvalues $\upsilon_1,\upsilon_3,\upsilon_4,\upsilon_5$ are negative, and the other two eigenvalues (υ_2 and υ_6) will be negative if $\alpha_1<\alpha_4 K+\alpha_5$ and $\theta_1<\theta_5$. Hence E_1 is locally asymptotically stable if $\alpha_1<\alpha_4 K+\alpha_5$ and $\theta_1<\theta_5$.

Now the Jacobian matrix for the Forest-Afforestation equilibrium point $E_2=(F_2,0,0,0,0,A_2)$ where $A_2>0$ and $F_2>0$ is

$$J(E_2)=\begin{pmatrix} \sigma_1 & -\beta_1 F_2 & -\beta_2 F_2 & -\beta_3 F_2 & -\beta_4 F_2 & \beta_5 F_2 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & \sigma_4 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & \sigma_5 & 0 \\ -\frac{\theta_2 A_2}{K} & 0 & -\theta_3 A_2 & -\theta_4 A_2 & 0 & \sigma_6 \end{pmatrix}$$

where $\sigma_1=-\frac{rF_2}{K},\sigma_2=\alpha_1-\alpha_4F_2-\alpha_5,\sigma_3=-(\gamma_2F_2+\gamma_3A_2+\gamma_4),$

$\sigma_4=-(\delta_3F_2+\delta_4A_2+\delta_5),\sigma_5=-(\eta_3A_2+\eta_4),\sigma_6=-\frac{\theta_1A_2}{A_{\max}}$

The characteristic equation is

$$(\lambda-\sigma_2)(\lambda-\sigma_3)(\lambda-\sigma_4)(\lambda-\sigma_5)\left[(\lambda-\sigma_1)(\lambda-\sigma_6)+\frac{\beta_5\theta_2F_2A_2}{K}\right]=0 \qquad (9)$$

The four eigenvalues are $\sigma_2=\alpha_1-\alpha_4F_2-\alpha_5$, $\sigma_3=-(\gamma_2F_2+\gamma_3A_2+\gamma_4)$, $\sigma_4=-(\delta_3F_2+\delta_4A_2+\delta_5)$, and $\sigma_5=-(\eta_3A_2+\eta_4)$
And the other two eigenvalues are the roots of the quadratic equation

$$\lambda^2-(\sigma_1+\sigma_6)\lambda+\left(\sigma_1\sigma_6+\frac{\beta_5\theta_2F_2A_2}{K}\right)=0 \qquad (10)$$

Since $A_2>0$, $F_2>0$ and $\sigma_1<0,\sigma_3<0,\sigma_4<0,\sigma_5<0,\sigma_6<0$. Also, the roots of the quadratic equation (10) have negative real parts if $(\sigma_1+\sigma_6)<0$ and $\left(\sigma_1\sigma_6+\frac{\beta_5\theta_2F_2A_2}{K}\right)>0$. Therefore, the equilibrium point E_2 is locally asymptotically stable if $\alpha_1<\alpha_4F_2+\alpha_5$ and $\left(\sigma_1\sigma_6+\frac{\beta_5\theta_2F_2A_2}{K}\right)>0$.

Local Stability of Interior Equilibrium E^*

Let $J^*=J(E^*)$ be the Jacobian matrix of the system (1) evaluated at the equilibrium point $E^*=(F^*,P^*,U^*,I^*,R^*,A^*)$. The characteristic equation of J^* is given by $\det(\lambda I-J^*)=0$. Since the system (1) is six dimensional therefore characteristic equation can be written as

$$\lambda^6+c_1\lambda^5+c_2\lambda^4+c_3\lambda^3+c_4\lambda^2+c_5\lambda+c_6=0 \qquad (11)$$

Where the coefficients c_1,c_2,c_3,c_4,c_5 and c_6 depend on the entries of J^* and the model parameters. In particular, $c_1=-tr(J^*)$ and $c_6=\det(-J^*)$.

For the above sixth-order polynomial, the Routh-Hurwitz array is

$$\begin{array}{c|cccc} \lambda^6 & 1 & c_2 & c_4 & c_6 \\ \lambda^5 & c_1 & c_3 & c_5 & 0 \\ \lambda^4 & b_1 & b_2 & b_3 & 0 \\ \lambda^3 & d_1 & d_2 & 0 & 0 \\ \lambda^2 & e_1 & e_2 & 0 & 0 \\ \lambda & g_1 & 0 & 0 & 0 \\ \lambda^0 & c_6 & 0 & 0 & 0 \end{array}$$

, where

$$b_1 = \frac{c_1c_2 - c_3}{c_1}, b_2 = \frac{c_1c_4 - c_5}{c_1}, b_3 = c_6,$$
$$d_1 = \frac{b_1c_3 - c_1b_2}{b_1}, d_2 = \frac{b_1c_5 - c_1b_3}{b_1},$$
$$e_1 = \frac{d_1b_2 - b_1d_2}{d_1}, e_2 = b_3, g_1 = \frac{e_1d_2 - d_1e_2}{e_1}$$

Therefore, by the Routh-Hurwitz criterion, all roots of the characteristic equation have negative real parts if $c_1 > 0, b_1 > 0, d_1 > 0, e_1 > 0, g_1 > 0$ and $c_6 > 0$.

Theorem 4: The interior equilibrium $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$ of the system (1) is locally asymptotically stable if the coefficients of the characteristic polynomial of J^* satisfy $c_1 > 0, b_1 > 0, d_1 > 0, e_1 > 0, g_1 > 0,\ c_6 > 0$.

Proof: The linearized system around E^* is given by $\dot{X} = J^* X$,
where $X = (F - F^*, P - P^*, U - U^*, I - I^*, R - R^*, A - A^*)^T$.

The eigenvalues of J^* are the roots of the characteristic equation (11). According to the Routh-Hurwitz criterion, all roots of this polynomial have negative real parts if the first column of the Routh-Hurwitz array is strictly positive. Hence, under the stated conditions, all eigenvalues of J^* lie in the left half of the complex plane. Therefore, the interior equilibrium $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$ is locally asymptotically stable.

Global Stability of Interior Equilibrium E^*

Theorem 5: Let $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$ be the biologically feasible interior equilibrium of the system (1). If the negative quadratic terms in the Lyapunov derivative dominate all mixed nonlinear terms in the feasible region Δ, then E^* is globally asymptotically stable in Δ.

Proof: Consider the following Lyapunov function:

$$W = \left(F - F^* - F^* \ln\frac{F}{F^*}\right) + a\left(P - P^* - P^* \ln\frac{P}{P^*}\right) + \frac{b}{2}(U - U^*)^2 + \frac{c}{2}(I - I^*)^2 + \frac{d}{2}(R - R^*)^2 + \frac{e}{2}(A - A^*)^2 \tag{12}$$

where the constants $a, b, c, d, e > 0$. Since $x - x^* - x^* \ln\frac{x}{x^*} \geq 0$ for $x > 0$, the function W is positive definite in the feasible region Δ. Also, $W = 0$ only at $(F, P, U, I, R, A) = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$.

Differentiating W with respect to t, we obtain

$$\frac{dW}{dt} = \left(1 - \frac{F^*}{F}\right)\frac{dF}{dt} + a\left(1 - \frac{P^*}{P}\right)\frac{dP}{dt} + b(U - U^*)\frac{dU}{dt} + c(I - I^*)\frac{dI}{dt} + d(R - R^*)\frac{dR}{dt} + e(A - A^*)\frac{dA}{dt}$$

Using system (1) and equilibrium conditions at E^*, the above expression can be written as

$$\frac{dW}{dt} \le -D_1(F-F^*)^2 - D_2(P-P^*)^2 - D_3(U-U^*)^2 - D_4(I-I^*)^2 - D_5(R-R^*)^2 - D_6(A-A^*)^2 + \Lambda$$

where $D_i > 0,\ i = 1,2,...,6$ and Λ contains mixed nonlinear cross-product terms generated by the coupling among F,P,U,I,R,A. By Young's inequality, $|xy| \le \frac{\varepsilon}{2}x^2 + \frac{1}{2\varepsilon}y^2, \varepsilon > 0$, each mixed term in Λ can be bounded by a linear combination of quadratic terms [XXVII]. Hence, there exist non-negative constants $q_1, q_2, q_3, q_4, q_5, q_6$ such that:

$$\Lambda \le q_1(F-F^*)^2 + q_2(P-P^*)^2 + q_3(U-U^*)^2 + q_4(I-I^*)^2 + q_5(R-R^*)^2 + q_6(A-A^*)^2 .$$

Therefore,

$$\frac{dW}{dt} \le -(D_1-q_1)(F-F^*)^2 - (D_2-q_2)(P-P^*)^2 - (D_3-q_3)(U-U^*)^2 - (D_4-q_4)(I-I^*)^2$$
$$-(D_5-q_5)(R-R^*)^2 - (D_6-q_6)(A-A^*)^2$$

If the dominance conditions $D_i > q_i, i = 1,2,...,6$ holds, then there exist positive constants $\kappa_i = (D_i - q_i) > 0, i = 1,2,...,6$, such that:

$$\frac{dW}{dt} \le -\kappa_1(F-F^*)^2 - \kappa_2(P-P^*)^2 - \kappa_3(U-U^*)^2 - \kappa_4(I-I^*)^2 - \kappa_5(R-R^*)^2 - \kappa_6(A-A^*)^2$$

Thus, $\frac{dW}{dt} < 0$ for all $(F,P,U,I,R,A) \ne E^*$. Hence, W is a strictly Lyapunov function in Λ. Therefore, by Lyapunov's direct method [XIX] and LaSalle's invariance principle, the interior equilibrium $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$ is globally asymptotically stable in the feasible region Δ.

IV. Results and Discussion

Numerical Simulation

To demonstrate the theoretical results obtained in previous sections, the numerical simulation of the proposed model was performed. The simulation aims to verify the existence and stability of the interior equilibrium as well as to demonstrate the system's long-term dynamical behaviour under various initial conditions. For this, we consider the parameter values as:

$$r = 0.6, K = 50, \beta_1 = 0.002, \beta_2 = 0.002, \beta_3 = 0.001, \beta_4 = 0.001, \beta_5 = 0.0003,$$
$$M = 100, \alpha_1 = 0.25, \alpha_2 = 0.002, \alpha_3 = 0.004, \alpha_4 = 0.001, \alpha_5 = 0.13, U_{\max} = 30,$$
$$\gamma_1 = 0.15, \gamma_2 = 0.003, \gamma_3 = 0.005, \gamma_4 = 0.004, \delta_1 = 0.120, \delta_2 = 0.070, \delta_3 = 0.005,$$
$$\delta_4 = 0.004, \delta_5 = 0.002, I_{\max} = 15, R_{\max} = 20, \eta_1 = 0.1, \eta_2 = 0.04, \eta_3 = 0.01, \eta_4 = 0.1,$$
$$A_{\max} = 12, \theta_1 = 0.2, \theta_2 = 0.03, \theta_4 = 0.007, \theta_5 = 0.1$$

Using the baseline parameter values, the interior equilibrium is obtained approximately as $E^* = (45,20,8,3,4,5)$. The Jacobian matrix J^* was evaluated at this equilibrium.

The corresponding eigenvalues are

$$\lambda_1 = -0.554103, \lambda_2 = -0.333271 + 0.038916i, \lambda_3 = -0.333271 - 0.038916i,$$
$$\lambda_4 = -0.204617, \lambda_5 = -0.092252, \lambda_6 = -0.015025$$

Since all eigenvalues have negative real parts, they satisfy Routh-Hurwitz stability conditions for the selected baseline parameter set. Therefore, the interior equilibrium point E^* is locally asymptotically stable.

To examine the long-term numerical behaviour of the system (1), the model was integrated using the MATLAB ODE45 solver over the time interval $t \in [0, 200]$. The simulations were carried out for different initial states of (F, P, U, I, R, A) given by, $A_0 = (40,15,6,2,3,4), B_0 = (48,25,10,4,5,6), C_0 = (35,30,12,5,6,4), D_0 = (46,18,7,2.5,3.5,7)$

The numerical trajectories corresponding to these initial states approach the same coexistence equilibrium E^*. This confirms that the numerical behaviour of the system is consistent with analytical stability results. Hence, under the selected parameter set, the system approaches a stable coexistence state in which forest resources, population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts attain a long-term balance.

The dominant eigenvalue is $\lambda_6 = -0.015025$. Since this eigenvalue has the largest real part and lies close to the imaginary axis, which represent the slowest mode of convergence toward the interior equilibrium. This indicates that the coupled forest-anthropogenic pressure-human-afforestation system may require a relatively long time to return to its coexistence state, even though the equilibrium state is locally asymptotically stable.

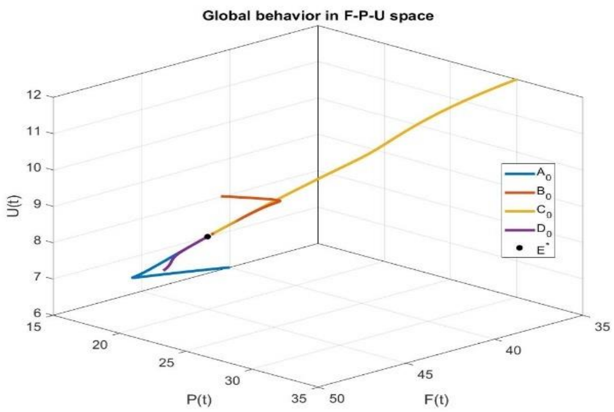

Fig. 1. Global Stability of E^* in F-P-U Space

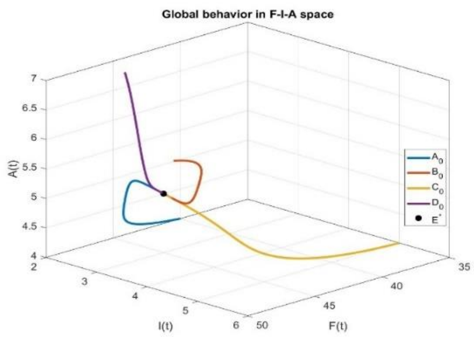

Fig. 2. Global Stability of E^* in *F-I-A* Space

The global behaviour in $F-P-U$ and $F-I-A$ spaces shows that all trajectories converge towards the same coexistence equilibrium point $E^* = \left(F^*, P^*, U^*, I^*, R^*, A^*\right)$ for the different initial conditions. This confirms the system's global asymptotic stability and suggest for the implementation of strong afforestation and forest conservation policies can restore the ecological balance and maintain sustainable forest resources in the long-term.

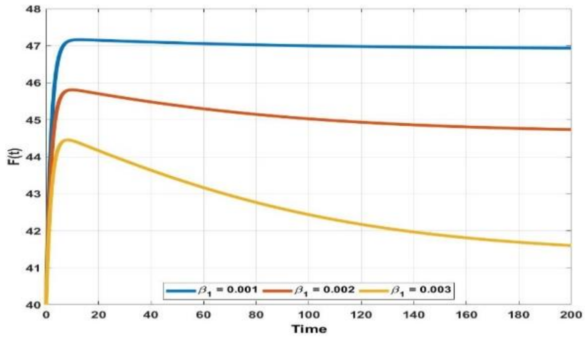

Fig. 3. Variation of $F(t)$ versus Time (t) for the different values of β_1

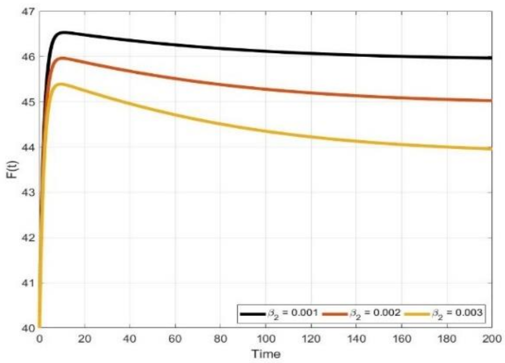

Fig. 4. Variation of $F(t)$ versus Time *(t)* for the different values of β_2

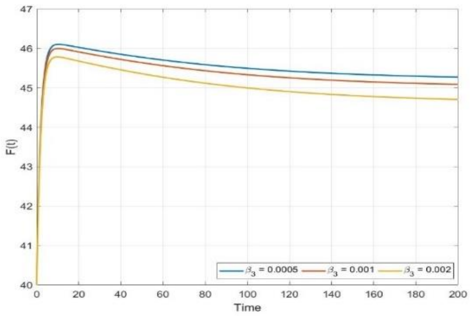

Fig. 5. Variation of $F(t)$ versus Time *(t)* for the different values of β_3

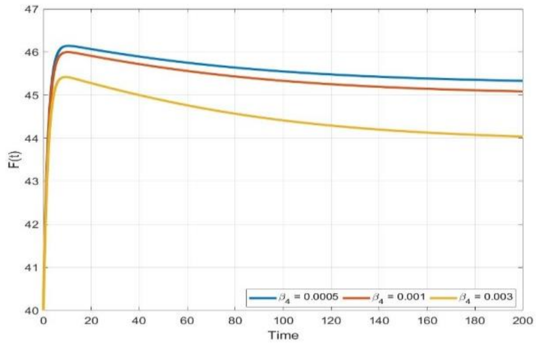

Fig. 6. Variation of $F(t)$ versus Time *(t)* for the different values of β_4

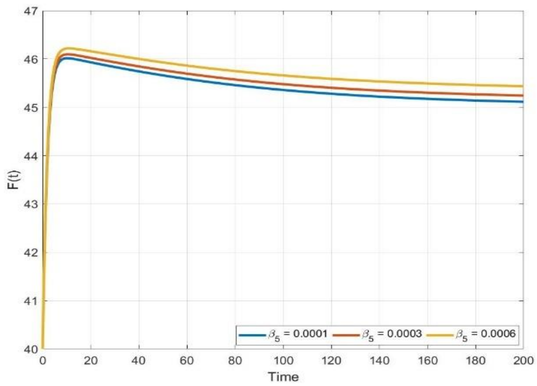

Fig. 7. Variation of $F(t)$ versus Time *(t)* for the different values of β_5

The parameter variation graphs clearly depict the effect of different anthropogenic factors on forest resources dynamics. The parameters $\beta_1, \beta_2, \beta_3$ and β_4 represent the rate at which forest resources are depleted due to population pressure, urbanization, industrialization, and residential land expansion, respectively. The graphical representation of parameter variation shows that increasing of these parameter values causes a noticeable decline in forest resource density. In particular, higher values of β_1 indicate that the rapid population pressure increases the demand for residential land, fuel, and forest products, resulting in more severe depletion of forest. Similarly, large values of β_2 and β_3 indicate that urban expansion and industrialization intensify the extraction of forest-based resources and land conversion. The greater value of β_4 further accelerates forest depletion because of residential land expansion continuously converts forest area into human settlements and infrastructure needs. In contrast, the afforestation parameter β_5 shows a positive contribution to forest recovery. The corresponding graph depicts that the larger values of the parameter significantly increase forest resources density and help the system approach a higher equilibrium state. Thus, afforestation efforts act as an effective mechanism to control the destructive effects of anthropogenic pressure on forest resources.

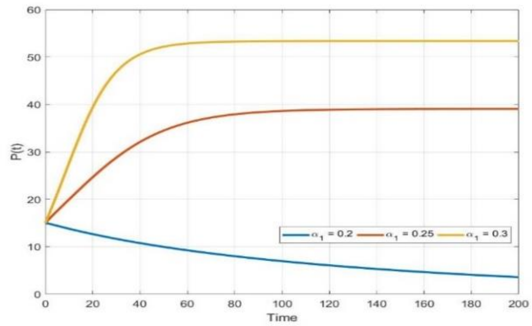

Fig. 8. Variation of $P(t)$ versus Time *(t)* for the different values of α_1

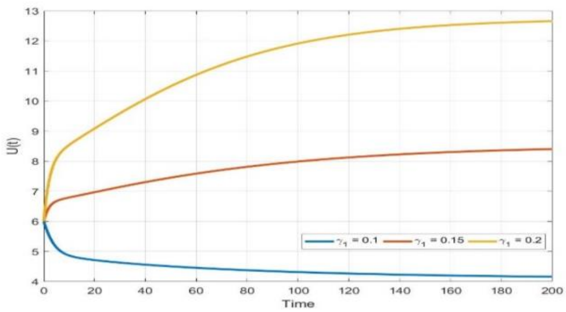

Fig. 9. Variation of $U(t)$ versus Time *(t)* for the different values of γ_1

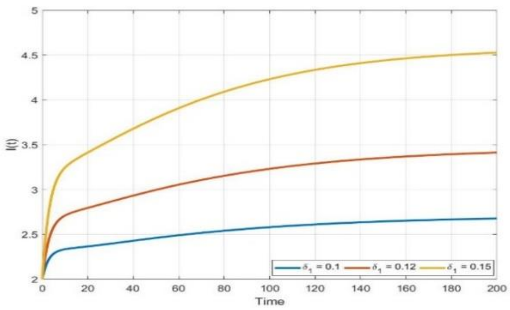

Fig. 10. Variation of $I(t)$ versus Time *(t)* for the different values of δ_1

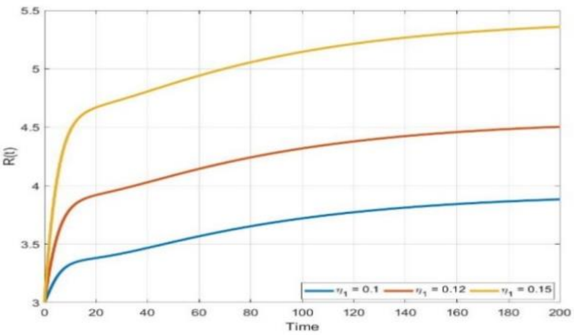

Fig. 11. Variation of $R(t)$ versus Time *(t)* for the different values of η_1

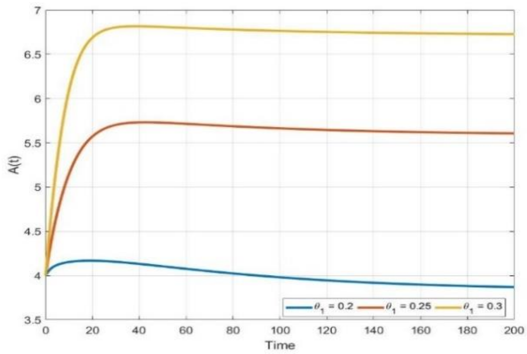

Fig. 12. Variation of $A(t)$ versus Time *(t)* for the different values of θ_1

The graphical representation corresponding to $\alpha_1, \gamma_1, \delta_1$ and η_1 parameters also indicates that an increase in growth rate of population, urbanization, industrialization, and residential land expansion indirectly puts pressure on long-term density of forest resources. Thus, the proposed nonlinear model is mathematically valid and ecologically meaningful.

Global Sensitivity and Uncertainty Analysis

The one-parameter-at-a-time simulations provide useful qualitative information about the effect of individual parameters. However, since system (1) is nonlinear and contains several coupled feedback interactions among F, P, U, I, R and A, therefore a global sensitivity analysis was performed to quantify the relative effect of selected parameters under simultaneous variations.

Latin Hypercube Sampling was used to generate $N = 1000$ parameter sets by varying the selected parameters $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \alpha_1, \gamma_1, \delta_1, \eta_1, \theta_1$ within $\pm 20\%$ of their baseline values, that is $p \in [0.8 p_0, 1.2 p_0]$, where p_0 denotes the baseline value. The system (1) was solved over $t \in [0, 200]$ for each sample. The output quantities considered were the long-term forest resource density $F(200)$ and the forest sustainability index $S_F = \dfrac{F(200)}{K}$. To identify the most influential parameters, Partial rank correlation coefficients (PRCC) were then computed. The results are presented in Table 1 and Fig. 13.

Table 1: PRCC Sensitivity indices for $F(200)$ and S_F

Parameter	PRCC with *F*(200)	p-value *F*(200)	PRCC with S_F	p-value S_F	Effect
β_1	-0.99496	0	-0.99496	0	Strong negative
β_2	-0.95735	0	-0.95735	0	Strong negative
β_3	-0.61659	7.7529e-105	-0.61659	7.7529e-105	Strong negative
β_4	-0.73832	2.3482e-171	-0.73832	2.3482e-171	Strong negative
β_5	-0.029427	0.35476	-0.029427	0.35476	Very weak/negligible
α_1	-0.87487	1.221e-313	-0.87487	1.221e-313	Strong negative
γ_1	-0.06891	0.030071	-0.06891	0.030071	Weak negative
δ_1	0.018075	0.5698	0.018075	0.5698	Very weak/negligible
η_1	0.013683	0.66704	0.013683	0.66704	Very weak/negligible
θ_1	-0.016541	0.603	-0.016541	0.603	Very weak/negligible

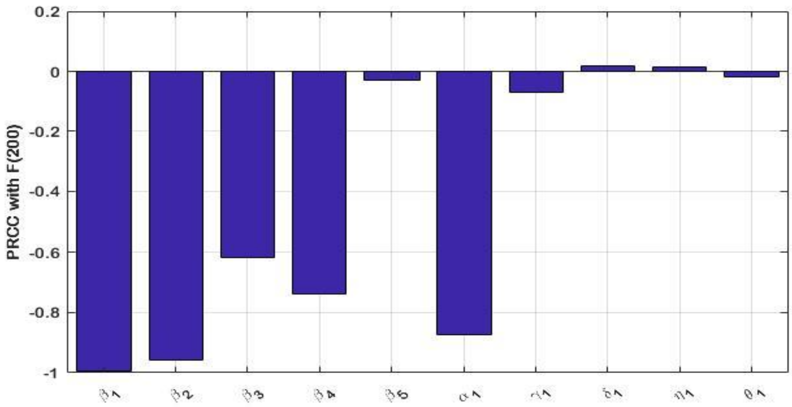

Fig.13. PRCC-based global sensitivity indices of selected parameters w.r.t $F(200)$

The PRCC results show that $\beta_1, \beta_2, \alpha_1, \beta_4$ and β_3 have the strongest negative effect on $F(200)$. This indicates that population pressure, urbanization, industrialization, and residential land expansion are the dominant drivers of long-term forest depletion and strongly reduce forest sustainability. The afforestation-related parameters β_5 and θ_1 show weak and statistically insignificant PRCC values under the selected uncertainty range. This does not contradict their positive role in model structure and one-parameter simulation. Rather, it indicates that their relative influence is weaker than the dominant anthropogenic depletion parameters for the chosen baseline parameter set.
To examine the effect of parameter uncertainty on long-term forest sustainability, the same Latin Hypercube samples used in the sensitivity analysis were used for uncertainty propagation. For each sampled parameter set, the system (1) was solved over the time interval $t \in [0,200]$ and the final forest resource density $F(200)$ and

sustainability index S_F were recorded. The uncertainty results are summarized in Table 2.

Table 2: Uncertainty Propagation results for $F(200)$ and S_F

Output Quantity	Mean Value	Standard Deviation	Minimum Value	Maximum Value	95% Confidence interval
$F(200)$	28.172	1.8881	23.642	32.775	[24.744, 31.571]
S_F	0.56345	0.037761	0.47283	0.6555	[0.49488, 0.63142]

The uncertainty analysis gives a mean value of 28.172 for $F(200)$ with standard deviation 1.8881. The minimum and maximum values are 23.642 and 32.775, respectively, and the 95% confidence interval is [24.744, 31.571]. For the sustainability index S_F, the mean value is 0.56345 with standard deviation 0.037761, and the 95% confidence interval is [0.49488, 0.63142]. These results show that simultaneous variation in ecological and anthropogenic parameters produces noticeable uncertainty in long-term forest prediction. Since the sensitivity analysis indicates that the dominant parameters are mainly depletion-related, the uncertainty results further support the need to control population pressure, urbanization, industrialization, and residential land expansion along with strengthening afforestation efforts.

V. Conclusion

In this study, a nonlinear dynamical model was developed to examine the combined effect of population pressure, urbanization, industrialization, residential land expansion, and afforestation efforts on forest resources dynamics. The model describes the interactions among six state variables and provides a mathematical framework for analysing forest depletion and recovery under anthropogenic pressure. Positivity and boundedness of solutions were established, ensuring that all state variables remain biologically feasible within the considered region. The equilibrium analysis identified coexistence equilibrium and the relevant steady states. Sufficient conditions were discussed for the feasibility and stability of the equilibrium states. Local stability was examined through the Jacobian matrix, while global stability was studied using an appropriate Lyapunov function under suitable parameter restrictions. Numerical simulations performed using MATLAB ODE45 solver supported the theoretical results and showed convergence of trajectories toward the coexistence equilibrium for different initial conditions. The numerical and sensitivity analysis results indicate that population pressure, urbanization, industrialization, and residential land expansion are the major contributors to forest resource depletion. In contrast, afforestation efforts support forest regeneration and help to improve ecological balance. The global sensitivity analysis further shows that controlling dominant anthropogenic depletion parameters is essential for long-term forest sustainability. Therefore, effective forest management requires a combined strategy involving regulation of population pressure, urban and Industrial expansion, residential land conversion, and strengthening of afforestation programs. The proposed model may be useful for environmental planners, policymakers, and resource managers in designing sustainable forest conservation strategies.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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