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AN OPTIMUM ALLOCATION IN MULTIVARIATE STRATIFIED SAMPLING UNDER NONLINEAR CONSTRAINTS

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Abstract

This study proposes a framework of a Multi-Constrained Nonlinear Programming Problem (MCNLPP) for optimum sample allocation in multivariate stratified sampling with simultaneous nonlinear time constraint and cost constraint. The proposed model uses the Karush–Kuhn–Tucker conditions and the Lagrange multiplier technique to minimize the weighted sum of variances with operational resource constraints. Logarithmic, piecewise, and quadratic time functions are included as well as the nonlinear cost functions, and the resulting models are used to implement optimum stratum allocations under different operational scenarios using LINGO. The proposed framework, compared with representative deterministic allocation methods under the same resource conditions, demonstrates a balanced approach to survey precision, cost of operations, and time to collect the data. Finally, sensitivity analyses show that allocations are robust to changes in resource constraints. Overall, the framework generalizes the current compromise allocation methods by explicitly including nonlinear operational time and cost in a single deterministic optimization model.

Keywords: multivariate stratified sampling, optimum allocation, nonlinear constraints, Survey optimization, Lagrange multipliers, (SDG 12 responsible consumption and production).

I. Introduction

Due to its high efficiency and representativeness, stratified sampling design is one of the most popular sampling techniques. The literature on sampling, beginning with (Tschuprow) and (Neyman, 1934), has long addressed the problem of finding an ideal distribution of sample sizes among various strata, producing a minimum variance of an estimator at some given cost of a survey, or producing a minimum cost at a given level of precision.

In multivariate studies, when an allocation that is optimal in one variable may not always be optimal in the optimality conditions of other variables, a single optimal allocation is typically a challenging task. Yates established the idea of compromise allocation, which maximizes a combined objective function that represents several variables to achieve a balanced allocation. As noted by (Aoyama), (Folks), (Kokan), (Chatterjee, 1968), (Ahsan, 1977), (Ahsan and Khan), (Melaku), (Jahan) and (Ansari) among others, other researchers extended or modified this criterion to create more successful compromise allocation techniques.

Despite these developments, researchers are still working on ways of improving the technique of optimal sample allocation in multivariate stratified sampling. Indicatively, (Haq et al.) designed a model that directly incorporates non-response as a part of the optimisation model by using response-rate parameters, where Flexible Fuzzy Goal Programming (FFGP) and Binary Goal Programming methods were used. The allocation problem developed by (Ali H. Ahmadini et al.) is a nonlinear programming model that was converted into a multi-objective problem to obtain a practical allocation, including the nonlinear cost function that models labour costs, as gamma distribution. In the same vein, (Jackson et al.) used stratified sampling, a linear cost function, and an advanced optimisation method to generate a robust framework of small-area mean estimation and applied the Lagrange multiplier method to calculate the optimal sample sizes and reduce the cost of the survey to the target precision. The two-stage multivariate sampling by (Ahmad, Ansari, et al.) suggested a trade-off of allocation based on dynamic programming to ensure that the weighted sum of variances is minimised under a quadratic travel cost constraint.

In a similar study, (Ahmad, Rabbani, et al.) also included quadratic travel costs in the multivariate stratified sampling model and demonstrated that the suggested research methods are better than the classical ones. A conceptual framework of the multivariate stratified sampling within a stochastic environment was introduced by (Alshqaq et al.) by considering the framework as a nonlinear stochastic multi-objective optimisation problem. Equally, (Mahfouz et al.) introduced a stochastic multi-objective model in which cost and variance parameters are randomly distributed and hence resulted in more effective and accurate sample assignments compared with the deterministic approaches. The optimal sample size and variance of multivariate stratified sampling with nonlinear time functions were also established by (AHMAD et al.) as the Lagrange multiplier technique. More recently, (Singh et al.) used the Lagrange multiplier method to optimize population mean estimation with generalised exponential estimators and compared their findings with classical models with a linear cost structure.

Table 1: Research review summary.

Authors	Cost structure	Time structure	Optimization framework	Main contribution	Relation to present study
Haq et al. (2020)	Linear	Not explicit	FFG programming	Non-response with FFG	Fuzzy optimization approach

Ahmadani et al. (2021)	Nonlinear	Not considered	Multi-objective NLP	Non-linear labour cost	Nonlinear cost only
Jackson et al. (2021)	Linear	Linear	Lagrange multiplier	Small-area estimation	Linear operational constraints
Ahmad et al. (2022)	Quadratic	Not considered	Dynamic programming	Two-stage sampling	Quadratic cost only
Mahfouz et al. (2023)	Stochastic	Stochastic	Stochastic optimization	Random cost & variance	Stochastic framework
Ahmad et al. (2023)	Linear	Nonlinear	Lagrange multiplier	Nonlinear sampling time	Nonlinear time only
Present study	Nonlinear	Nonlinear	MCNLPP	Unified compromise allocation	Integrated Nonlinear cost & time.

Table 1 demonstrates that the recent research work has been mainly on individual issues of the allocation problem, which has led to the development of an integrated optimization framework that is capable of simultaneously addressing nonlinear operational constraints.

While significant work has been done in the field of multivariate stratified allocation using nonlinear, stochastic, and fuzzy optimization models, the work has typically concentrated on individual operational aspects without considering the overall problem. The prevailing studies generally examine either nonlinear cost, nonlinear time, or stochastic uncertainty, or fuzzy decision environments individually. In this context, however, little attention has been paid to a single deterministic optimization framework, which also integrates the nonlinear operational time and nonlinear cost in the classical compromise allocation paradigm without losing the analytical tractability of the Karush–Kuhn–Tucker optimality conditions.

The present study proposes a Multi-Constrained Nonlinear Programming Problem (MCNLPP) to solve this problem by optimum sample allocation in multivariate stratified sampling. The proposed framework is also nonlinear in both the operational time as well as the nonlinear cost constraints, and the minimisation of the weighted sum of variances is carried out using the Karush-Kuhn-Tucker optimisation framework. These operational constraints are incorporated into a deterministic compromise allocation model so that the proposed approach can be used to balance the different aspects of survey precision, cost, and time under the same resource conditions. The framework is further evaluated alongside representative deterministic allocation methods to show its practical performance under similar optimization settings.

II. Problem formulation

Let a multivariate stratified population have L number of strata and p characteristics on each population unit. Let N_h denote the sizes of the h^{th} stratum and n_h units to be drawn without replacement.

then;

N_h : population size in h^{th}stratum
n_h : sample size to be determined in h^{th}stratum
S_{ih}^2 : population variance of variable i^{th}characteristics in h^{th}stratum
w_h : stratum weight
α_i : variance weight of i^{th} characteristics
$var\left(\hat{Y}_i\right)$: variance of the estimate

Objective function:

Our objective is to minimize the total weighted variance of the i^{th} characteristics

$$z = \min(\textstyle\sum_{i=1}^{p} \alpha i \ var\left(\hat{Y}_i\right)) = \min\left(\sum_{i=1}^{p} \alpha i \ \left(\sum_{h=1}^{L} {w_h^2 S_{ih}^2}/{n_h}\right)\right) \tag{1}$$

Ideally, $var\left(\hat{Y}_i\right) = \sum_{h=1}^{L} {w_h^2 S_{ih}^2}/{n_h} - \sum_{h=1}^{L} {w_h^2 S_{ih}^2}/{N_h}$ for large N_h, the second term will be negligible, or it could be ignored as it is independent of n_h.

$$\text{Let } k_h \equiv \sum_{i=1}^{p} \alpha i \, w_h^2 S_{ih}^2 > 0$$

$$\text{Now}, z = \sum_{h=1}^{L} \frac{k_h}{n_h}$$

The optimization criterion employed is the weighted sum of variances, which aligns with the traditional compromise allocation framework. While this formulation does not overtly include inter-characteristic covariance components, its prevalence can be attributed to its analytical manageability and practical applicability. The integration of the complete variance-covariance framework in the context of nonlinear time and cost constraints would significantly escalate computational complexity and would transcend the boundaries of the current investigation. This constraint is duly recognized and presents a prospective avenue for subsequent scholarly inquiry. In practice, survey variables are often weakly correlated or analysed separately, making the adopted criterion suitable for many applied settings.
Subject to Constraints:

- Nonlinear Time: $T\ (n) = \sum_{h=1}^{L} t_h(n_h) \leq T_{max}$

  t_h: sampling time per unit in stratum h

  T: total available time.

- Cost: $C(n) = \sum_{h=1}^{L} c_h n_h \leq C_{max}$

  c_h: monetary cost per unit sample in stratum h.

  C: Total available cost budget

- $2 \leq n_h \leq N_h$, for all h, are added to take care of oversampling and to provide an estimate of strata variances S_{ih}^2.

Lagrangian function:

To jointly handle these constraints, the Lagrange multiplier technique provides an elegant analytical framework for deriving optimum stratum allocations under complex nonlinear dependencies.

Introducing the Lagrangian multipliers.

- $\lambda \geq 0$ for time.
- $\gamma \geq 0$ for cost. This is the dual feasibility condition of the KKT.

$$L(n_h, \lambda, \gamma) = \sum_{h=1}^{L} \frac{k_h}{n_h} + \lambda \left(\sum_{h=1}^{L} t_h(n_h) - T \right) + \gamma \left(\sum_{h=1}^{L} c_h n_h - C \right)$$

Karush-Kuhn-tucker conditions

Under appropriate regularity assumptions, the optimum allocation satisfies the following conditions:

- Stationarity condition

For each stratum h: $\frac{\delta L}{\delta n_h} = 0$

- Primal feasibility

The optimum allocation must satisfy:

$$\sum_{h=1}^{L} t_h(n_h) \leq T$$

$$\sum_{h=1}^{L} c_h n_h \leq C$$

and $2 \leq n_h \leq N_h$

- Dual feasibility

The Lagrange multipliers satisfy: $\lambda \geq 0, \gamma \geq 0$.

- Complementary slackness

The complementary slackness conditions are:

$$\lambda\left(\sum_{h=1}^{L} t_h(n_h) - T\right) = 0$$

and

$$\gamma\left(\sum_{h=1}^{L} c_h n_h - C\right) = 0$$

Under the differentiability of the objective and constraint functions together with the satisfaction of a suitable constraint qualification (e.g., the Linear Independence Constraint Qualification), the KKT conditions constitute necessary conditions for optimality. These conditions are used to derive the optimum stratum allocations presented below.

Let's start with the stationary condition. Take the partial derivative of L w.r.t n_h and set to zero.

$$\frac{\delta \mathrm{L}}{\delta \mathrm{n_h}} = \frac{\delta}{\delta \mathrm{n_h}}[\sum_{h=1}^{L}\frac{k_h}{n_h} + \lambda\left(\sum\nolimits_{h=1}^{L} t_h(n_h) - T\right) + \gamma(\sum\nolimits_{h=1}^{L} c_h n_h - C] = 0$$

$$\frac{\delta \mathrm{L}}{\delta \mathrm{n_h}} = \frac{-k_h}{\mathrm{n}_h^2} + \lambda \mathrm{t}_h^1(n_h) + \gamma c_h = 0 \qquad \text{Stationarity condition.}$$

$$\frac{k_h}{\mathrm{n}_h^2} = \lambda \mathrm{t}_h^1(n_h) + \gamma c_h$$

Solving for n_h gives:

$$n_h^2 = \frac{k_h}{\lambda \mathrm{t}_h^1(n_h) + \gamma c_h}$$

By taking the square roots of both sides, we have:

$$n_h = \left(\frac{k_h}{\lambda \mathrm{t}_h^1(n_h) + \gamma c_h}\right)^{\frac{1}{2}} \tag{2}$$

Equation (2) provides a general analytical expression for the optimum stratum sample sizes under arbitrary differentiable time functions. Specific nonlinear time functions can therefore be incorporated through their first derivatives.

Determination of the Optimal Lagrange Multipliers

The optimum stratum sample sizes are implicit functions of the Lagrange multipliers λ and γ as given in equation (2). As these multipliers are not known a priori, they must be solved simultaneously with the optimum allocations by enforcing the active resource constraints. If both the cost and time constraints are binding at the optimum, the complementary slackness conditions reduce to;

$$\sum\nolimits_{h=1}^{L} t_h(n_h) \le T \text{ and } \sum\nolimits_{h=1}^{L} c_h n_h \le C.$$

Substituting the allocation expression in (2) into the above constraints yields the coupled nonlinear system.

$$F_1(\lambda,\gamma) = \sum\nolimits_{h=1}^{L} t_h(n_h(\lambda,\gamma)) - T = 0 \tag{3}$$

$$F_2(\lambda,\gamma) = \sum\nolimits_{h=1}^{L} c_h n_h(\lambda,\gamma) - C = 0 \tag{4}$$

Now,

$$n_h(\lambda,\gamma) = \left(\frac{k_h}{\lambda \mathrm{t}_h^1(n_h) + \gamma c_h}\right)^{\frac{1}{2}}$$

The above equations constitute a coupled nonlinear system determining the optimum Lagrange multipliers. The sample allocations enter implicitly through the nonlinear time functions, and thus closed-form solutions are generally not available. The multipliers and the corresponding optimum allocations are thus obtained simultaneously by a nonlinear programming solver.

Existence and Uniqueness of the Lagrange Multiplier solution

A feasible solution exists if the cost and time budgets set for the prescribed survey are feasible, as well as the sampling bounds $2 \le n_h \le N_h$. If the feasible region is not empty, the objective and constraint functions are continuously differentiable and the standard Karush–Kuhn–Tucker regularity conditions are satisfied, then the coupled nonlinear system provides the basis for obtaining the Lagrange multipliers numerically. There are typically no closed-form solutions, though, because the sample allocations are implicit in the nonlinear time functions. The numerical optimization software is used to calculate the value of the multipliers and the optimum sample size for each stratum.

Incorporation of the time functions into the framework.

The following nonlinear time functions and their derivatives are considered.

- Logarithmic time: $t_h(n_h) = p_h \log(1 + q_h n_h)$

  Then $\quad t_h^1(n_h) = \frac{p_h q_h}{1+q_h n_h}$

  Substituting $t_h^1(n_h) = \frac{p_h q_h}{1+q_h n_h}$ in equation (2) above, we have;

$$n_h = \left(\frac{k_h}{\lambda(\frac{p_h q_h}{1+q_h n_h})+\gamma c_h}\right)^{\frac{1}{2}} \tag{5}$$

- Piecewise time: $t_h(n_h) = a_h n_h + b_h \sqrt{n_h}$

  Then $\quad t_h^1(n_h) = a_h + \frac{b_h}{2\sqrt{n_h}}$,

  Substituting $t_h^1(n_h) = a_h + \frac{b_h}{2\sqrt{n_h}}$ in equation (2) above, we have;

$$n_h = \left(\frac{k_h}{\lambda(a_h+\frac{b_h}{2\sqrt{n_h}})+\gamma c_h}\right)^{\frac{1}{2}} \tag{6}$$

The linear time function presupposes a constant duration allocated per observation, thereby suggesting a uniform resource allocation. The quadratic time function encompasses escalating marginal time, such as interviewer fatigue. The logarithmic time function illustrates instances where the duration designated for each observation increases at a declining pace due to the impact of learning and enhancements in efficiency as the fieldwork advances. Finally, the piecewise time function incorporates a stepwise change in the time rate, such as a zone-wise workload variation.

Extension to Quadratic cost and quadratic time.

Objective function:

Our objective is to minimize the sum of weighted variance of the i^{th} characteristics.

$$z = \min\left(\sum_{i=1}^{p} \alpha i \ var\,(\hat{Y}_i)\right) = \min\,(\sum_{i=1}^{p} \alpha i \ (\sum_{h=1}^{L} {w_h^2 S_{ih}^2}/{n_h}))$$

$$\text{Let } k_h \equiv \sum_{i=1}^{p} \alpha i \, w_h^2 S_{ih}^2 > 0$$

$$\text{Now}, z = \sum_{h=1}^{L} \frac{k_h}{n_h}$$

Using the same objective function, together with the cost and time constraints introduced above, the quadratic cost model is formulated as follows.

$$L(n_h, \lambda, \gamma) = \sum_{h=1}^{L} \frac{k_h}{n_h} + \lambda\left(\sum_{h=1}^{L} t_h(n_h) - T\right) + \gamma\left(\sum_{h=1}^{L} c_h(n_h) - C\right) = 0$$

$$\frac{\delta \mathrm{L}}{\delta \mathrm{n_h}} = \frac{-k_h}{\mathrm{n}_h^2} + \lambda \mathrm{t}_h^1(n_h) + \gamma c_h^1(n_h) = 0$$

$$\frac{k_h}{\mathrm{n}_h^2} = \lambda \mathrm{t}_h^1(n_h) + \gamma c_h^1(n_h)$$

Rearranging the stationarity condition yields:

$$n_h^2 = \frac{k_h}{\lambda \mathrm{t}_h^1(n_h) + \gamma C_h^1(n_h)}$$

Solving for n_h gives:

$$n_h = \left(\frac{k_h}{\lambda \mathrm{t}_h^1(n_h) + \gamma c_h^1(n_h)}\right)^{\frac{1}{2}} \tag{7}$$

Equation (7) provides the general analytical framework.

Hence, we examine some specific nonlinear time and cost functions commonly encountered in survey operations.

- Quadratic time: $t_h(n_h) = a_h n_h + b_h n_h^2$

  Then $\mathrm{t}_h^1(n_h) = a_h + 2b_h n_h$

- Quadratic cost: $c_h(n_h) = \alpha_h n_h + \beta_h n_h^2$

  Then $c_h^1(n_h) = \alpha_h + 2\beta_h n_h$

Substituting $\mathrm{t}_h^1(n_h)$ and $c_h^1(n_h)$ in equation 5 above, we have;

$$n_h = \left(\frac{K_h}{\lambda(a_h + 2b_h n_h) + \gamma(\alpha_h + 2\beta_h n_h)}\right)^{\frac{1}{2}} \tag{8}$$

The suggested optimization framework has been implemented in LINGO by formulating the entire nonlinear programming model including the objective function and all operational constraints. The software simultaneously determines optimum sample allocations and corresponding Lagrange multipliers for each of the nonlinear time specifications.

Numerical Solution Procedure.

Although the optimum allocation is derived analytically using the Lagrange multiplier technique, its implementation in real survey settings is straightforward. In practice, stratum-level quantities such as N_h, S_{ih}^2, unit cost c_h, and unit time t_h are estimated from pilot surveys or historical data. The appropriate nonlinear time or cost function is then selected based on field conditions, such as learning effects, interviewer fatigue, or zone-wise operational constraints. The resulting multi-constrained nonlinear programming problem is solved using standard optimization software such as LINGO, GAMS, or MATLAB. The computed stratum sample sizes n_h are rounded to the nearest integers and adjusted, if necessary, to satisfy feasibility constraints. Finally, sensitivity analysis with respect to budget and time limits can be performed to assess the robustness of the allocation before field implementation.

The entire optimization model was directly programmed in LINGO. The objective function, nonlinear cost and time constraints, and sampling bounds were written as shown. LINGO simultaneously optimizes the allocation of samples and the Lagrange multipliers for the resulting allocation. At the solution, the Lagrange multipliers satisfy the Karush-Kuhn-Tucker conditions of optimality. All allocations were evaluated for feasibility with a complete listing of operational constraints prior to reporting.

III. Numerica1 illustrations:

(Raghav et al.) discussed the following example having a population (N = 2050) splits into 5 strata with 3 characteristics. The proposed nonlinear programming model was implemented and solved using *LINGO* developed by *LINDO Systems Inc*. It is assumed that the total available survey cost budget (C_{max}) = 9500 units, Total Time budget (T_{max}) = 4500 units, and a target variance (V_0) = 0.5. The values $N_h, S_{1h}^2, S_{2h}^2, S_{3h}^2, t_h, c_h$ and w_h are shown in table1. While Tables 2 to 5 present the results from the findings.

Table 2: The Data

h	N_h	S^2_{1h}	S^2_{2h}	S^2_{3h}	t_h	c_h	w_h
1	395	35	1005	20	4.0	7	0.19
2	412	107	67	35	4.5	9	0.20
3	438	53	533	19	5.0	10	0.21
4	421	204	805	33	5.5	12	0.21
5	384	91	101	26	6.0	13	0.19

To assess the relative performance of the proposed nonlinear allocation framework, the results are compared with classical multivariate allocation models that assume linear cost structures and do not explicitly account for sampling time. Such classical approaches include Proportional, Neyman (optimum), and Cochran allocations, which remain standard benchmarks in the literature.

Table 3: Results when C_{max} = 9500, T_{max} = 4500 and V_0 = 0.5

Model	Strata					Total sample size	Total time	Total cost	Weighted variance
	1	2	3	4	5				
Proportional	188	196	208	200	183	975	4872	9939	0.245
Neyman	250	119	220	277	110	976	4819	9775	0.181
Cochran	293	123	216	249	95	976	4745	9541	0.191
Proposed (Log time)	301	123	211	243	97	975	4731	9500	0.196
Proposed (Piecewise time)	264	115	198	238	95	910	4442.5	8954	0.209
Proposed (Quadratic time/cost)	259	115	196	235	95	900	4396.0	8863	0.210

In contrast to Ahmad et al. (2023), where the nonlinear formulation considers only nonlinear sampling time, the proposed framework allows for simultaneous nonlinear time and nonlinear cost while preserving the classical weighted-variance objective. Although Neyman and Cochran allocations yield lower weighted variances in Table 2, this happens at the expense of high survey cost and time beyond the allocated budget, which is not feasible within the operational limits of the study. The proposed models achieve a competitive variance while simultaneously satisfying cost, time, and target variance constraints, thereby offering a more practical allocation for real survey operations. Sensitivity analysis illustrates how the proposed model with different time functions gives distinct allocations.

Table 4: KKT verification.

Constraint	Slack	Dual price
Time	0.0000	0.00002068
Cost	4492.123	0.0000
Lower bounds	>0	0

- KKT verification mathematically.

$$\min f(n) = \sum_{h=1}^{5} \frac{k_h}{n_h}$$

Subject to:

$$g_1(n) = Ttime - 4500 \leq 0$$
$$g_2(n) = Tcost - 9500 \leq 0$$

The KKT conditions require:

- Primal feasibility:

$$g_1(n) = 4371 - 4500 \leq 0$$
$$g_2(n) = 9500 - 9500 \leq 0$$
$$g_1(n) = -129 \ and \ g_2(n) = 0, \text{Satisfied.}$$

- Dual feasibility: $\lambda_1 \geq 0 \ and \ \lambda_2 \geq 0$

  Satisfied because $\lambda_1 = 0.00002068 > 0$ and $\lambda_2 = 0$.
- Complementary slackness: $\lambda_i g_i(n) = 0$

For cost: $0.00002068 * 0 = 0$ satisfied.
For time: $0 * 4492.123 = 0$ satisfied.

The zero slack for the time constraint indicates that the available survey time is fully used at the optimum, and the positive slack for the cost constraint indicates that the optimal solution is still within the budget. The dual prices are KKT-optimal.

Sensitivity analysis.

Table 5: Results when C_{max} = 10000 and T_{max} = 5000

Functions	Total sample size	Total time	Total cost	Weighted variance
Log time	1026	4978.5	9998	**0.186**
Piecewise time	1010	4930.0	9936	0.188
Quadratic time/cost	1003	4873.5	9795	0.191

Table 6: Results when C_{max} = 10500 and T_{max} = 5500.

Functions	Total sample size	Total time	Total cost	Weighted variance
Log time	1078	5231.0	10505	0.177
Piecewise time	1078	5231.0	10505	0.178
Quadratic time/cost	1052	5111.5	10273	0.182

Table 7: Results when C_{max} = 11000 and T_{max} = 6000.

Functions	Total sample size	Total time	Total cost	Weighted variance
Log time	1128	5474.5	10995	0.169
Piecewise time	1127	5468.5	10982	0.170
Quadratic time/cost	1052	5345.5	10744	0.174

IV. Discussion

The data used in this study are summarized in Table 2, and the overall optimum allocation results for the proportional, Neyman, and Cochran allocation frameworks as well as the proposed logarithmic, piecewise, and quadratic cost/time allocations are reported in Table 3. The same fiscal and temporal parameters (C_{max} = 9500, T_{max} = 4500 and V_0 = 0.5) respectively are used to evaluate all the allocations. While the Neyman and Cochran allocations result in smaller weighted variances in Table 3, the cost and time of the surveys are in excess of the allocated budget, so they are not feasible in the operational constraints of the study. The models proposed here would result in competitive variance and fulfil the cost, time, and target-variance constraints and would provide a more practical allocation for actual survey operations. The sensitivity analysis also demonstrates the different allocation patterns that can be obtained by the proposed model for different time-function specifications.

The KKT verification is included in Table 4, and the sensitivity analysis is included in Tables 5, 6, and 7. Table 5 is the one that is obtained when the cost and time budgets are relaxed to C_{max} = 10000, T_{max} = 5000 and V_0 = 0.5 was used throughout the sensitivity analysis. Given these loose constraints, the time functions with the largest allocations are the logarithmic time function; the piecewise time function and the quadratic time/cost function have the smallest allocation. All of the weighted variances in Table 4 are less than those in Table 3, which indicates the increased precision possible when using the extra resources available.

The results are reported in Table 6 when the budgets are further raised to C_{max} = 10500, T_{max} = 5500. The nearly equal weighted variances and total sample allocations, total time, and total costs of the piecewise and logarithmic time functions at this level demonstrate that the piecewise function is approximately as effective as the logarithmic function. Total cost for both functions is just over the budget, suggesting a slight overspend on the resource constraint. The quadratic time/cost function, on the other hand, results in a smaller total sample size, total time, and total cost, but has a higher weighted variance.

As the budget grows to C_{max} = 11000, T_{max} = 6000, allocations get higher as well (see Table 7), so budget expansion is also a positive factor for the precision attainable. The logarithmic time function again yields a larger overall sample size than the piecewise function, but the weighted variances of the two are very close while the logarithmic function uses more resources. As noted in the previous tables, the quadratic time/cost function gives a slightly higher weighted variance, which translates to a smaller sample size and lower overall use of resources.

The combined results show that the choice of the trade-off between cost, time, and precision is an intrinsic characteristic of the allocation problem. The various time-function specifications lead to different allocations of resources and allocation patterns, which consequently influence the efficiency and precision of the survey; especially the precision of the estimated parameters and the allocation of resources by strata.

V. Conclusion

This paper presents a mathematical model of optimum sample allocation in a multivariate stratified sampling problem that is prepared with simultaneous consideration of time and cost constraints through the Lagrange multiplier technique. The allocation problem is stated as a multi-constrained nonlinear programming problem (MCNLPP) to provide a unified optimization framework to reconcile survey precision and constraints on operational resources. The allocation patterns obtained from the three time/cost functions – logarithmic, piecewise, and quadratic – are compared and show differences in allocation patterns, and they represent the inherent trade-offs between precision and survey cost and between the time spent in sampling versus the number of samples allocated. This proposed framework is unlike many of the papers that have been published, which assume that survey time is proportional to the cost or assume a secondary consequence of the cost constraint by making the distinction of survey time as an independent operational constraint. This finding is another testament to the validity of the joint modelling of time and cost, to the detriment of its derivative, which cannot be replicated using existing deterministic methods.

The paper has a drawback in that the nonlinear time functions were not determined from historical survey or pilot-study data, so no consideration was given to parameter uncertainty or possible alternative time functions. The solutions to these problems are helpful in the practical implementation of the proposed framework and will be a future research direction.

This study is an addition to the literature, including consideration of non-linear time constraints in addition to non-linear cost structures, in a multi-constrained compromise allocation. When the time effect is not accounted for in a nonlinear fashion and/or resource requirements are not accurately estimated, budget is not effectively utilized, and precision is low. This framework therefore emphasizes the need to consider time and cost together when designing surveys using sampling methods, and offers a solid basis for developing more comprehensive optimum allocation models that include more realistic operational restrictions.

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Conflict of Interest:

The authors declare that there was no relevant conflict of interest regarding this paper.

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