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IMPROVED ANALYTICAL SOLUTIONS OF THE HELMHOLTZ DUFFING OSCILLATOR BY THE EXTENDED ITERATION METHOD

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Abstract

The Helmholtz–Duffing oscillator is a type of nonlinear system that frequently occurs in mechanical, electrical, and physical models incorporating nonlinear restoring forces. It combines the features of Helmholtz and Duffing nonlinearities. Due to the presence of the combined quadratic and cubic nonlinear terms, it is still difficult to obtain exact analytical solutions. This paper proposes an Extended Iteration Method (EIM) to provide better analytical solutions for the Helmholtz-Duffing oscillator. The derived analytical solutions are compared with numerical solutions and those found in the literature to verify the accuracy of the method. Numerical solutions and with those available in the literature. The findings show that the suggested EIM outperforms existing analytical methods in terms of accuracy and simplicity, producing outstanding agreement with numerical solutions. The approach may be applied to many additional severely nonlinear models and offers a robust and effective foundation for researching nonlinear oscillatory systems.

Keywords: Helmholtz Duffing Oscillator, Iteration Method, Extended Iteration Method, Fourier cosine series.

Nomenclature

a	The oscillator's initial amplitude
x	Dimensionless Displacement (m)
$F_a, F_b, G_a, G_b, H_a, H_b$	Fixed Parameters
t	Time (s)
F, f, H	Nonlinear Functions

x_{a1}, x_{a2}	First and second periodic solutions of the Helmholtz–Duffing oscillator for the positive direction, respectively.
x_{b1}, x_{b2}	First and second periodic solutions of the Helmholtz–Duffing oscillator for the negative direction, respectively.
Error (%)	Percentage Error

Greek letters

β	Constant Parameter
Ω	Oscillation frequency
$\Omega_0, \Omega_1, \Omega_2$	The Helmholtz-Duffing oscillator's first, second, and third approximate frequencies, respectively.
$\Omega_{a_0}, \Omega_{a_1}, \Omega_{a_2}$	The Helmholtz-Duffing oscillator's first, second, and third approximation frequencies for the positive direction, respectively.
$\Omega_b, \Omega_{b_1}, \Omega_{b_2}$	The Helmholtz-Duffing oscillator's first, second, and third approximation frequencies for the negative direction, respectively.
Ω_{exact}	Exact frequency

I. Introduction

Nonlinear oscillatory systems play a fundamental role in diverse fields of science and engineering, including mechanical vibrations, electrical circuits, plasma physics, and structural dynamics. Among these systems, the Helmholtz–Duffing oscillator has attracted considerable attention due to its ability to model complex nonlinear phenomena such as self-excitation, resonance, and amplitude-dependent frequency shifts. The governing equation of this oscillator combines the linear restoring force of the Helmholtz oscillator with the cubic nonlinearity of the Duffing type, making its analytical treatment challenging. Traditional analytical and numerical methods, such as the Hamiltonian Approach (HA) [I], Coupled Homotopy Variational (CHV) [I] Formulation, Energy Balance Method (EBM) [II], He's Frequency–Amplitude Formulation (HFAF) [II], He's Max–Min Method (HMMM) [III], Mickens Iteration Method (MIM) [IV, V, XIII, XIV, XV], Mickens Extended Iteration Method (MEIM) [VI, VII], Modified Harmonic Balance Method (MHBM) [VIII], Modified Extended Iteration Method (MdEIM) [IX], Homotopy Perturbation Method (HPM) [X], Iterative Homotopy Harmonic Balance Method (IHHBM) [XI], Harmonic Balance Method (HBM) [XII], Second-Order Hamiltonian Approach (SHA) [XVI], First-Order Global Error Minimization Method (FGEM) [XVI, Perturbation Method (PM) [XVII, XVIII], have been applied to approximate solutions of the nonlinear dynamical systems. Although these methods have some limitations, such as the perturbation method, which is powerful and widely used, it has several important limitations. One major limitation is that it relies on the presence of a small parameter; if such a parameter does not exist or is not sufficiently small, the method may yield inaccurate or divergent results. Additionally, perturbation solutions are typically valid only for weakly nonlinear systems and may fail to describe strongly nonlinear behavior or large-amplitude motions. The Energy Balance Method (EBM), though simple and effective for analyzing nonlinear oscillatory systems, has several limitations. One major limitation is that it provides only approximate solutions, which may become inaccurate for systems with strong nonlinearities or large-amplitude oscillations. The Harmonic Balance Method (HBM) is a powerful analytical technique for approximating periodic solutions of nonlinear differential equations, but it has several important limitations. First, it assumes that the system's response can be represented by a finite number of harmonics, which may not be accurate for strongly nonlinear systems where many higher-order harmonics are significant. Moreover, the accuracy of HBM depends on the correct estimation of harmonic terms and initial assumptions, which may not always be straightforward.

In recent years, several analytic procedures have been developed to find the analytical solutions of the Helmholtz–Duffing oscillator including the Hamiltonian Approach (HA) [I], Coupled Homotopy Variational (CHV) [I] Formulation, Energy Balance Method (EBM) [II], He's Frequency–Amplitude Formulation (HFAF) [II], Modified Harmonic Balance Method (MHBM) [VIII], Homotopy Perturbation Method (HPM) [X], second-Order Hamiltonian Approach (SHA) [XVI], First-Order Global Error Minimization Method (FGEM) [XVI], Second-Order Global Error Minimization Method (SGEM) [XVI]. However, these techniques often face limitations, including restricted validity for strong nonlinearities, slow convergence, and dependence on small parameters. To overcome these challenges, iteration-based methods have gained prominence for their simplicity, flexibility, and accuracy in solving nonlinear differential equations without the need for linearization or perturbative assumptions. Building on this progress, the present study introduces an Extended Iteration Method (EIM) to obtain improved analytical solutions of the Helmholtz–Duffing oscillator. The EIM refines the conventional iteration framework by incorporating higher-order correction terms and optimizing the iteration process, thereby achieving superior accuracy across a wider range of nonlinear parameters.

The primary objective of this paper is to demonstrate the effectiveness and reliability of the Extended Iteration Method in deriving analytical expressions for displacement and frequency characteristics of the Helmholtz–Duffing oscillator. The proposed approach not only simplifies the analytical process but also provides results that are in strong agreement with numerical simulations. While most approaches work well for first-order solutions but struggle with higher-order ones, the suggested approach produces exceptional results for higher-order solutions. This study contributes to advancing analytical techniques for nonlinear dynamical systems and broadens the understanding of the complex behavior inherent in the Helmholtz–Duffing model.

II. The Methodology

Assume that the general form of the nonlinear oscillator is

$$\ddot{x} + F(\ddot{x}, x) = 0, \text{ with } x(0) = a, \dot{x}(0) = 0, \tag{1}$$

$\Omega^2 x$ added to both sides, we get

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x) \cong H(x) \tag{2}$$

Mickens' iteration method as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = H(x_k, \Omega_k); k = 0,1,2,3, \dots \tag{3}$$

and the method of extended iteration as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = H(x_{k-1}, \Omega_k) + H_x(x_{k-1}, \Omega_k)(x_k - x_{k-1}) \tag{4}$$

together with the initial guess

$$x_0(t) = a \ \cos(\Omega_0 t). \tag{5}$$

Hence, x_{k+1} satisfies the initial conditions

$$x_{k+1}(0) = a, \quad \dot{x}_{k+1}(0) = 0. \tag{6}$$

At each level of the iteration, frequencies of the oscillator are defined by the criterion that the secular terms (Nayfeh [XVII]) should not exist in the full solution of $x_{k+1}(t)$. The preceding approach provides the sequence of solutions:$x_0(t), x_1(t), x_2(t), \cdots$

III. Solution procedure

The Helmholtz-Duffing oscillator

$$\ddot{x} + x + (1-\beta)x^2 + \beta x^3 = 0 \tag{7}$$

where the asymmetric parameter is denoted by β.

Equation (1) is a Helmholtz oscillator with a single-well potential when $\beta = 0$ and a Cube-Duffing oscillator when $\beta = 1$. In both positive and negative directions, an asymmetric oscillator's characteristics will differ. In this case, Eq. (1) can be expediently considered in two parts

$$\ddot{x} + x + (1-\beta)x^2 \mathrm{sgn}(x) + \beta x^3 = 0 \text{ , for } x \geq 0 \text{ with } x(0) = a, \dot{x}(0) = 0 \tag{8}$$

$$\ddot{x} + x - (1-\beta)x^2 \mathrm{sgn}(x) + \beta x^3 = 0 \text{ , for } x \leq 0 \text{ with } x(0) = b, \dot{x}(0) = 0 \tag{9}$$

In which $sgn(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$ is a sign function.

An asymmetric limit zone for oscillation of the aforementioned system can be taken to be $[-b, a]$, for positive a and b. The turning points are represented by both $x = a$ and $x = -b$.

Introducing $\Omega^2 x$ in Eq (9), we get,

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x - (1-\beta)x^2 - \beta x^3 \cong H(x, \Omega) \tag{10}$$

Where $\quad H(x, \Omega) = \Omega^2 x - x - (1-\beta)x^2 - \beta x^3 \qquad (11)$

and $\qquad H_x = \frac{\partial H}{\partial x} = \Omega^2 - 1 - 2(1-\beta)x - 3\beta x^2 \qquad (12)$

Using the approximation method, we obtain from (5)

$$\begin{aligned}\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} &= \Omega_k^2 x_{k-1} - x_{k-1} - (1-\beta){x_{k-1}}^2 - \beta {x_{k-1}}^3 \\ &\quad + \left(\Omega_k^2 - 1 - 2(1-\beta)x_{k-1} - 3\beta {x_{k-1}}^2\right)(x_k - x_{k-1})\end{aligned} \tag{13}$$

Equation (4) is rewritten as

$$x_{a0} = x_{a0}(t) = \mathrm{acos}\theta \tag{14}$$

where $= \Omega_a \mathrm{t}$, for $k = 0$, the equation (4) becomes

$$\ddot{x}_{a1} + \Omega_{a0}^2 x_{a1} = \Omega_{a0}^2 x_{a0} - x_{a0} - (1-\beta){x_{a0}}^2 - \beta {\mathrm{x}_{a0}}^3 \tag{15}$$

The elementary function equation (15) is expanded into a cosine series by substituting the right-hand side of equation (16). After eliminating the secular term, we obtain

$$\Omega_{a0} = \sqrt{1 + \frac{3a^2\beta}{4}} \tag{16}$$

In a similar procedure, we can achieve the negative direction

$$x_{b0} = x_{b0}(t) = \mathrm{bcos}\theta, \text{ where } \theta = \Omega_b \mathrm{t} \tag{17}$$

$$\Omega_{b0} = \sqrt{1 + \frac{3b^2\beta}{4}} \tag{18}$$

The oscillator's first approximate frequency is

$$\Omega_0 = \frac{\Omega_{a0} + \Omega_{b0}}{2} \tag{19}$$

The first approximate solution for the positive direction

$$x_{a1} = \frac{a(32a - 32a\beta - 3a^2\beta + 96\Omega^2)\mathrm{Cos}[t\Omega]}{96\Omega^2} - \frac{a^2(-1+\beta)(-3+\mathrm{Cos}[2t\Omega])}{6\Omega^2} + \frac{a^3\beta\mathrm{Cos}[3t\Omega]}{32\Omega^2} \tag{20}$$

In a similar procedure, we can take $x_b(t) = \mathrm{bcos}\theta$, where $\theta = \Omega_b \mathrm{t}$ (21)

in the opposite direction, and the initial approximation of Eq. (10) is

$$x_{b1} = \frac{b(-32b + 32b\beta - 3b^2\beta + 96\Omega^2)\mathrm{Cos}[t\Omega]}{96\Omega^2} + \frac{b^2(-1+\beta)(-3+\mathrm{Cos}[2t\Omega])}{6\Omega^2} + \frac{b^3\beta\mathrm{Cos}[3t\Omega]}{32\Omega^2} \tag{22}$$

In the oscillator's second iteration, we obtain

For $k = 1$ from (5), we get

$$\begin{aligned} \ddot{x}_{a2} + \Omega_{a_1}^2 x_{a2} &= \left(\Omega_{a_1}^2 x_{a0} - x_{a0} - (1-\beta){\mathrm{x}_{a0}}^2 - \beta {\mathrm{x}_{a0}}^3\right) \\ &+ (\Omega_{a1}^2 - 1 - 2(1-\beta)x_{a0} - 3\beta {\mathrm{x}_{a0}}^2)(x_{a1} - x_{a0}) \end{aligned} \tag{23}$$

The elementary function equation (23) is expanded into a cosine series by substituting the right-hand side of equation (26). After eliminating the secular term, we obtain

$$\Omega_{a_1} = \sqrt{\frac{a(64(-1+\beta) + a(160 + \beta(-314 + 160\beta + 9a(-16 + (16+a)\beta)))) - 48(4 + 3a^2\beta)\Omega^2}{64a(-1+\beta) + 6a^2\beta - 192\Omega^2}} \tag{24}$$

With the same method, we can achieve the negative direction

$$\Omega_{b_1} = \sqrt{\frac{b(-64(-1+\beta) + b(160 + \beta(-314 + 160\beta + 9b(16 + (-16+b)\beta)))) - 48(4 + 3b^2\beta)\Omega^2}{-64b(-1+\beta) + 6b^2\beta - 192\Omega^2}} \tag{25}$$

The oscillator's second approximate frequency is

$$\Omega_1 = \frac{\Omega_{a_1}+\Omega_{b_1}}{2} \tag{26}$$

Equation (9) has a second approximate solution that is

$$x_{a2} = F_a \mathrm{Cos}[\Omega_{a_1}t] + G_a \mathrm{Cos}[3\Omega_{a_1}t] + H_a \mathrm{Cos}[5\Omega_{a_1}t] \tag{27}$$

where

$$F_a = a - \frac{a^5\beta^2}{1024\Omega^2\Omega_{a_1}^2} - \frac{a^3(64+(64-96a+9a^2)\beta^2+4\beta(-29+24a+24\Omega^2-3\Omega_{a_1}^2))}{3072\Omega^2\Omega_{a_1}^2}, \tag{28}$$

$$G_a = \frac{a^3(64+(64-96a+9a^2)\beta^2+4\beta(-29+24a+24\Omega^2-3\Omega_{a_1}^2))}{3072\Omega^2\Omega_{a_1}^2}, H_a = \frac{a^5\beta^2}{1024\Omega^2\Omega_{a_1}^2}. \tag{29}$$

Similarly, the second approximate solution of (10) in the negative direction is

$$x_{b2} = F_b \mathrm{Cos}[t\Omega_{b_1}] + G_b \mathrm{Cos}[3t\Omega_{b_1}] + H_b \mathrm{Cos}[5t\Omega_{b_1}] \tag{30}$$

where

$$F_b = b - \frac{b^5\beta^2}{1024\Omega^2\Omega_{b_1}^2} - \frac{b^3(64+(64+96b+9b^2)\beta^2-4\beta(29+24b-24\Omega^2+3\Omega_{b_1}^2))}{3072\Omega^2\Omega_{b_1}^2}, \tag{31}$$

$$G_b = \frac{b^3(64+(64+96b+9b^2)\beta^2-4\beta(29+24b-24\Omega^2+3\Omega_{b_1}^2))}{3072\Omega^2\Omega_{b_1}^2}, H_b = \frac{b^5\beta^2}{1024\Omega^2\Omega_{b_1}^2}. \tag{32}$$

For the third iteration of the oscillator (5), we get

For $k = 2$, we get,

$$\ddot{x}_{a3} + \Omega_{a_2}^2 x_{a3} = \left(\Omega_{a_2}^2 x_{a1} - x_{a1} - (1-\beta){x_{a1}}^2 - \beta {x_{a1}}^3\right)$$

$$+\Omega_{a_2}^2 - 1 - 2(1-\beta)x_{a1} - 3\beta {x_{a1}}^2)(x_{a2} - x_{a1}) \tag{33}$$

The elementary function equation (35) is expanded into a cosine series by substituting the right-hand side of equation (38). After eliminating the secular term, we obtain

$$\Omega_{a_2} = \frac{1}{64\sqrt{6}}(\sqrt{(}-((-864a^9(-1+\beta)\beta^4 - 297a^{10}\beta^5 + 18874368\Omega^2\Omega_{a_1}^6 + 65536a^3(-1+\beta)\Omega_{a_1}^2(-80(-1+\beta)^2\Omega^2 + 9\beta\Omega_{a_1}^2) + 384a^6\beta(-60(-1+\beta)^2(16+\beta(-29+16\beta-24\Omega^2)) + \beta(-1028+\beta(2083-1028\beta+108\Omega^2))\Omega_{a_1}^2 - 351\beta^2\Omega_{a_1}^4) + 12288a^4\Omega_{a_1}^2(2(-1+\beta)^2(16+\beta(-29+16\beta-608\Omega^2)) - 3\beta(18+\beta(-31+18\beta+24\Omega^2))\Omega_{a_1}^2 + 9\beta^2\Omega_{a_1}^4) - 12a^8\beta^3(3344+\beta(-6661+3344\beta+72\Omega^2-891\Omega_{a_1}^2)) + 1152a^7(-1+\beta)\beta^2(464+\beta(-931+464\beta+3\Omega_{a_1}^2)) + 24576a^2\Omega_{a_1}^4(-16+\beta(29-16\beta+3\Omega_{a_1}^2+24\Omega^2(-1+24\Omega_{a_1}^2))) + 24576a^5(-1+\beta)\beta(320(-1+\beta)^2\Omega^2 + \Omega_{a_1}^2(-8+\beta(19-8\beta+33\Omega_{a_1}^2))))/(\Omega_{a_1}^4(a^2(16+\beta(-29+16\beta+3a(8+(-8+a)\beta)+24\Omega^2)) - 3(a^2\beta+256\Omega^2)\Omega_{a_1}^2)))))) \tag{34}$$

With the same method, we can achieve the negative direction

$$\Omega_{b_2} = \frac{1}{64\sqrt{6}}(\sqrt{(}-((864b^9(-1+\beta)\beta^4 - 297b^{10}\beta^5 + 18874368\Omega^2\Omega_{b_1}^6 + 65536b^3(-1+\beta)\Omega_{b_1}^2(80(-1+\beta)^2\Omega^2 - 9\beta\Omega_{b_1}^2) + 384b^6\beta(-60(-1+\beta)^2(16+\beta(-29+16\beta-24\Omega^2)) + \beta(-1028+\beta(2083-1028\beta+108\Omega^2))\Omega_{b_1}^2 -$$

$$351\beta^2\Omega_{b_1}^4) + 12288b^4\Omega_{b_1}^2(2(-1+\beta)^2(16+\beta(-29+16\beta-608\Omega^2)) - 3\beta(18+\beta(-31+18\beta+24\Omega^2))\Omega_{b_1}^2 + 9\beta^2\Omega_{b_1}^4) - 12b^8\beta^3(3344+\beta(-6661+3344\beta+72\Omega^2-891\Omega_{b_1}^2)) - 1152b^7(-1+\beta)\beta^2(464+\beta(-931+464\beta+3\Omega_{b_1}^2)) + 24576b^2\Omega_{b_1}^4(-16+\beta(29-16\beta+3\Omega_{b_1}^2+24\Omega^2(-1+24\Omega_{b_1}^2))) - 24576b^5(-1+\beta)\beta(320(-1+\beta)^2\Omega^2+\Omega_{b_1}^2(-8+\beta(19-8\beta+33\Omega_{b_1}^2))))/(\Omega_{b_1}^4(b^2(16+\beta(-29+16\beta+3b(-8+(8+b)\beta)+24\Omega^2)) - 3(b^2\beta+256\Omega^2)\Omega_{b_1}^2))))) \tag{35}$$

The oscillator's third approximate frequency is

$$\Omega_2 = \frac{\Omega_{a_2} + \Omega_{b_2}}{2} \tag{36}$$

IV. Results and Discussions

The Extended Iteration Method (EIM) has been successfully applied to obtain improved analytical approximations for the periodic solutions of the Helmholtz–Duffing oscillator. The results derived through this method demonstrate remarkable accuracy and convergence when compared with existing analytical approaches such as the Hamiltonian Approach (HA) [I], Coupled Homotopy Variational (CHV) [I] Formulation, Energy Balance Method (EBM) [II], He's Frequency–Amplitude Formulation (HFAF) [II], Modified Harmonic Balance Method (MHBM) [VIII], Homotopy Perturbation Method (HPM) [X], second-Order Hamiltonian Approach (SHA) [XVI], First-Order Global Error Minimization Method (FGEM) [XVI], which shows in Table 2 and Table 3. Table 1 shows that the approximate frequencies obtained by the proposed method are compared with the exact values of frequencies for $\beta = 0.9$, with the percentage errors. The oscillators by $\Omega_0, \Omega_1 \, and \, \Omega_2,$ respectively. The third approximate frequencies are very close to the exact results, which are comparatively closer to the exact result than the others. Figure 1 compares the exact and second-order approximate solutions of (7) for $a = 10$ and $\beta = 0.9$. The error produced by the recommended approach is contrasted with the errors achieved by a number of existing techniques in Figure 2. Figure 3 shows the phase plane maps that represent system stability and the impact of parameter β, plotted for various values of this parameter

We have the percentage error (%) according to the definition:

Error= $\left|\frac{\Omega_{exact}-\Omega_k}{\Omega_{exact}}\right| \times 100\%; k = 0,1,2, ...$

Where,

Ω_k =Optained approximate frequency of the oscillator

Ω_{exact} =Exact frequency of the oscillator

From Tables 1 and 2, it can be concluded that EIM demonstrates superior performance compared to the existing techniques within the parameter range of $0.5 \leq \beta \leq 0.9$.

Table 1: The approximate frequencies obtained by the proposed method are compared with the exact values of frequencies for $\beta = 0.9$, with the percentage errors.

a/b	Ω_0 Er (%)	Ω_1 Er (%)	Ω_2 Er (%)	$\Omega_{[8]}^{exact}$
0.01	1.00003365 0.000029	1.00003365 0.000029	1.00003365 0.000029	1.00003336
0.1	1.00336040 0.001073	1.00335897 0.000930	1.00335897 0.000930	1.00334964
0.5	1.08090914 0.109295	1.08042866 0.153697	1.08043056 0.153522	1.08209182
1.0	1.29380212 0.309771	1.28932580 0.654681	1.28936835 0.651402	1.29782241
5.0	4.22758564 1.293255	4.14504483 0.684428	4.14537270 0.676572	4.17361022
10	8.27631381 1.786028	8.09982331 0.384534	8.10027475 0.378982	8.13109020
50	41.09132917 2.142131	40.18921466 0.100285	40.19105928 0.095700	40.22955916
100	82.16445296 2.181946	80.35897149 0.063399	80.36263339 0.058845	80.40995076

Table 2: Comparison between the approximate frequencies obtained by the proposed method and existing and exact values of frequencies for $\beta = 0.5$

a/b	$\Omega_{[exact]}^{[VIII]}$	$\Omega_{3rd\ order}^{[Proposed]}$ Er (%)	$\Omega_{FGEM}^{[XVI]}$ Er (%)	$\Omega_{SHA}^{[XVI]}$ Er (%)	$\Omega_{HPM}^{[X]}$ Er (%)
0.01	1.00000832	**1.00001874** **0.001041**	1.00002109 0.001277	1.00211995 0.211161	1.00000848 0.000016
0.1	1.00087042	**1.00186799** **0.099670**	1.00210639 0.123489	1.02262932 2.173998	1.00088302 0.001259
10	6.25708076	**6.00706579** **3.995712**	6.29382740 0.587282	6.49403352 3.786953	6.39046888 2.131794
50	30.1675046 8	**29.95087866** **0.718077**	31.05019406 2.925960	30.75976838 1.963250	30.83621494 2.216657
100	60.1139223 1	**59.89666551** **0.361408**	62.07396104 3.260540	61.15210265 1.727021	61.44816332 2.219520

Table 3: Comparison between the approximate frequencies obtained by the proposed method and existing and exact values of frequencies for $\beta = 0.9$

a/b	Ω_{exact}[VIII]	$\Omega_{3rd\ order}^{[Proposed]}$ Er (%)	Ω_{CHV}[I] Er (%)	Ω_{MHBM}[VIII] Er (%)	Ω_{EBM}[II] Er (%)	Ω_{HPM}[X] Er (%)
0.01	1.00003336	**1.00003374** **0.000037**	1.00002949 0.000386	1.00003365 0.000028	1.00003365 0.000002	1.00003365 0.000002
0.1	1.00334964	**1.00336802** **0.001831**	1.00336713 0.001743	1.00335865 0.000927	1.00335001 0.000035	1.00335106 0.000140
0.5	1.08209182	**1.08046111** **0.150699**	1.08094092 0.106358	1.08044075 0.152581	1.08256404 0.043640	1.080258341 0.045430
1.0	1.29782241	**1.28874760** **0.699233**	1.29387477 0.304173	1.28954738 0.637608	1.30241057 0.353527	1.30244785 0.356400
5.0	4.17361022	**4.14375352** **0.715368**	4.22768490 1.295633	4.15285021 0.497411	4.25577625 1.968703	4.25579387 1.969126
10	8.13109020	**8.09945508** **0.389063**	8.27606073 1.782916	8.11709681 0.172097	8.30612634 2.152677	8.30613546 2.152789
50	40.2295591 6	**40.19155808** **0.094460**	41.09342908 2.147351	40.27851570 0.121692	41.12172027 2.217675	41.12172212 2.217680
100	80.4099507 6	**80.36393107** **0.057231**	82.13314127 2.143006	80.53776157 0.158948	82.19486707 2.219770	82.19486800 2.219771

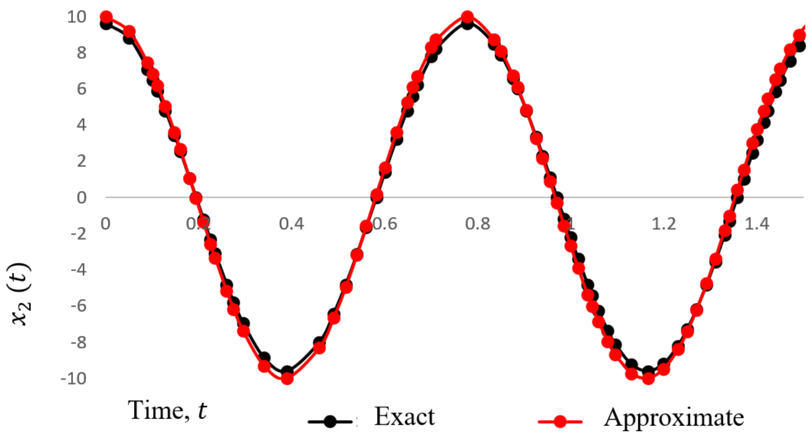

Fig.1. The second-order approximation solutions of equation (8) for $a = 10$ and $\beta = 0.9$ are compared with the corresponding exact solutions.

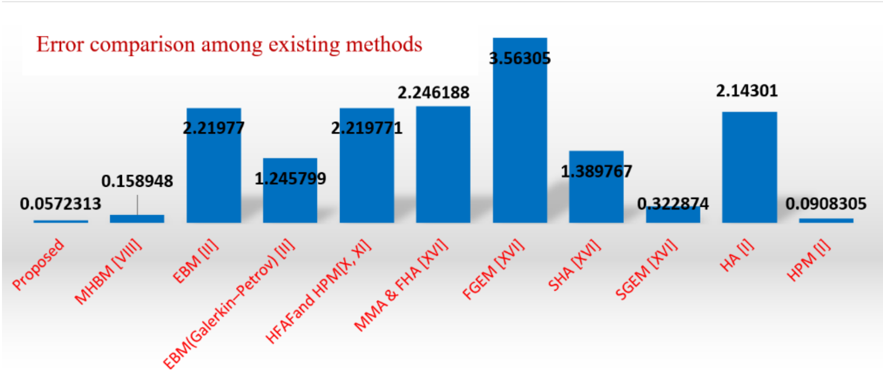

Fig. 2. Error comparison of equation (8) for $a = 10$ and $\beta = 0.9$ among the proposed method and the existing methods.

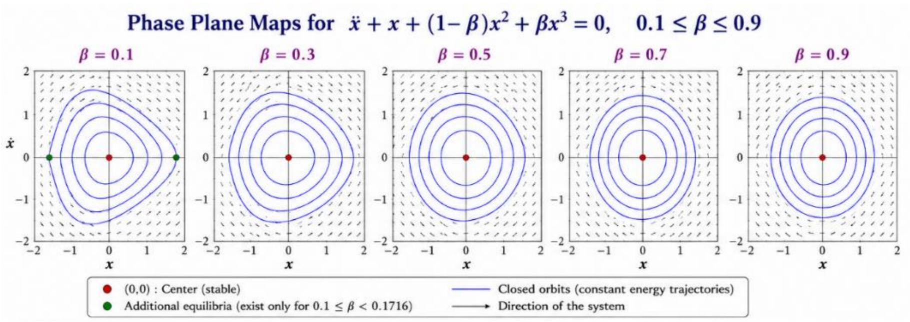

Fig. 3. The phase plane maps that represent the system stability and the impact of β on $\ddot{x} + x + (1-\beta)x^2 + \beta x^3 = 0$, $0.1 \le \beta \le 0.9$

V. Convergence analysis

Iterative approaches aim to provide a set of convergent solutions (x_k) and frequencies (Ω_k).

$$x_{ex} = \lim_{k\to\infty} x_k \text{ or } \Omega_{ex} = \lim_{k\to\infty} \Omega_k$$

(x_{ex}) represents the exact solution of the specified nonlinear oscillator. The current strategy reduces errors in each iterative step, resulting in $|\Omega_2 - \Omega_{ex}| = |80.36263339 - 80.40995076| < \varepsilon$, where ε is a small positive number. The initial amplitude $a = 100$ and the nonlinearity parameter $\beta = 0.9$. This clearly shows that the adopted approach is convergent.

VI. Conclusion

In this study, the Extended Iteration Method (EIM) has been developed and successfully applied to derive improved analytical solutions for the Helmholtz–Duffing oscillator. The proposed method effectively overcomes the limitations of conventional analytical approaches such as the Hamiltonian Approach, Coupled Homotopy

Variational Formulation, Energy Balance Method (EBM), He's Frequency–Amplitude Formulation, Modified Harmonic Balance Method, Homotopy Perturbation Method, second-Order Hamiltonian Approach, First-Order Global Error Minimization Method, Second-Order Global Error Minimization Method, which often rely on small parameters or exhibit limited accuracy for strong nonlinearities.

The results demonstrate that the EIM provides highly accurate approximations for both small and large initial amplitudes, whereas the majority of methods result in greater inaccuracy for bigger initial amplitude values and less for lower ones. The suggested method maintains excellent agreement with the numerical Runge–Kutta solutions. The proposed method achieves excellent results for higher-order solutions, whereas most existing methods produce adequate results for first-order solutions but not for higher-order solutions. The method shows superior convergence properties, reduced computational effort, and broader applicability compared to previous iterative techniques.

Overall, the Extended Iteration Method proves to be an efficient, straightforward, and powerful analytical tool for solving nonlinear oscillators. Its adaptability and precision make it suitable for application to other nonlinear dynamical systems in physics and engineering. Future work may focus on extending the method to multi-degree-of-freedom systems, forced oscillations, and damping effects to further explore its potential in complex nonlinear modeling.

Competing Interests :

The authors claim to have no conflicting interests.

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