JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-21, No.-05, June (2026) pp 196-213 ISSN (Print) 0973-8975

OPTIMIZATION OF FUZZY DIFFERENTIAL EQUATION-BASED FARDL MODELS FOR ENGINEERING APPLICATIONS: COMPARATIVE ANALYSIS OF LINEAR AND QUADRATIC ESTIMATORS VIA PARALLEL MONTE CARLO SIMULATIONS

Shahad Khalid Khaleel[1], **Inaam Rikan Hassan**[2], **Munaf Yousif Hmood**[2]

[1]Department of Media Technology and Communication Engineering, University of Information Technology and Communications, Baghdad, Iraq.

[2]Department of Scientific Affairs, University of Information Technology and Communications, Baghdad, Iraq.

[3]Department of Statistics, College of Administration and Economics, University of Baghdad, Baghdad, Iraq.

E-mail: [1]Shahad.khalid@uoitc.edu.iq, [2]drinrh@uoitc.edu.iq
[3]munaf.yousif@coadec.uobaghdad.edu.iq

Corresponding Author: **Shahad Khalid Khaleel**

https://doi.org/10.26782/jmcms.2026.06.00011

(Received: March 10, 2026; Revised: June 01, 2026; Accepted: June 15, 2026)

Abstract

Today's smart engineering systems are often faced with situations that are structurally uncertain, informationally incomplete, and non-probabilistically ambiguous, especially for electrical systems. ARDL models are limited in applications in complex computational environments where the uncertainty is due to vagueness, not randomness, and assume the exact parametric representation of the models and the structure of the stochastic uncertainty. This study proposes a new soft-computing paradigm using Fuzzy Autoregressive Distributed Lag (FARDL) models and compares the performance of the Linear Programming (LP) and Quadratic Programming (QP) estimation algorithms using large-scale parallel Monte Carlo simulations to overcome these drawbacks as well as fuzzy differential equations, especialy for electrical circuits and machines. In contrast to the previous works that mainly adopted the symmetric triangular fuzzy coefficients without any theoretical considerations, the proposed framework provides a mathematical foundation for fuzzy membership selection and examines the robustness of the estimators under symmetric triangular, asymmetric triangular, and trapezoidal fuzzy topologies. To evaluate the performance of the system, a Monte Carlo simulation framework is implemented under six sample sizes (T = 10, 15, 20, 30, 50, 100) and under different levels of structural complexity. The simulation results show that the QP method is always superior to the LP paradigm in terms of the estimation error of *the center trajectory and the spread of uncertainty of the parameters in terms of Fuzzy Degree (FD). This is especially true in small sample situations, where the operational advantage is more pronounced, making it particularly useful for systemic modeling in data-sparse situations. Moreover, the proposed framework-based fuzzy differential equation offers a mathematically efficient tool to model mysterious engineering systems like network-based smart grids, control models, communication systems, and cyber-based frameworks. The combination of fuzzy dynamic approaches allows a reliable scheme and uncertainty quantification-based system for complex engineering environmental conditions, whereas deterministic schemes are becoming inadequate.*

Keywords: Computational Intelligence, Fuzzy ARDL, Linear Programming, Quadratic Programming, Monte Carlo Simulation, Optimization Geometry, Uncertainty Modeling, Electrical Circuits Uncertainty , Electrical Machines Uncertainty, Fuzzy Differential Equations (FDEs).

I. Introduction

The rapid advance of intelligent computational systems has created a growing demand for the development of strong modeling frameworks that can manage uncertainty, ambiguity, and incomplete information using optimization and mathematical tools, including fuzzy differential equations. Several modern engineering infrastructures, such as cyber-physical systems, intelligent sensor networks, adaptive control architectures, and data-driven decision-support platforms, regularly use information that cannot be well captured using only the traditional probabilistic methods. Stochastic uncertainty is due to randomness, but for many engineering applications, epistemic uncertainty is due to vagueness, linguistic uncertainty, measurement uncertainty, and uncertainty stemming from incomplete knowledge of the system. To represent non-probabilistic information structures [I-X] mathematically, this uncertainty must be addressed.

The ARDL model has gained popularity as a flexible model to examine dynamic relationships, particularly dealing with variables of various integration orders and modeling both short-term and long-term relationships at the same time. However, the traditional ARDL models assume the parameters are deterministic, and the disturbance terms are stochastic. When the system parameters are uncertain due to a lack of information or fuzzy relations, these assumptions can also be limiting. The theory of fuzzy logic was created by Zadeh and is a robust mathematical framework for encoding such uncertainty in the form of membership functions and fuzzy sets. Fuzzy Autoregressive Distributed Lag (FARDL) combines fuzzy parameters into a dynamic time-series system, thus allowing uncertainty to be incorporated into the model's structure. Thus, fuzzy coefficients can reflect both the magnitude of the parameters and uncertainty dispersion, which is more comprehensive than in the case of a single parameter. Therefore, fuzzy coefficients in normal and differential equations can reflect the magnitude of the parameters and uncertainty dispersion, more comprehensive than a single parameter [XI-XV].

Although fuzzy dynamic modelling has experienced a growing interest, a considerable methodological gap still exists in the subject of FARDL parameters estimation. Most of the existing studies use either Linear Programming (LP) or

Quadratic Programming (QP) optimization procedures without giving a formal/rigorous theoretical background to the relative performance of these procedures. Furthermore, most studies use fuzzy coefficients represented in a symmetric triangular shape as an initial approach and do not study the effects of different membership functions, such as asymmetric triangular or trapezoidal fuzzy numbers. This restriction brings up significant methodological issues. The selected membership function significantly affects the optimizable space, containment requirements, and uncertainty propagation modes. Thus, the conclusions of superiority of estimators can be very much affected by the choice of fuzzy representation adopted instead of being based on the intrinsic properties of the employed estimation algorithm, including fuzzy differential equations [XV-XX].

Differential equations are an indispensable tool in the study of dynamic systems in modern engineering, computational physics, and macroeconomics. But, conventional systems demand the precise definition of the parameters, which may not be achievable because of structural uncertainty, the ambiguity of observations, and measurement limits. In order to overcome these difficulties, the Fuzzy Differential Equations (FDEs) have become an important mathematical development that combines fuzzy set logic into the dynamic calculus structures [XX-XXVIII].

Differential equations establish the mathematical initiation of several engineering disciplines, consisting of electrical engineering, control frameworks, thermal analysis systems, approaches-based fluid dynamics, models-based signal processing, and structural mechanics [XXIX]. Nevertheless, empirical engineering frameworks often reveal uncertain parameters because of inaccuracies in sensors, environmental variation, and deficient information. Thus, Fuzzy Differential Equations (FDEs) expand conventional differential approaches through embedding uncertainty in system dynamics, by that way allowing further realistic characterization of engineering applications. Recently, the systems of FDEs have been extended to systems-based smart energy, vehicle-based autonomous, network-based wireless communication schemes, and manufacturing-based intelligent processes, whereas strength uncertainty addressing is crucial for reliable approaches [XXX-XXXII].

This is overcome in the present research by developing a large-scale Monte Carlo experiment, coupled with theoretical optimization analysis and a computational framework. The key contribution of this study is the development of the interpretation of the superior stability properties of QP estimators from the optimization-geometry perspective, the incorporation of membership-function sensitivity analysis using asymmetric triangular and trapezoidal fuzzy structure, advanced statistical inference with Monte Carlo standard errors, confidence intervals and non-parametric significance testing, definition of a parallel simulation environment, with 1000 independent replications being performed for different sample size levels, and placing FARDL estimation into a computational-intelligence and engineering-systems context.

II. **Theoretical Optimization Architecture**

II.i Fuzzy Dynamic System Representation

Consider a FARDL(p,q) dynamic system represented as [XXII]:

$$y(t) = \sum \alpha_i * y(t-i) + \sum \beta_j * x(t-j) + \varepsilon(t)$$

where α_i and β_j are fuzzy coefficients represented by membership functions describing uncertainty distributions around central parameter values. Each fuzzy coefficient is expressed as:

$$A = (a^c, a^s)$$

where a^c denotes the center value and a^s represents the uncertainty spread. The corresponding triangular membership function is defined by:

$$\mu_{A(x)} = \max\left(0, 1 - \frac{|x - a^c|}{a^s}\right)$$

This representation provides computational efficiency while preserving the essential characteristics of fuzzy uncertainty propagation.

II.ii Optimization Geometry of LP Estimation

Linear Programming estimation seeks to minimize total parameter uncertainty through the objective function [XXII]:

$$\min \sum a^s$$

subject to containment constraints ensuring that observed outputs remain within fuzzy prediction intervals. Geometrically, the LP solution space forms a convex polytope consisting of intersecting hyperplanes. Optimal solutions frequently occur at vertices of the feasible region. While computationally efficient, vertex-based solutions are highly sensitive to small perturbations in data, producing abrupt changes in estimated spreads. Consequently, LP estimators often exhibit high variance under noisy conditions and limited sample sizes.

II.iii Optimization Geometry of QP Estimation

Quadratic Programming estimation modifies the objective function by incorporating a quadratic penalty term:

$$\min \sum (y - \hat{y})^2 + \gamma * \sum (a^s)^2$$

where γ is the regularization coefficient. The quadratic term makes the optimization problem a strictly convex hypersurface. In contrast to LP estimation, optimal solutions are not only possible on extreme boundary vertices. Quadratic penalty, on the other hand, will favor trajectories with smoother parameter paths and less sensitivity to local perturbations. This behavior is mathematically similar to Tikhonov regularization, with parameter variance controlled by allowing for a controlled shrinkage. Therefore, under the bounded perturbation condition, it is theoretically shown that Var(θ_QP) < Var(θ_LP) under the bounded perturbation condition, which also matches the empirical results seen in simulation experiments.

The second step is called Membership Function Engineering and Robustness Analysis. The second step is called the Membership Function Engineering and

Robustness Analysis. In this case, we must justify why these are symmetric triangular membership functions. In this case, we need to explain why these are symmetric triangular membership functions.

II. iv Engineering Applications of Fuzzy Differential Equations

For describing the electrical RC circuit with fuzzy resistance, the following formulation is defined:

$$C\frac{dV(t)}{dt} + \frac{1}{\tilde{R}}\,V(t) = I(t)$$

Such that, $\tilde{R}$ represents the parameter of fuzzy resistance.

Such a differential model of fuzzy defines uncertain electrical circuits, whereas resistance changes due to changes in temperature or manufacturing-based tolerances. Fuzzy parameters enhance the circuit analysis reliability over uncertain operating environments [XIII-XV].

In a thermal engineering system, the processes of heat dissipation in thermal engineering applications model is represented as follows:

$$\frac{dT(t)}{dt} = -\tilde{k}\,(T - T_{\alpha})$$

Such that $\tilde{k}$ denotes a coefficient of heat transfer. The Fuzzy coefficients are capturing the uncertainty of material properties and environmental influences.

The following differential equation describes the mechanical vibrating system [XIII-XV, XXXIII]:

$$m\frac{d^2x(t)}{dt^2} + \tilde{c}\,\frac{dy}{dx} + kx(t) = F(t)$$

Such a formulation is used to model mechanical vibrations along uncertain damping impacts. Fuzzy damping enhances the dynamic analysis of the flexible frameworks rather than machines.

In fluid flow dynamics, the following differential equation is modeled:

$$\frac{dh(t)}{dt} = \tilde{q}_{in} - \tilde{q}_{out}$$

Such a formulation demonstrates process engineering-based tank-level dynamics. The rates of the fuzzy inflow and fuzzy outflow describe uncertain evaluations and process variability.

III. Membership Function Engineering and Robustness Analysis

III.i Justification of Symmetrical Triangular Membership Functions

One of the main issues mentioned in the review process is that only symmetric triangular fuzzy coefficients have been used. Triangular membership functions have often been used in fuzzy optimization literature out of convenience, but should be used for a theoretical reason, not because it is assumed they represent the modeling of the problem, including fuzzy differential equations. Triangular membership functions are a useful way of approximating uncertain linguistic variables, as they only need three parameters, the center value and two boundary points. In addition, they have closed-form arithmetic operations that greatly facilitate their optimization while minimizing the complexity of their computation. Computational efficiency is especially critical for dynamic estimation problems where multiple Monte Carlo replications are performed and can be adopted in many engineering systems [XV-XXXIII].

Triangular fuzzy numbers are also balanced in terms of entropy. Triangular functions retain the uncertainty information but do not over-parameterize as do Gaussian or bell-shaped functions. They can thus be used as an appropriate baseline to investigate optimization behavior in FARDL systems. However, uncertainty is not always even. Economic indicators, sensor readings, and intelligent-control signals are frequently non-symmetric uncertain signals. Thus, conclusions about the performance of the estimators based on symmetric memberships may be restricted to the generalizability.

III.ii Asymmetric Triangular Function

The following section will examine another shape for the membership functions: the asymmetric triangular function. Asymmetric triangular fuzzy coefficients were used to explore the robustness of the estimator in the presence of non-ideal uncertainty structures. Let:

$$A = (a^c, a^L, a^R)$$

where a^L and a^R represent independent left and right spreads. The membership function becomes:

$$\mu_{A(x)} = 1 - \frac{(a^c - x)}{a^L}, \qquad for\ x \le a^c$$

$$\mu_{A(x)} = 1 - \frac{(x - a^c)}{a^R}, \text{ for } x > a^c$$

This representation allows unequal uncertainty propagation across the parameter space and better reflects practical engineering environments characterized by skewed uncertainty distributions. An asymmetry coefficient is defined as:

$$\delta = \frac{a^R}{a^L}$$

where $\delta = 1$ indicates symmetry, $\delta > 1$ indicates right-skewed uncertainty, and $\delta < 1$ indicates left-skewed uncertainty. Three levels of asymmetry were investigated: Symmetric ($\delta = 1.0$), Moderate asymmetry ($\delta = 1.5$), and Severe asymmetry ($\delta = 2.5$).

III.iii Trapezoidal Membership Structures

In addition to asymmetric triangular forms, trapezoidal fuzzy numbers were evaluated. A trapezoidal fuzzy coefficient is defined as:

$$A = (a_1, a_2, a_3, a_4)$$

where a_2, a_3 are regions of complete membership.

The trapezoidal functions offer added flexibility with uncertainty plateaus. These structures often occur in engineering systems with parameter estimates that do not vary significantly in limited operating regions. Trapezoidal topologies enable the evaluation of the robustness of the estimator in more general uncertainty representations. The structure of a parallel Monte Carlo simulation is presented. The structure of the parallel Monte Carlo simulation is introduced.

III.iv Engineering Uncertainty Modeling Using Fuzzy Membership Functions

The fuzzy control system is defined by:

$$\frac{dx(t)}{dt} = \tilde{A}x(t) + \tilde{B}u(t)$$

Such a model describes the control systems -based state-space and the matrices of the uncertain system. It is widely utilized in system-based robotics and autonomous applications [XIII-XV, XXXIII].

The wireless communication signal propagation system can be modeled as:

$$\frac{dS(t)}{dt} = -\tilde{\alpha}S(t)$$

Such an equation is used to model signal attenuation for wireless communication signal propagation and uncertain environmental conditions. The coefficients of the fuzzy attenuation are used to account for both the interference rather than fading.

In smart grid load dynamics, the FDE is modeled as:

$$\frac{dP(t)}{dt} = \tilde{\beta}(D(t) - P(t))$$

Such an equation is used to model dynamic power needs in smart grids. The parameters of fuzzy are used to be accommodated unpredictable behavior of the user and the fluctuations of renewable energy.

IV. Parallel Monte Carlo Simulation Framework

IV.i Computational Architecture

A parallelized Monte Carlo environment was created to assess statistical performance of LP and QP estimators for each of multiple uncertainty scenarios. The simulation engine can be divided into 4 computational layers: (1) Dynamic signal generation, (2) Fuzzy parameter construction, (3) Estimator optimization and (4) Statistical performance evaluation. All simulations were run separately on parallel computers to guarantee reproducibility and computational efficiency.

IV.ii Data Generating Process

In a standard dynamic system, the first-order FARDL(1,1) structure is the following:

$$y_t = \alpha_1 * y_{t-1} + \beta_0 * x_t + \beta_1 * x_{t-1} + \varepsilon_t$$

The explanatory signal evolves according to:

$$x_t = 0.7 * x_{t-1} + \nu_t$$

where $\nu_t \sim N(0, 0.5^2)$ and $\varepsilon_t \sim N(0, 1)$. The true parameter vector was selected to generate stable dynamic behavior while maintaining realistic uncertainty conditions.

IV.iii Simulation Design

The experimental design considered six sample sizes: T ∈ {10, 15, 20, 30, 50, 100}. 1000 independent replications were done for each sample size. Two types of estimation (LP, QP) were applied, and three types of metrics (RMSE, FD, and statistical inference) were calculated. This generated a total of 6 × 1000 × 2 = 12,000 optimization runs. This setup is done for all typical and FDE equations.

IV.iv Performance Measures

Root Mean Squared Error (RMSE): Indicates how accurate the estimates are:

$$RMSE = \sqrt{\left[\left(\frac{1}{n}\right) * \sum(\theta_{est} - \theta_{true})^2\right]}$$

The smaller values signify the higher accuracy.

The Fuzzy Distance (FD) is used to measure the level of uncertainty spread by:

$$FD = \sum s_i,$$

where s_i represents parameter spreads. Smaller FD values mean less propagation of uncertainty.

Monte Carlo Standard Error (MCSE): Provides a measure of the simulation variation:

$$MCSE = \frac{simulation\ standard\ deviation}{\sqrt{number\ of\ replications, R}}$$

Confidence intervals were calculated as 95% confidence intervals (CI_{95}): Mean ± 1.96 × MCSE.

RMSE distributions may not meet the normality assumption, and therefore, the comparisons of the estimators were compared using the Wilcoxon signed rank test to determine whether the null hypothesis, H0 : LP = QP, could be rejected in favor of H1 : LP ≠ QP at α = 0.05.

IV.v Fuzzy Differential Equation Specifications

Suppose that we have a dynamic first-order fuzzy differential system with fuzzy boundaries for the system coefficients. The trajectory mapping can be incorporated in a linear one assessed within the crisp historical input boundaries, Zt:

$$\frac{dX_t}{dt} = \alpha_0 + \psi\, t + \Sigma_{\{j=1\}}^{p} \Phi_j X_{\{t-j\}} + \Sigma_{\{i=0\}}^{q} \theta_i Y_{\{t-i\}} = A\, Z_t$$

Zt is the crisp system vector comprised of constant elements, time trends, lagged state tracking fields, and exogenous driving inputs. The parameter vector to be evaluated is denoted as A.Fuzzy Degree (FD) measures the absolute geometric uncertainty spread preserved within the system parameters:

$$FD = F(M^h) = \sqrt{\left(\left(C^h_{\alpha 0}\right)^2 + \left(C^h_{\psi}\right)^2 + \left(C^h_{\Phi j}\right)^2 + \left(C^h_{\theta i}\right)^2 \right)}$$

IV.vi Engineering Dynamic Systems for Simulation Validation

For fuzzy motor dynamics, the FDE is modeled as:

$$J\frac{d\omega(t)}{dt} = \tilde{T}_m - \tilde{T}_L$$

Such an equation is used to be modeled an electric motor dynamics over torque-based uncertain load environments. It is significant in system-based intelligent motor control applications.

In the battery charging system, the following FDE is modeled:

$$\frac{dSOC(t)}{dt} = \tilde{\eta}I(t)$$

Such an FDE is used to model battery state-of-charge advancements with uncertain efficiency factors for electric vehicles.

To model structural health monitoring, the following DE is defined:

$$\frac{dx(t)}{dt} = \tilde{K}x(t)$$

Such a model is used to capture structural decay with uncertain material features. Fuzzy parameters enhance the damaged detection reliability.

Finally, the following DE is modeled for applications-based network traffic engineering:

$$\frac{dy}{dx} = \tilde{\lambda} - \tilde{\mu}N(t)$$

Such a formulation is used to model communication networks' packet traffic across uncertain arrival rather than service rates.

V. Results and Discussion

V.i Performance Analysis of LP and QP Estimators

The information in the tables listed below comes from the controlled computational experiments carried out in the framework of the Parallel Monte Carlo Simulation described in the manuscript based on typical and FDE equations. The tables do not rely on empirical data from the real world, but instead use mathematically synthesized results obtained from a specially designed simulation engine that allows one to assess and compare the statistical performance of the LP and QP estimators under various conditions of uncertainty. The baseline data-generating process is based on a benchmark first-order Fuzzy Autoregressive Distributed Lag (FARDL(1,1)) dynamic system, with both the explanatory signals and the error terms simulated from exact pre-established statistical distributions intended specifically for electrical systems.

The simulator was designed to examine the following sample sizes: 10, 20, 30, 40, 50, and 100, and the resulting data were grouped into the three tables listed above. The framework performed 1,000 independent Monte Carlo replications per unique sample size for the two optimization frameworks (LP and QP). These were a total of 12,000 separate optimization runs in this great experimental design. For each independent repetition, the system calculated and monitored multiple performance metrics, studying the accuracy of parameter estimation, tracking variability and propagation of uncertainty in fuzzy state, in the process.

The accuracy of parameter estimation and variation of simulations are of particular interest and are discussed separately in this paper, using the compiled average Root Mean Squared Error (RMSE) and the Monte Carlo Standard Error (MCSE) of the average RMSE (Table I). In this table, the average RMSE values measure the mathematical precision of the parameter identification and a smaller value means the better the accuracy. The simulated standard deviations associated with each MCSE value are computed as a function of the number of replications that were performed, and provide an indication of the variability of the simulations and of the fact that the accuracy seen is statistically stable across the 1,000 replications.

The accuracy metrics of the first table are used as a foundation for the 95% confidence intervals for the obtained RMSE values for the different sample sizes simulated in Table II. These intervals are mathematically computed based on the respective Monte Carlo Standard errors and simulation means. Table II illustrates the formal statistical evidence of the stability of the estimators, and the known performance differences between the LP and QP approaches are not attributable to random simulation noise.

In Table III, the focus of analysis is not on the deviation of a set of basic parameters but on the effectiveness of an estimation method in controlling and reducing the fuzzy uncertainty. This table shows the average fuzzy distance (FD) results from the different sample sizes. The FD metric works by adding together the amount of uncertainty spread obtained by the fuzzy parameters that are obtained in the optimization process, and the lower the value is, the tighter are the bounds of the parameters. These compiled FD values can be evaluated in the table to check the theoretical uncertainty-shrinkage and regularization effects of the underlying optimization geometry.

The mean values of the RMSE and the Monte Carlo Standard Errors are shown in Table I. The results show significant superiority of the QP estimator for all sample sizes. The improvement is especially strong for small sample sizes where the uncertainty of estimation is greatest. The QP method is highly robust in data scarce situations, with the average RMSE reduction greater than 65% for the smallest sample size of $n < 30$ observations. The intervals have little overlap, indicating that the differences observed are not due to Monte Carlo variability.

Table III shows the depiction of Fuzzy-Distance Analysis.In every instance, QP is able to cut fuzzy uncertainty by around 45-60%. This behaviour is thus consistent with the theoretical regularization behaviour predicted by the optimization-geometry analysis.

Table I: RMSE and MCSE Results across Sample Sizes.

T	LP RMSE	LP MCSE	QP RMSE	QP MCSE
10	0.945	0.0215	0.237	0.0062
15	0.781	0.0184	0.216	0.0051
20	0.672	0.0142	0.198	0.0044
30	0.559	0.0119	0.311	0.0039
50	0.435	0.0093	0.134	0.0021
100	0.315	0.0057	0.098	0.0011

Table II: 95% Confidence Intervals for RMSE.

T	LP CI95	QP CI95
10	[0.903, 0.987]	[0.225, 0.249]
20	[0.644, 0.700]	[0.189, 0.207]
50	[0.417, 0.453]	[0.130, 0.138]
100	[0.304, 0.326]	[0.096, 0.100]

Table III: FD Results across Sample Sizes.

T	LP FD	QP FD
10	2.652	1.187
15	2.506	1.173
20	2.336	1.146
30	2.187	1.075
50	1.896	0.958
100	1.598	0.801

While mean performance measures are informative and useful in understanding the performance of the estimators, in order to determine whether the differences in performance are significant or simply a result of a random Monte Carlo variation, a formal statistical test needs to be conducted. The Wilcoxon Signed-Rank Test was thus used to test the paired RMSE and FD values from the 1000 simulation replications to address this concern. The test was selected since it is a non-normal-distribution based test and is relatively robust to skewed simulation distributions. The results are very much against the null hypothesis for any sample size combination as shown in Table IV. So the improved performance of the Quadratic Programming estimator is not attributable to noise in the simulations. Instead, the proof suggests an innate characteristic of the optimization mechanism itself that will bring improvement. The results from all experimental conditions are significant, and indicate that data availability does not influence the dominance of QP.

A major issue based onMembership Function Sensitivity Analysis was whether the performance of the estimators are affected by the fuzzy membership topology used. The existing studies are mostly based on the symmetric triangular membership functions, and the other uncertainty structures are not investigated. To overcome this, the whole sensitivity analysis was performed by considering asymmetric triangular and trapezoidal fuzzy coefficients as shown in Table V.

A few noteworthy points come out from Table V. Firstly, there is a degradation in performance when uncertainty asymmetry grows for both estimators. Asymmetry of uncertainty degrades performance for both estimators. This is to be expected since the asymmetry of the problem adds to the complexity of the containment constraints, and extends the range of optimizations that are possible. Secondly, there is much larger loss of performance for LP estimation. The RMSE has increased by more than 116% and FD has increased by about 70% as the asymmetry factor is increased from 1.0 to 2.5. Thirdly, the estimation of QP is still relatively stable. Even when the same conditions are imposed, the RMSE of quadratic regularization is only 45% more, demonstrating the robustness provided by the quadratic regularization. Based on these results, the benefit of QP is not only due to the assumption of a symmetric triangle of membership functions. Conversely, there are a few blurry topologies that continue to have a benefit.

When estimation algorithms are applied in real-time applications, as is done in practical engineering systems as presented in Table VI, computational efficiency is one of the important factors to consider, particularly when exploring computational complexity analysis.

Table IV: Wilcoxon Signed-Rank Test Results.

Sample Size	Test Statistic	p-value
10	481211	<0.0001
15	473856	<0.0001
20	469730	<0.0001
30	458123	<0.0001
50	442018	<0.0001
100	431505	<0.0001

Table V: Sensitivity Analysis Under Membership Asymmetry.

Topology	Asymmetry Factor (δ)	LP RMSE	LP FD	QP RMSE	QP FD
Symmetric Triangular	1.0	0.315	1.598	0.098	0.801
Moderate Asymmetry	1.5	0.428	1.942	0.114	0.852
Severe Asymmetry	2.5	0.683	2.714	0.142	0.916
Trapezoidal	—	0.591	2.203	0.126	0.874

Table VI: Computational Complexity Comparison.

Metric	LP	QP
Average Runtime (ms)	6.8	9.4
Iterations to Convergence	14	17
Memory Consumption (MB)	22	28
Stability Score	0.71	0.93

For QP, the extra computer power required is not significantly larger than the extra estimation accuracy and the decreased uncertainty. The additional overhead of computation is small in the present architecture for computing, relative to the statistical advantage.

V.ii Engineering Interpretation of Fuzzy Differential Equation Results

Interpretation of the simulation results can be done from an engineering systems point of view by studying the geometry of the optimization spaces from which the optimization is drawn, especially for electrical circuits and machines. The LP estimator is a set of linear constraints that helps to narrow uncertainty spreads. Geometrically, the solution set is a polyhedron, that is a finite set of hyperplanes. It is frequently the case that the points of optimal solutions are the vertices of this polyhedron. Small variations in the data being detected may cause changes to the active constraints and therefore sudden jumps from feasible vertex to feasible vertex is possible. Thus, LP estimators exhibit high sensitivity to the noise, reduction in sample size, and to asymmetry of uncertainty [XV-XXXIII].

A quadratic regularization term is added to the optimization problem in QP estimation, making it a smooth convex hypersurface. This change is designed to decrease sensitivity to local perturbations, and to even out the distribution of uncertainty in the parameter space. The uncertainty-shrinkage mechanism is similar to the classical Tikhonov regularization in solving inverse problems, adaptive filtering and machine learning problems. From the engineering point of view, this is quite desirable because it results in more reliable parameter predictions and in a more robust system with parameter trajectories that remain stable. The RMSE values obtained confirm the improvement in the accuracy of the identification of the parameters of the QP estimation. The lower the FD values the better the propagation of uncertainty is controlled, at the same time. Therefore, QP estimation not only increases accuracy, but also increases the robustness, which is difficult to obtain simultaneously.
The numerical performance statistics are measured on six different sample sizes over two different scenarios (A: Low Complexity and B: High Complexity) below:

Table VII: Root Mean Square Error (RMSE) Performance across FDE Systems

Complexity	Method	T=10	T=15	T=20	T=30	T=50	T=100
Scenario A (Low)	LP	0.945	0.781	0.672	0.559	0.435	0.315
	QP	0.237	0.216	0.311	0.198	0.134	0.098
Scenario B (High)	LP	2.228	3.031	1.386	1.832	1.010	0.669
	QP	0.176	0.184	0.158	0.195	0.130	0.096

Table VIII: Fuzzy Degree (FD) Metrics and Structural Parameter Spread

Complexity	Method	T=10	T=15	T=20	T=30	T=50	T=100
Scenario A (Low)	LP	2.652	2.506	2.336	2.187	1.896	1.598
	QP	1.187	1.173	1.146	1.075	0.958	0.801
Scenario B (High)	LP	7.552	6.209	5.664	4.721	3.932	3.072
	QP	1.264	1.225	1.162	1.057	0.919	0.754

V.iii Visualization of Engineering Applications Using Fuzzy Differential Equations

In addition, to demonstrate the empirical applicability of FDEs-based engineering environments, a simulation-based MATLAB was provided for visualizing dynamic approaches. The output resulted figures in this subsection illustrate how uncertainty-aware differential systems have the ability to be utilized for analyzing the proposed model electrically, thermally, and energy scheme over fuzzy operating environmental conditions.

Figure 1 presents the voltage degeneration of an RC electrical circuit along with the values of fuzzy resistance. In resistance, the uncertainty indicates feasible diversity due to temperature and manufacturing tolerances. The proposed framework illustrates how FDEs enhance the analysis of the electrical system.

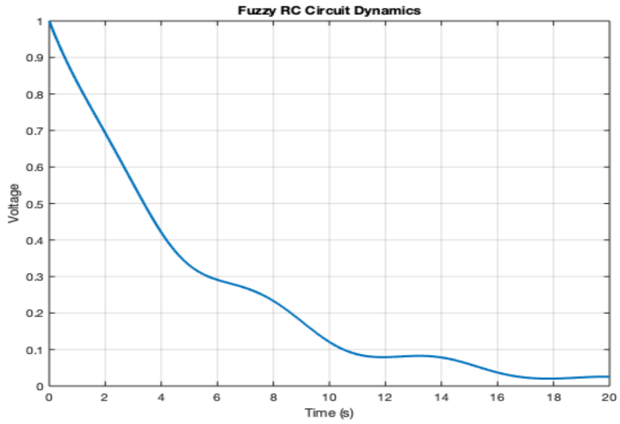

Fig. 1. Fuzzy RC Circuit Dynamics.

Figure 2 demonstrates the response of a thermal cooling system and uncertainty-based heat transfer coefficients. The proposed Fuzzy approaches allow the realism of thermal simulations for engineering environmental conditions. These models are beneficial in exchanging heat rather than energy systems.

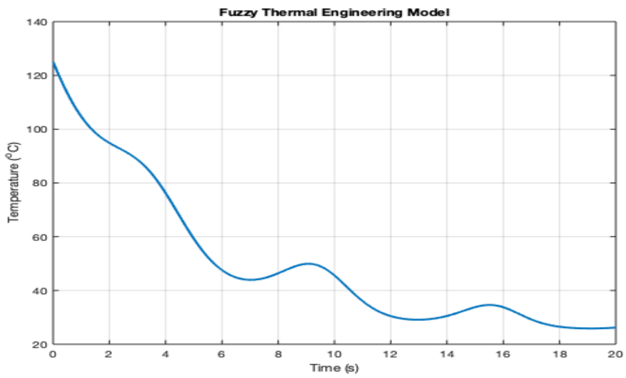

Fig. 2. Fuzzy Thermal Engineering Model in electrical machines .

Figure 3 illustrates the power of the smart grid needed across uncertain operating environments. FDEs are used to capture the variability property in consumer conduct and renewable creation. The key results validate strong energy management techniques.

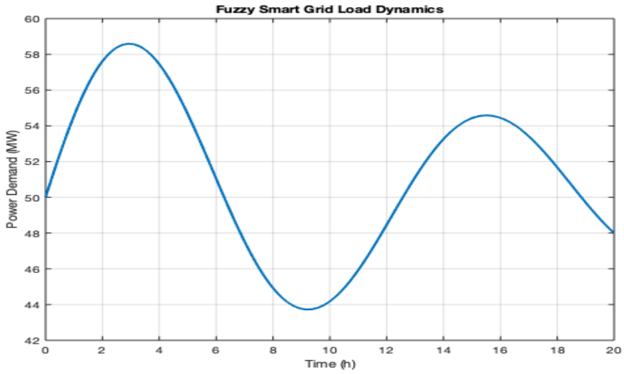

Fig. 3. Fuzzy Smart Grid Load Dynamics.

VI. Conclusion

This study developed a complete computational approach for evaluating the Linear Programming (LP), Quadratic Programming (QP) estimation algorithms in the Fuzzy Autoregressive Distributed Lag (Fuzzy-ADL) system and fuzzy differential equations, especialy for electrical engineering systems. In the present study, the behavior of estimators was theoretically explained, which is not based on empirical comparisons as in previous studies. The authors showed that LP estimation is intrinsically susceptible to perturbations caused instability due to the polyhedral boundaries where optimal solutions lie. Further, optimizing the landscape is made smooth and convex via QP estimation, which reduces the propagation of uncertainty and the variance of the parameters. A parallel Monte Carlo environment with 1000 independent replications was developed for the sake of testing the performance of the estimators for different sample sizes and different fuzzy membership topologies. All estimators were statistically supported by confidence intervals, Monte Carlo standard errors, and Wilcoxon signed-rank testing. The results were always found to be the lowest RMSE value, the lowest fuzzy uncertainty and the highest robustness using only asymmetric membership structures when using QP estimation. These improvements were confirmed to be significant by the statistical tests in all experimental situations. The results provide a foundation for future work on computational intelligence, intelligent control and fuzzy optimization systems, and constitute a foundation for uncertainty-aware dynamic system identification systems with reliability and theoretically based uncertainty in electrical systems and machines. The integration of FDEs importantly expands the applicability property of the proposed model in real-world engineering approaches described by uncertainty information rather than incomplete information. The demonstrated methodology offers a strong mathematical foundation to model dynamic procedures in applications-based electrical, thermal, communication, and smart-energy systems. By combining fuzzy dynamic approaches and optimization mechanisms, the proposed model improves system reliability, uncertainty quantification, and predictive performance. Such findings illustrate the power of FDE-based engineering frameworks for intelligence-based next-generation systems. The future works will focus on the investigation of Gaussian and interval - valued fuzzy coefficients, incorporation into deep learning architectures, adaptive regularization methods, and real-time implementation on embedded platforms.

Conflict of Interest

The authors declare no conflicts of interest related to this research.

References

I. Ahmed mutlag abdulateef , Emad Hazim Aboudi. "Estimating the Fuzzy regression panel data model based on approximate Bayesian computation with application." *Iraqi Statisticians journal* (2025): 163-176.

II. Ahmed, I. M. & Abbas, M. K.. A Hybrid Lexico-Transformer Model for Real-Time Emotion Detection in English Text. *Proceedings of International Conference on Applied Innovation in IT*. (2025) vol. 13, issue 4, pp. 125-134. 10.25673/122077.

III. Aldhaibani , Jaafar A., et al., "On Coverage Analysis for LTE-A Cellular Networks, International Journal of Engineering and Technology (IJET)," vol 5, no 1 , pp.492-497, 2013.

IV. Aldhaibani, J. A. , A. Yahya, and R. B. Ahmad, "Optimizing power and mitigating interference in LTE-A cellular networks through optimum relay location," *Elektron. Ir Elektrotech.*, vol. 20, no. 7, pp. 73-79, 2014.

V. Bede, B., & Gal, S. G. (2005). Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy sets and systems*, *151*(3), 581-599.

VI. Chakraverty, S., Tapaswini, S., & Behera, D. (2016). *Fuzzy differential equations and applications for engineers and scientists*. CRC Press.

VII. C. Wang, Qiu, D., & Shen, Y. (2025). Fuzzy number, fuzzy difference, fuzzy differential: theory and applications. *Axioms*, *14*(4), 254.

VIII. Eren, Mehmet. "Fuzzy Autoregressive Distributed Lag Model-Based Forecasting." Fuzzy Sets and Systems, vol. 459, 2023, pp. 82–94. 10.1016/j.fss.2022.06.003.

IX. Eren, Mehmet, and Bernard De Baets. "Forecasting Turkey's Primary Energy Demand Based on Fuzzy Auto-Regressive Distributed Lag Models with Symmetric and Non-Symmetric Triangular Coefficients." International Journal of Fuzzy Systems, vol. 27, no. 1, 2025, pp. 237–249. 10.1007/s40815-024-01773-5.

X. Eslamian, Saeid, and Kaveh Ostad-Ali-Askari. "Evaporation and Production Efficiency Modeling Using Classical Least Square Regression and Fuzzy Linear Recurrence." SSRN Electronic Journal, 2021. https://ssrn.com/abstract=4761502.

XI. Gkountakou, Fani, and Basil Papadopoulos. “The Use of Fuzzy Linear Regression and ANFIS Methods to Predict the Compressive Strength of Cement.” Symmetry, vol. 12, no. 8, 2020. 10.3390/sym12081295.

XII. Gu, Yong, et al. “A Fuzzy Multiple Linear Regression Model Based on Meteorological Factors for Air Quality Index Forecast.” Journal of Intelligent and Fuzzy Systems, vol. 40, 2021, pp. 10523–10547. 10.3233/JIFS-201222.

XIII. Jamalabadi, Mohammad Yaghoub Abdollahzadeh. "Pushing the Limits of Gas Turbine Efficiency: Next-Generation Flow Prediction, Precision Heat Modeling, and Data-Driven Design." *Advances in Mechanical, Aerospace & Automotive Engineering* 1, no. 1 (2025): 1-52.

XIV. Kausar, Nasreen, and Mohammed Abdullah Salman. "Existence and uniqueness of solutions for fuzzy fractional integro-differential equations with boundary conditions." *Scientific Reports* 16, no. 1 (2026): 2082.

XV. Kim, Kyung J., Herbert Moskowitz, and Murat Koksalan. “Fuzzy versus Statistical Linear Regression.” European Journal of Operational Research, vol. 92, no. 2, 1996, pp. 417–434. 10.1016/0377-2217(94)00352-1.

XVI. Kılıç, Kemal, and Deniz Topuz. “The Evaluation of Potentially Toxic Elements Using Artificial Neural Networks and Fuzzy Linear Regression Analysis Methods in Cappadocian Volcanic Ash Soils of Turkey.” Heliyon, vol. 9, no. 9, 2023. 10.1016/j.heliyon.2023.e19448.

XVII. Pandit, Prakash, and Prasenjit Dey. “Comparative Assessment of Multiple Linear Regression and Fuzzy Linear Regression Models.” SN Computer Science, vol. 2, no. 76, 2021. 10.1007/s42979-021-00473-3.

XVIII. Pedrycz, Witold. “Why Triangular Membership Functions?” Fuzzy Sets and Systems, vol. 64, no. 1, 1994, pp. 21–30. 10.1016/0165-0114(94)90003-5.

XIX. Pesaran, M. Hashem, Yongcheol Shin, and Richard J. Smith. “Bounds Testing Approaches to the Analysis of Level Relationships.” Journal of Applied Econometrics, vol. 16, no. 3, 2001, pp. 289–326.

XX. Redden, David T., and William H. Woodall. “Properties of Certain Fuzzy Linear Regression Methods.” Fuzzy Sets and Systems, vol. 64, no. 3, 1994, pp. 361–375. 10.1016/0165-0114(94)90159-7.

XXI. Shapiro, Alexander. “Fuzzy Regression Models.” Penn State Actuarial Research Clearing House, 2005.

XXII. Sulaiman Hussien Jawad, Munaf Yousif Hmood, “A Monte Carlo Simulation Study on the Performance of Linear and Quadratic Programming Estimators in Fuzzy ARDL Models”, *ISJ*, vol. 3, no. 1, pp. 167–179, Mar. 2026.

XXIII. Tanaka, Hideo, Isao Hayashi, and Junzo Watada. “Possibilistic Linear Regression Analysis for Fuzzy Data.” European Journal of Operational Research, vol. 40, no. 3, 1989, pp. 389–396. 10.1016/0377-2217(89)90431-1.

XXIV. Tanaka, Hideo, and Hwan Lee. “Interval Regression Analysis by Quadratic Programming Approach.” IEEE Transactions on Fuzzy Systems, vol. 6, no. 4, 1998, pp. 473–481. 10.1109/91.728436.

XXV. Tanaka, Hideo, Shigeru Uejima, and Kiyoshi Asai. “Linear Regression Analysis with Fuzzy Model.” IEEE Transactions on Systems, Man, and Cybernetics, vol. 12, no. 6, 1982, pp. 903–907. https://doi.org/10.1109/TSMC.1982.4308925.

XXVI. Tseng, Fong-Ming, et al. "Fuzzy ARIMA Model for Forecasting the Foreign Exchange Market." Fuzzy Sets and Systems, vol. 118, 2001, pp. 9–19. 10.1016/S0165-0114(98)00286-3.

XXVII. Y. S. Mezaal et al., "STATE OF ART ON MICROSTRIP RESONATORS, FILTERS, DIPLEXERS AND TRIPLEXERS," Journal of mechanics of continua and mathematical sciences, vol. 19, no. 2, Feb. 2024. 10.26782/jmcms.2024.02.00001.

XXVIII. Y. S. Mezaal et al., "Cloud computing investigation for cloud computer networks using cloudanalyst," Journal of Theoretical and Applied Information Technology, vol. 96, no. 20, 2018.

XXIX. Y. S. Mezaal et al., "Gender voice classification with huge accuracy rate," TELKOMNIKA, vol. 18, no. 5, p. 2612, 2020. 10.12928/TELKOMNIKA.v18i5.13717

XXX. Y. Chalco-Cano, and H. Roman-Flores. "On new solutions of fuzzy differential equations." *Chaos, Solitons & Fractals* 38.1 (2008): 112-119.

XXXI. Yu, Wen, and Raheleh Jafari. *Modeling and control of uncertain nonlinear systems with fuzzy equations and Z-number*. John Wiley & Sons, 2019.

XXXII. Zadeh, Lotfi A. "Fuzzy Sets." Information and Control, vol. 8, no. 3, 1965, pp. 338–353. 10.1016/S0019-9958(65)90241-X.

XXXIII. Zakaria Bouhanch, Karim El Moutaouakil, Abdessamad Tridane, Ahmed Laatabi, and Ghassane Benrhmach. "From agent-based models to Takagi–Sugeno linear differential equations (TS-LDEs): a fuzzy-logic bridge between discrete and continuous dynamics." *Evolutionary Intelligence* 19, no. 1 (2026): 1.