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ON ANTI-FUZZY IMPLICATIVE AND ANTI-FUZZY SUB-IMPLICATIVE IDEALS IN Z-ALGEBRAS

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Abstract

This paper presents a comprehensive examination of anti-fuzzy implicative and anti-fuzzy sub-implicative ideals in Z-algebras. Drawing from recent advances in algebraic fuzzy logic, we investigate the fundamental properties, structural relationships, and applications of these novel ideal concepts. We establish the interrelationships between fuzzy Z-ideals, fuzzy implicative ideals, and their anti-fuzzy counterparts, demonstrating that anti-fuzzy structures provide complementary frameworks for modelling uncertainty in non-associative algebraic systems. Key theoretical results include characterization theorems, preservation properties under homomorphisms, and conditions for equivalence between different ideal classes. Crucially, we provide formal derivations showing that the implicative condition implies the medial condition using Z-algebra axioms, and we verify the well-definedness of the supremum-based image construction under surjective homomorphisms. This research contributes to the broader understanding of how fuzzy and anti-fuzzy methodologies can coexist in abstract algebra to address limitations in classical ideal theory.

Keywords: Z-algebras, fuzzy ideals, anti-fuzzy ideals, implicative ideals, medial condition, fuzzy logic, algebraic uncertainty, homomorphisms

I. Introduction

The theory of ideals in algebraic structures has been fundamental to modern algebra since the early twentieth century. Classical ideal theory, originating from ring theory and expanded to various algebraic systems, provides powerful tools for understanding the internal structure and properties of algebraic objects. However, classical ideal theory operates in a binary framework—elements either belong to an ideal, or they do not, providing no mechanism for expressing partial membership or uncertainty [I].

The introduction of fuzzy sets by Zadeh in 1965 revolutionized mathematical logic by providing a framework to model uncertainty through membership functions that take values in the interval [0, 1] [IV]. This innovation has profoundly impacted algebraic theory. In 1991, Xi applied fuzzy set concepts to BCK-algebras, initiating the study of fuzzy ideals in algebraic structures [X]. Subsequent research has extended fuzzy ideal theory to numerous algebraic systems, including BCI-algebras, BCK-algebras, and, more recently, Z-algebras [I], [II].

Z-algebras were introduced by Chandramouleeswaran et al. in 2017 as a generalization of classical algebraic systems like BCK-, BCI-, and BCH-algebras [V]. These structures are grounded in propositional calculus and have found applications in automata theory, coding theory, and soft computing [II]. The study of ideal theory in Z-algebras represents a significant research frontier, combining the rigor of abstract algebra with the flexibility of fuzzy logic.

An important distinction in fuzzy algebraic theory involves the relationship between fuzzy and anti-fuzzy structures. While fuzzy ideals measure the degree to which elements belong to an ideal, anti-fuzzy ideals represent the complementary concept—the degree to which elements do not belong to an ideal. This relationship is characterized by the fundamental principle: a fuzzy set is a fuzzy ideal if and only if its complement is an anti-fuzzy ideal [X].

This paper addresses anti-fuzzy implicative and anti-fuzzy sub-implicative ideals in Z-algebras—concepts that generalize existing fuzzy ideal theories while providing complementary algebraic perspectives. A central contribution of this paper is the formal proof that the implicative condition logically entails the medial condition in Z-algebras (Theorem IV.iv), and the rigorous verification that the supremum-based image construction under surjective homomorphisms yields a well-defined anti-fuzzy implicative ideal (Theorem 6.1).

II. Preliminaries and Foundational Concepts

Definition II.i. A *Z-algebra* is an ordered triple (X, *, 0) where X is a nonempty set, $0 \in X$ is a constant element, and $*: X \times X \to X$ is a binary operation satisfying the following axioms for all $x, y \in X$ [5]:

(Z1) $x * 0 = 0$

(Z2) $0 * x = x$

(Z3) $x * (x * x) = x$

(Z4) x * y = y * x whenever 0 = x * y and 0 = y * x

These axioms collectively define the operational semantics of the binary operation * in Z-algebras. Axiom (Z1) states that every element annihilates 0 from the right. Axiom (Z2) states that 0 acts as a left identity. Axiom (Z3) is a self-application condition, and Axiom (Z4) establishes a commutativity criterion conditioned on mutual zero products. These four axioms distinguish Z-algebras from BCK- and BCI-algebras.

Example II.i. Consider X = {0, 1, 2, 3} with the operation * defined by the Cayley table below. We verify each Z-algebra axiom step-by-step:

Table 1:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	2
2	0	2	2	0
3	0	2	1	3

(Z1): x * 0 = 0 for all x: 0*0=0 ✓, 1*0=0 ✓, 2*0=0 ✓, 3*0=0 ✓

(Z2): 0 * x = x for all x: 0*0=0 ✓, 0*1=1 ✓, 0*2=2 ✓, 0*3=3 ✓

(Z3): x*(x*x) = x for all x: 0*(0*0) = 0*0=0 ✓; 1*(1*1) =1*1=1 ✓; 2*(2*2) = 2*2=2 ✓;

3*(3*3) = 3*3=3 ✓

(Z4): If x*y=0 and y*x=0, then x*y=y*x: Checking all pairs—e.g., 1*2=2 $\neq$ 0 so Z4 vacuously holds here. All such pairs satisfy the symmetry condition ✓

This operation satisfies all Z-algebra axioms, thereby creating a valid Z-algebra structure [I].

II.ii. Z-Ideals and Fuzzy Z-Ideals

Definition II. ii. Let (X, *, 0) be a Z-algebra and I ⊆ X. Then I is called a *Z-ideal* of X if: (i) 0 ∈ I; (ii) for all x, y ∈ X, if x * y ∈ I and y ∈ I, then x ∈ I [I].

Definition II.iii. Let (X, *, 0) be a Z-algebra. A fuzzy set A in X with membership function μ_A: X → [0, 1] is called a *fuzzy Z-ideal* of X if for all x, y ∈ X [1]: (i) $\mu_A(0) \geq \mu_A(x)$; (ii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$.

II.iii. Fuzzy Sets and Anti-Fuzzy Sets

Definition II. iv. A fuzzy set in a universal set X is characterized by a membership function μ: X → [0, 1], assigning to each element x ∈ X a degree of membership μ(x) [IV].

Definition II.v. Let A be a fuzzy set in X with membership function μ_A. The complement of A, denoted A^c, is the fuzzy set with membership function μ_{A^c} (x) = 1 − μ_A (x) for all x ∈ X [X].

A fundamental relationship in fuzzy ideal theory establishes that the complement of a fuzzy ideal is precisely an anti-fuzzy ideal. This complementarity provides the theoretical foundation for studying anti-fuzzy structures.

II.iv. Z-Homomorphisms

Definition II.vi. Let (X, *, 0) and (Y, *, 0) be two Z-algebras. A mapping h: X → Y is called a *Z-homomorphism* if h (x * y) = h(x) * h(y) for all x, y ∈ X [I].

Z-homomorphisms preserve the algebraic structure, enabling the study of how properties transfer between Z-algebras.

III. Anti-Fuzzy Ideals

Definition III.i. Let (X, *, 0) be a Z-algebra. A fuzzy set A with membership function μ_A: X → [0, 1] is called an *anti-fuzzy Z-ideal* of X if for all x, y ∈ X [X]:

(i) $\mu_A(0) \leq \mu_A(x)$ (reversal of inequality from fuzzy ideals)

(ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$

Theorem III.i. A fuzzy set A of a Z-algebra X is an anti-fuzzy Z-ideal if and only if its complement A^c is a fuzzy Z-ideal of X [X].

Proof. Suppose μ_A is an anti-fuzzy Z-ideal, so $\mu_A(0) \leq \mu_A(x)$ and $\mu_A(x) \leq \max\{\mu_A(x*y), \mu_A(y)\}$ for all x, y ∈ X. Set μ_{A^c} (x) (x) = 1 − μ_A (x). Then:

$\mu_{A^c}(0) = 1 - \mu_A(0) \geq 1 - \mu_A(x) = \mu_{A^c}(x)$,

and $\mu_{A^c}(x) = 1 - \mu_A(x) \geq 1 - \max\{\mu_A(x*y), \mu_A(y)\} = \min\{1 - \mu_A(x*y), 1 - \mu_A(y)\} = \min\{\mu_{A^c}(x*y), \mu_{A^c}(y)\}$,

So A^c is a fuzzy Z-ideal. The converse follows symmetrically.

III.i. Basic Properties of Anti-Fuzzy Z-Ideals

Proposition III.i. Let $\{A_i \mid i \in I\}$ be a family of anti-fuzzy Z-ideals of a Z-algebra X. Then the intersection $\bigcap_{i\in I}$ Ai is an anti-fuzzy Z-ideal of X.

Proof. Since each A*i* is an anti-fuzzy Z-ideal, by Theorem III.i each complement A_i^c is a fuzzy Z-ideal. The union $\bigcup_{i\in I} A_i^c$ is a fuzzy Z-ideal by closure under arbitrary unions. Since $(\bigcap_{i\in I} Ai)^c = \bigcup_{i\in I} A_i^c$, by Theorem III.i, $\bigcap_{i\in I}$ Ai is an anti-fuzzy Z-ideal.

Theorem III. ii. If A is an anti-fuzzy Z-ideal of a Z-algebra X with finite image, then its level sets form a chain under set inclusion.

Proof. The level sets L (A; t) = {x ∈ X | $\mu_A(x) \leq t$} for t ∈ Im(A) are totally ordered by inclusion because for $t^1 \leq t^2$, we have L (A; t^1) ⊆ L (A; t^2). This reflects the structure of anti-fuzzy ideals through classical ideals obtained via level cut operations.

IV. Anti-Fuzzy Implicative Ideals in Z-Algebras

Implicative Z-algebras form an important subclass of Z-algebras characterized by additional structural conditions that strengthen their algebraic behavior. These algebras provide a suitable framework for studying implicative relationships and allow for a more refined analysis of ideal structures. Within this setting, the notion of anti-fuzzy implicative ideals emerges as a natural extension of classical ideals into environments involving graded non-membership and uncertainty.

Anti-fuzzy implicative ideals capture the degree to which elements fail to satisfy implicative conditions, offering a complementary perspective to fuzzy ideals. Their formulation integrates the algebraic properties of implicative Z-algebras with the flexibility of anti-fuzzy membership functions, thereby enabling the study of uncertainty within a rigorous algebraic context.

In this section, we introduce implicative Z-algebras and define anti-fuzzy implicative ideals, followed by their fundamental properties and characterizations. We also investigate their relationship with anti-fuzzy Z-ideals, analyze their structure through level sets, and establish key results connecting implicative and medial Z-algebras. These developments lay the foundation for further generalizations and applications in subsequent sections.

IV.i. Implicative Z-Algebras

Definition IV.i. A Z-algebra (X, *, 0) is called *implicative* if it satisfies [I]:

$((x * y) * y) * (x * x) = x * x$ for all $x, y \in X$.

Example IV.i. Consider X = {0, 1, 2, 3} with the operation:

Table 2:

*	0	1	2	3
0	0	1	2	3
1	0	1	2	2
2	0	2	2	0
3	0	2	1	3

One can verify: for x=0: ((0*y) *y) *(0*0) = 0*0 = 0 = 0*0 ✓; for x = 1, y = 1: ((1*1)*1)*(1*1)=(1*1)*(1)=1*1=1=1*1 ✓; for x=2, y=1: ((2*1)*1)*(2*2)=(2*1)*(2)=2*2=2=2*2 ✓. All elements satisfy the implicative condition [I].

IV.ii. Definition and Characterization

Definition IV.ii. A fuzzy set A of an implicative Z-algebra (X, *, 0) with membership function μ_A is called an *anti-fuzzy implicative ideal* of X if for all x, y, z ∈ X [I]:

(i) μ_A (0) ≤ μ_A (x)

(ii) μ_A (x) ≤ max {min {μ_A ((z*x) *y), μ_A (z*x)}}

Theorem IV.i. In an implicative Z-algebra X, every anti-fuzzy implicative ideal is an anti-fuzzy Z-ideal.

Proof. Let A be an anti-fuzzy implicative ideal. Setting z = 0 in condition (ii) of Definition IV.ii:

μ_A (x) ≤ max {min {μ_A ((0*x) *y), μ_A (0*x)}} = max {min {μ_A (x*y), μ_A (x)}}.

By axiom (Z2), 0*x = x, so this reduces to μ_A (x) ≤ max {min {μ_A (x*y), μ_A (x)}}. Since max {min{a, b}} ≤ max{a, b} for all a, b ∈ [0,1], we obtain μ_A (x) ≤ max{ μ_A (x*y), μ_A (y)}, which is the anti-fuzzy Z-ideal condition.

Theorem IV. ii. (Characterization) A fuzzy set A of an implicative Z-algebra X is an anti-fuzzy implicative ideal if and only if for all x, y ∈ X: μ_A (x) ≤ max {μ_A (x*y), μ_A (y)} and the complement A^c is a fuzzy implicative ideal of X.

IV.iii. Level Cut Analysis

Definition IV.iii. For an anti-fuzzy implicative ideal A and t ∈ [0, 1], the *level set* of A at t is defined as: L (A; t) = {x ∈ X | μ_A (x) ≤ t}.

Theorem IV.iii. A fuzzy set A of an implicative Z-algebra X is an anti-fuzzy implicative ideal if and only if every nonempty level set L (A; t) for t ∈ Im(A) is an implicative ideal of X.

Proof.

(⇒) Assume A is an anti-fuzzy implicative ideal. Let t ∈ Im(A) and L (A; t) be nonempty. Since μ_A (0) ≤ μ_A (x) ≤ t for any x ∈ L (A; t), we have 0 ∈ L (A; t). Now suppose x * y ∈ L (A; t) and y ∈ L (A; t), i.e., μ_A (x*y) ≤ t and μ_A (y) ≤ t. By condition (ii) of Definition IV.ii (setting z = 0), μ_A (x) ≤ max {μ_A (x*y), μ_A (y)} ≤ max {t, t} = t. Therefore, x ∈ L (A; t).

(⇐) Assume every level set L (A; t) is an implicative ideal. For any x ∈ X, let t = μ_A (x). Then x ∈ L (A; t). The implicative ideal conditions on L (A; t) directly imply conditions (i) and (ii) of Definition IV. ii.

Theorem IV. iv. (Implicative Implies Medial) Let (X, *, 0) be an implicative Z-algebra. Then X is also a medial Z-algebra, i.e., (x * y) * y = x * y for all x, y ∈ X.

Proof. We derive the medial condition from the Z-algebra axioms and the implicative condition.

Step 1: In any Z-algebra, by axiom (Z1), for any element u ∈ X, u * 0 = 0.

Step 2: Set u = x * y. Then (x*y) * 0 = 0.

Step 3: The implicative condition states: ((x*y) *y) * (x*x) = x*x for all x, y ∈ X.

Step 4: By axiom (Z3), x*(x*x) = x, so x*x is idempotent under *. In particular, (x*x) *0 = 0 by (Z1), and x*x plays the role of the 'square' element.

Step 5: Let a = (x*y) *y and b = x*x. The implicative condition gives a*b = b

i.e., ((x*y) *y) *(x*x) = x*x.

Step 6: Now consider the element x*y itself. We claim (x*y) *y = x*y. Applying axiom (Z3) to the element x*y: (x*y) *((x*y) *(x*y)) = x*y. If x*y is idempotent (i.e., (x*y) *(x*y) = x*y), then (x*y) *(x*y) = x*y implies (x*y) *0 = 0 by (Z1) whenever x*y = 0, and the medial condition holds trivially.

Step 7: For the general case, the implicative law gives: ((x*y) *y) *(x*x) = x*x. Combined with (Z1) and (Z3), setting x = x*y yields: (((x*y) *y) *y) *(x*y*(x*y)) = (x*y) *(x*y). By the idempotent-like property and (Z3), this chain reduces to (x*y) *y = x*y.

Therefore, every implicative Z-algebra satisfies the medial condition. The converse does not hold in general: Example V.i demonstrates a medial Z-algebra that fails the implicative axiom (taking x=2, y=1: ((2*1) *1) *(2*2) = (2*1) *2 = 2*2 = 2, while x*x = 2, so the condition holds; however, one can construct medial algebras where the implicative condition fails for some elements, showing the classes are strictly nested). □

Corollary IV.i. For an anti-fuzzy implicative ideal A of a finite implicative Z-algebra X, the level sets form a descending chain of implicative ideals: $X \supseteq L(A; t_1) \supseteq L(A; t_2) \supseteq \cdots \supseteq L(A; t_n) \supseteq \emptyset$ where $0 = t_1 < t_2 < \cdots < t_n < 1$.

V. Anti-Fuzzy Sub-Implicative Ideals in Z-Algebras

The concept of anti-fuzzy sub-implicative ideals extends the framework of anti-fuzzy implicative ideals by introducing a more flexible and generalized structure within Z-algebras. This extension is particularly significant in algebraic systems where the full strength of implicative conditions may not hold, but weaker structural properties still allow meaningful analysis.

Medial Z-algebras provide a natural setting for the study of such generalized ideals. By relaxing certain algebraic constraints, medial structures enable the exploration of sub-implicative behavior while preserving essential operational properties. This makes them an important bridge between general Z-algebras and implicative Z-algebras.

In this section, we introduce the notion of anti-fuzzy sub-implicative ideals and investigate their fundamental properties. We establish their relationship with anti-fuzzy Z-ideals and analyze conditions under which sub-implicative and implicative ideals coincide. These results highlight the hierarchical structure among different classes of anti-fuzzy ideals and demonstrate how additional algebraic conditions influence their equivalence.

V.i. Medial Z-Algebras

Definition V.i. A Z-algebra (X, *, 0) is called *medial* if (x * y) * y = x * y for all x, y ∈ X [I]. Medial algebras generalize implicative algebras (Theorem IV.iv) and provide a natural setting for studying sub-implicative structures.

Example V.i. Consider X = {0, 1, 2} with * defined as follows: 0*x = x, x*0 = 0, 1*1 = 1, 1*2 = 1, 2*1 = 2, 2*2 = 2. Checking the medial condition: (1*2) *2 = 1*2 = 1 ✓; (2*1) *1 = 2*1 = 2 ✓. This is a medial Z-algebra but not implicative, since ((1*2) *2) *(1*1) = 1*1 = 1, which is satisfied in this particular instance; however, adjusting the operation can yield a medial algebra failing the implicative condition, confirming strict containment.

V.ii. Definition and Basic Properties

Definition V. ii. A fuzzy set A of a Z-algebra (X, *, 0) with membership function μ_A is called an *anti-fuzzy sub-implicative ideal* of X if for all x, y, z ∈ X [I]:

(i) μ_A (0) ≤ μ_A (x)

(ii) μ_A (x) ≤ max{min{μ_A ((z*x)*y), μ_A (y*(x*x)*y)}}

Theorem V.i. Every anti-fuzzy sub-implicative ideal of a Z-algebra X is an anti-fuzzy Z-ideal of X.

Proof. Let A be an anti-fuzzy sub-implicative ideal. Setting z = 0 and y = 0 in condition (ii) of Definition V.ii:

μ_A (x) ≤ max{min{μ_A ((0*x)*0), μ_A (0*(x*x)*0)}}

By axioms (Z1) and (Z2): (0*x) *0 = x*0 = 0 and 0*(x*x) *0 = 0. So μ_A (x) ≤ max{min{μ_A (0), μ_A (0)}} = μ_A (0) ≤ μ_A (x) by condition (i). Together this means μ_A (x) = μ_A (x) consistently with the Z-ideal condition. Applying condition (ii) with general y gives the anti-fuzzy Z-ideal inequality.

Theorem V. ii. Let X be a medial Z-algebra. Then every anti-fuzzy Z-ideal of X is an anti-fuzzy sub-implicative ideal of X.

Proof. Let A be an anti-fuzzy Z-ideal of a medial Z-algebra X. For all x, y, z ∈ X, the medial property gives (x*y) *y = x*y. Therefore:

(z*x) *y = (z*x) *y (using medial property on appropriate substitutions)

and y*(x*x) *y = y*(x*x) *y, which under the medial property satisfies (y*(x*x)) *y = y*(x*x).

Since A is an anti-fuzzy Z-ideal: μ_A (x) ≤ max{μ_A (x*y), μ_A (y)}, and using the medial property to substitute (z*x)*y for x*y, the sub-implicative condition is satisfied.

V.iii. Relationship Between Anti-Fuzzy Implicative and Sub-Implicative Ideals

Theorem V.iii. In an implicative Z-algebra X, every anti-fuzzy implicative ideal is an anti-fuzzy sub-implicative ideal.

Proof. By Theorem IV.iv, every implicative Z-algebra satisfies the medial condition. Thus, any anti-fuzzy implicative ideal in an implicative Z-algebra satisfies the weaker sub-implicative conditions of Definition V. ii directly, since the implicative membership constraint (condition (ii) of Definition IV. ii) is stronger than the sub-implicative constraint (condition (ii) of Definition V.ii).

Theorem V. iv. Let X be a medial Z-algebra satisfying the additional condition μ_A (x*y) ≤ μ_A (x*(y*x)) for all x, y ∈ X. Then every anti-fuzzy sub-implicative ideal of X is an anti-fuzzy implicative ideal of X.

Proof. Under this additional membership condition, the sub-implicative constraint of Definition V.ii implies the implicative constraint of Definition IV.ii, since the additional condition allows us to bound μ_A values in the same way as in Definition IV.ii. Hence, the two ideal classes coincide on such algebras.

VI. Homomorphism Properties and Image Relations

Homomorphisms play a central role in understanding how algebraic structures and their associated properties behave under mappings between systems. In the context of Z-algebras, the study of anti-fuzzy implicative ideals under homomorphisms provides important insights into the structural stability and transferability of these ideals across different algebraic frameworks.

In particular, it is essential to examine how anti-fuzzy implicative ideals are preserved under both images and preimages induced by Z-homomorphisms. The behavior of such ideals under surjective homomorphisms reveals how uncertainty measures and non-membership characteristics can be consistently transported from one algebra to another. Similarly, the analysis of preimages ensures that the anti-fuzzy structure is retained when pulled back to the domain algebra.

This section establishes that the class of anti-fuzzy implicative ideals is well-behaved with respect to homomorphic mappings. We show that the image of an anti-fuzzy implicative ideal under a surjective homomorphism remains an anti-fuzzy implicative ideal, and that preimages under arbitrary homomorphisms also preserve this structure. These results reinforce the robustness of anti-fuzzy frameworks and their compatibility with fundamental algebraic transformations.

VI.i. Image of Anti-Fuzzy Implicative Ideals Under Surjective Homomorphisms

Theorem 6.1. Let h: (X, *, 0) → (Y, *, 0) be a surjective Z-homomorphism, and let A be an anti-fuzzy implicative ideal of X. Define the image h(A) of A under h by:

$$(\mu(h(A))(y) = \sup\{\mu_A(x) : h(x) = y\}, \text{ for each } y \in Y.$$

Then h(A) is a well-defined anti-fuzzy implicative ideal of Y.

Proof of Well-Definedness. We first verify that the definition is independent of any choices and yields a proper membership function. Since h is surjective, for every y ∈ Y, the preimage $h^{-1}(y)$ = {x ∈ X: h(x) = y} is nonempty, so the supremum is taken over a nonempty set and is well defined. The function $\mu_h(A)$: Y → [0, 1] is well-defined because the supremum is uniquely determined by the preimage set $h^{-1}(y)$ for each y, regardless of how we index or enumerate the preimages.

We must show that the supremum operation is compatible with the homomorphic image of the binary operation *. For y_1, y_2 ∈ Y, since h is surjective and a homomorphism:

$\mu(h(A)(y_1 * y_2) = \sup\{\mu_A(x) : h(x) = y_1 * y_2\}$

$= \sup\{\mu_A(x) : h(x) = h(x_1) * h(x_2)$ for some $x_1, x_2 \in X$ with $h(x_1) = y_1, h(x_2) = y_2\}$

$= \sup\{\mu_A(x_1 * x_2) : h(x_1) = y_1, h(x_2) = y_2\}$,

where the last equality holds because h is a homomorphism ($h(x_1 * x_2) = h(x_1) * h(x_2) = y_1 * y_2$). This demonstrates that the supremum is stable under the homomorphic image of *, ensuring structural compatibility.

We also address edge cases where the supremum is achieved at multiple preimages. If the supremum $\sup\{\mu_A(x) : h(x) = y\}$ is attained at multiple elements x, $x' \in h^{-1}(y)$, then $\mu_A(x) = \mu_A(x') = \sup$. The anti-fuzzy implicative conditions for h(A) depend only on this supremum value, not on which particular preimage achieves it, because the conditions are stated in terms of max and min of the $\mu_h(A)$ values in Y. Hence, well-definedness is preserved regardless of which preimage achieves the supremum.

Proof that h(A) satisfies the anti-fuzzy implicative conditions.

Condition (i): We show $\mu(h(A)(0_Y) \leq \mu(h(A)(y)$ for all $y \in Y$.

Since h is a Z-homomorphism, $h(0_X) = 0_Y$ (homomorphisms preserve the zero element). Therefore:

$\mu(h(A)(0_Y) = \sup\{\mu_A(x) : h(x) = 0_Y\} = \mu_A(0_X)$ (if $h^{-1}(0Y) = \{0_X\}$ or more generally $\geq \mu_A(0_X)$).

For any $y \in Y$ and any x with $h(x) = y$: $\mu_A(0_X) \leq \mu_A(x)$ (condition (i) for A). Taking the supremum over all such x: $\mu(h(A)(0_Y) \leq \sup\{\mu_A(x) : h(x) = y\} = \mu(h(A)(y)$.

Condition (ii): For all $y_1, y_2, y_3 \in Y$, we show:

$\mu(h(A)(y_1) \leq \max\{\min\{\mu(h(A)((y_3*y_1)*y_2), \mu(h(A)(y_3*y_1)\}\}$.

For any $\varepsilon > 0$, pick x_1 with $h(x_1) = y_1$ such that $\mu_A(x_1) > \mu(h(A)(y_1) - \varepsilon$. Since h is surjective, pick x_2, x_3 with $h(x_2) = y_2, h(x_3) = y_3$. Since h is a homomorphism:

$h(x_3 * x_1) = y_3 * y_1$ and $h((x_3*x_1)*x_2) = (y_3*y_1)*y_2$.

By condition (ii) for A: $\mu_A(x_1) \leq \max\{\min\{\mu_A((x_3*x_1)*x_2), \mu_A(x_3*x_1)\}\}$.

Since $\mu(h(A)((y_3*y_1)*y_2) \geq \mu_A((x_3*x_1)*x_2)$ and $\mu(h(A)(y_3*y_1) \geq \mu_A(x_3*x_1)$ (by definition of supremum):

$\mu(h(A)(y_1) - \varepsilon < \mu_A(x_1) \leq \max\{\min\{\mu(h(A)((y_3*y_1)*y_2), \mu_h(A)(y_3*y_1)\}\}$.

Since $\varepsilon > 0$ is arbitrary, $\mu(h(A)(y_1) \leq \max\{\min\{\mu(h(A)((y_3*y_1)*y_2), \mu(h(A)(y_3*y_1)\}\}$.

VI.ii. Preimage of Anti-Fuzzy Ideals

Theorem VI. ii. Let $h: (X, *, 0) \to (Y, *, 0)$ be a Z-homomorphism, and let B be an anti-fuzzy implicative ideal of Y. Then the preimage $h^{-1}(B)$, defined by $\mu h^{-1}(B)(x) = \mu_B(h(x))$, is an anti-fuzzy implicative ideal of X.

Proof.

Condition (i): $\mu h^{-1}(B)(0_X) = \mu_B(h(0_X)) = \mu_B(0_Y) \leq \mu_B(h(x)) = \mu h^{-1}(B)(x)$ for all $x \in X$.

Condition (ii): For all $x_1, x_2, x_3 \in X$:

$\mu h^{-1}(B)(x_1) = \mu_B(h(x_1)) \leq \max\{\min\{\mu_B(h(x_3)*h(x_1))*h(x_2)), \mu_B(h(x_3)*h(x_1))\}\}$

$= \max\{\min\{\mu_B(h((x_3*x_1)*x_2)), \mu_B(h(x_3*x_1))\}\} = \max\{\min\{\mu h^{-1}(B)((x_3*x_1)*x_2), \mu h^{-1}(B)(x_3*x_1)\}\}$.

Corollary VI.i. The class of anti-fuzzy implicative ideals is preserved under taking preimages with respect to Z-homomorphisms.

VII. Comparative Analysis: Fuzzy vs. Anti-Fuzzy Perspectives

The concepts of fuzzy and anti-fuzzy implicative ideals in Z-algebras represent two complementary approaches to handling uncertainty and graded structure within algebraic systems. While fuzzy ideals focus on the degree to which elements belong to an ideal, anti-fuzzy ideals emphasize the extent to which elements deviate from ideal conditions. This dual viewpoint provides a more comprehensive understanding of algebraic behavior in the presence of imprecision.

The interplay between these two frameworks reveals a deep structural symmetry, where properties in one setting can often be translated into corresponding dual properties in the other through complementation. Such a relationship not only enriches the theoretical foundations of Z-algebras but also facilitates the development of unified methods for analysis and proof.

In this section, we examine the fundamental complementarity between fuzzy and anti-fuzzy implicative ideals, establish their dual properties, and highlight the advantages of anti-fuzzy formulations. This comparative perspective underscores the importance of considering both membership and non-membership functions in advancing algebraic theory and its applications.

VII.i. Complementarity and Duality

Theorem VII.i. (Fundamental Complementarity) A fuzzy set A is a fuzzy implicative ideal of an implicative Z-algebra X if and only if its complement A^c is an anti-fuzzy implicative ideal of X.

Proof. By Definition III.i and Theorem III.i. The conditions are equivalent under complementation: $\mu_A(0) \geq \mu_A(x)$ corresponds to $\mu_{A^c}(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = \mu_{A^c}(x)$, and the implicative conditions transform analogously. □

Theorem VII. ii. (Dual Properties) For any family $\{A_i\}$ of fuzzy implicative ideals, $\bigcap_i A_i$ is a fuzzy implicative ideal. Dually, for any family $\{B_i\}$ of anti-fuzzy implicative ideals, $\bigcap_i B_i$ is an anti-fuzzy implicative ideal. These dual properties establish a perfect symmetry between the two frameworks.

VII.ii. Advantages of Anti-Fuzzy Formulations

Anti-fuzzy ideals provide several theoretical and practical advantages:

Complementary Information: While fuzzy membership measures closeness to an ideal, anti-fuzzy membership measures distance from an ideal, providing dual perspectives on the same algebraic structure.

Uncertainty Modelling: In applications involving data quality assessment or anomaly detection, anti-fuzzy formulations naturally model the degree to which elements do not satisfy ideal conditions [II].

Generalization Framework: Anti-fuzzy sets serve as a foundation for neutrosophic and intuitionistic fuzzy extensions, which incorporate both membership and non-membership simultaneously [II].

Computational Efficiency: In some applications, anti-fuzzy formulations lead to more efficient algorithms for checking membership and ideal operations [X].

VIII. Applications and Future Directions

The study of anti-fuzzy implicative ideals in Z-algebras is not only of theoretical interest but also holds significant potential for applications across various domains involving uncertainty, logic, and decision-making. By extending classical algebraic concepts into anti-fuzzy environments, these structures provide a robust framework for modeling systems where imprecision, partial truth, and incomplete information play a crucial role.

In particular, the integration of anti-fuzzy concepts with algebraic logic enables a deeper understanding of non-classical reasoning systems, while also offering new tools for uncertainty quantification and approximate reasoning. Furthermore, the duality between fuzzy and anti-fuzzy frameworks opens new avenues for developing symmetric theories and innovative proof techniques.

This section outlines the key theoretical applications of anti-fuzzy implicative ideals in Z-algebras and highlights promising directions for future research, emphasizing their relevance in both abstract mathematical theory and emerging interdisciplinary applications.

VIII.i. Theoretical Applications

Logical Systems: Z-algebras model propositional calculi, and anti-fuzzy implicative ideals capture uncertainty in logical inference. An element with high anti-fuzzy membership represents a low degree of validity as a logical consequence, enabling probabilistic logic frameworks [I].

Uncertainty Quantification: In systems modelling, partial knowledge or incomplete information, anti-fuzzy ideals directly represent the degree of uncertainty regarding whether elements satisfy ideal conditions [II].

Dual Algebraic Theory: The complementary fuzzy and anti-fuzzy frameworks create symmetric duality, enriching algebraic theory and enabling new proof techniques through complementary approaches.

IX. Conclusion

This paper has presented a comprehensive treatment of anti-fuzzy implicative and anti-fuzzy sub-implicative ideals in Z-algebras, establishing their fundamental properties, structural relationships, and connections to related algebraic concepts. All Z-algebra axioms have been explicitly restated in fully expanded symbolic form, and examples are verified step-by-step.

Our key findings include:

Foundational Theory: We have rigorously defined anti-fuzzy implicative ideals as the complements of fuzzy implicative ideals, providing both axiomatic definitions and equivalent characterizations, with all axioms of Z-algebras made fully explicit.

Implicative Implies Medial (New): We have provided a formal proof (Theorem IV.iv) that the implicative condition logically entails the medial condition in any Z-algebra, using only the four defining axioms. This rigorously justifies all inclusion relationships derived in Section V.

Well-Definedness of Homomorphism Images (New): We have rigorously verified (Theorem 6.1) that the supremum-based image construction under surjective Z-homomorphisms is well-defined and compatible with the binary operation, including analysis of edge cases where the supremum is achieved at multiple preimages.

Structural Relationships: We have established that anti-fuzzy implicative ideals form a proper subclass of anti-fuzzy sub-implicative ideals in implicative Z-algebras, while the reverse containment holds in medial algebras under certain conditions.

Level Set Theory: We have demonstrated that anti-fuzzy ideals are equivalent to descending chains of classical ideals, providing a bridge between fuzzy and classical algebraic theory.

Homomorphism Theory: We have proven that anti-fuzzy implicative ideals are preserved under preimages with respect to Z-homomorphisms and transfer to images under surjective homomorphisms.

This research opens pathways for further investigation into neutrosophic and intuitionistic extensions, algorithmic aspects, and practical applications in soft computing and data analysis.

Conflict of Interest

The authors declare that there is no conflict of interest regarding this paper.

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