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GENERALIZED LOGARITHMIC SERIES AND THEIR CONNECTIONS TO POLYLOGARITHMS

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Abstract

This study develops a broad extension of logarithmic series and presents exact formulas for their sums. By reformulating the series through suitable integral and functional representations, the work uncovers direct links between these generalized series and polylogarithmic functions. The approach yields several transformation identities that streamline the evaluation of such series and reveal a unified structure underlying many classical logarithmic and alternating forms. Illustrative special cases and numerical checks highlight the accuracy and versatility of the derived results, demonstrating their usefulness in analytic methods and computational applications.

Keywords: Logarithmic series, Alternating series, Generalization, Summability

Nomenclature

p	natural number
q	natural number
n	natural number
F	function to generalized define series
k	positive integer

I. Introduction

Logarithmic-type series arise naturally in many areas of mathematics, particularly in problems involving frequency distributions, analytic number theory, and the study of special functions. Their historical relevance can be traced to biological and ecological modeling, where Fisher and collaborators introduced the logarithmic series distribution to describe species–abundance patterns in random samples of animal populations [VII]. Since then, logarithmic series have found applications in a wide range of analytical settings, including the evaluation of definite integrals, transformation formulas, and the development of summability methods.

Classical approaches to infinite series—summation, acceleration, and transformation—are well established in the literature through the foundational works of Knopp [XI, XII], Hardy [X], and Apostol [XVIII, I]. In recent decades, these methods have been complemented by specialized studies focusing on generalized harmonic functions, alternating series, and functional relations involving zeta- and polylogarithmic-type expressions [II, IV, V, VIII, XX]. Such results demonstrate the growing significance of functional identities in deriving closed forms or simplified representations for series that traditionally resist direct summation.

In the context of generalizing classical series, prior contributions have explored extensions of alternating and harmonic-type series, as well as their acceleration and convergence properties [XIII-XVI]. Many of these generalizations rely on integral representations, change-of-variable techniques, or the use of polylogarithm functions, particularly the dilogarithm, which frequently appears in the exact summation of series with logarithmic structure.

Motivated by these observations, the present manuscript develops a general framework for analyzing logarithmic-type series through integral representations and polylogarithmic identities. By decomposing the rational components of the series and exploiting substitutions that naturally lead to the polylogarithm. This yields explicit formulas that unify and extend earlier results appearing in the works of Modi and Ranabhatt [XIII-XVI], Apostol [XVIII, I], and recent contributions involving generalized harmonic and alternating series [II, V, VIII]. The main results established in Section 2 provide (i) a general representation formula for a broad class of logarithmic series, (ii) an evaluation of the associated integral function in terms of polylogarithms, and (iii) exact closed-form expressions for the sums of these generalized series. In addition, several limiting cases are derived, showing how classical logarithmic and alternating series emerge as natural consequences of the generalized framework. These results form the foundation for the theorems proved in the next section, where the method is systematically applied to obtain explicit summation identities and verifiable numerical examples.

II. Main Results

The study of logarithmic-type series has long been connected with classical problems in summability, analytic continuation, and the evaluation of special functions. Foundational treatments of infinite series by Knopp [XI, XII] and Hardy [X] emphasize the importance of transforming slowly convergent series into analytically manageable forms. In more recent developments, series involving logarithmic and harmonic components have been examined through functional and integral techniques, often leading to connections with polylogarithmic functions and related special functions (see Apostol [I, II]; Chen & Chen [IV]; Zheng [XX]).

A recurring structure in these investigations is the series for $|x|<1,\ k \in N$

$$\sum_{n=1}^{\infty} \frac{x^n}{n(k+n)} ,$$

which arises naturally when rational functions are expanded into power series or when logarithmic integrals are decomposed into elementary fractions. Earlier generalizations of alternating and harmonic-type series by Modi and Ranabhatt [XIII-XVI] indicate that such expressions often admit closed-form evaluations once their integral representations are properly identified.

The present work advances this direction by establishing a systematic method for expressing generalized logarithmic series through integral transforms and polylogarithmic identities. A key step is the representation of the term $1/n(k+n)$ by partial fractions, which converts the original series into an integral involving a rational function with simple poles. This leads to the auxiliary function:

$$F_k(x) = \int_0^x \frac{t^{k-1}}{1-t}\, dt$$

whose evaluation is closely related to the polylogarithm function $Li_s(x)$, a central object in analytic number theory and special function theory. The role of the dilogarithm and higher polylogarithms in summing series of this type has also been highlighted in several modern studies, including Gluzman & Yukalov [VIII] and Fan & Chu [VI].

By combining the integral representation of the generalized logarithmic series with known identities for $Li_s(x)$, we derive explicit closed-form expressions for the sums under consideration. These formulas not only generalize classical logarithmic series but also connect them with functional relations appearing in the broader literature on infinite series and special functions (e.g., Varin [XIX]).

The generalized logarithmic series can be reformulated into an equivalent integral representation using standard analytic techniques. This transformation establishes a direct connection between discrete series expressions and continuous integral forms, thereby enabling more flexible analysis and evaluation.

The results presented in this section provide a unified framework for evaluating generalized logarithmic series. They extend several known identities and reveal their intrinsic connection with polylogarithmic functions, offering a systematic and coherent approach to the study of such series.

Theorem 2.1

Let $0 < p < q$. Then the series $F(p,q) = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{p}{q}\right)^k \left(\frac{k \ln p - 1}{k^2}\right)$ admits the integral representation $F(p,q) = \int_0^p \frac{\ln t}{t+q}\, dt$, where the interchange between summation and integration is justified.

Proof:

We begin with the identity

$$\frac{1}{t+q} = \frac{1}{q}\frac{1}{1+\frac{t}{q}} = \frac{1}{q}\sum_{k=0}^{\infty}(-1)^k \left(\frac{t}{q}\right)^k = \sum_{k=0}^{\infty}(-1)^k \frac{t^k}{q^{k+1}} \quad \text{for } 0 \le t \le p < q$$

which follows from the geometric series expansion. This series converges absolutely for $\left|\frac{t}{q}\right| < 1$

Multiplying both sides by $\ln t$, this yields $\frac{\ln t}{t+q} = \sum_{k=0}^{\infty}(-1)^k \frac{t^k \ln t}{q^{k+1}}$

We now integrate over $(0, p)$

$$\int_0^p \frac{\ln t}{t+q}\, dt = \int_0^p \left(\sum_{k=0}^{\infty}(-1)^k \frac{t^k \ln t}{q^{k+1}} \right) dt$$

To justify the interchange of summation and integration, we verify the conditions of Tonelli's theorem (absolute integrability).

Consider $\sum_{k=0}^{\infty}\int_0^p \left|(-1)^k \frac{t^k \ln t}{q^{k+1}}\right| dt = \sum_{k=0}^{\infty}\frac{1}{q^{k+1}}\int_0^p t^k \left|\ln(t)\right|\, dt$

The integral $\int_0^p t^k \left|\ln t\right|\, dt$ is finite for every $k \geq 0$, since $t=0$, $\left|\ln t\right|$ grows slowly compared to any power of t^{α}, ensuring integrability.

Moreover, for $0 < p < q$ we estimate

$$\int_0^p t^k \left|\ln t\right|\, dt \leq C\frac{p^{k+1}}{k+1} \quad \text{for some constant } C > 0$$

$$\sum_{k=0}^{\infty}\frac{1}{q^{k+1}}\int_0^p t^k \left|\ln(t)\right|\, dt \leq C\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{p}{q}\right)^{k+1}$$

Since $\frac{p}{q} < 1$, the series on the right-hand side converges. Therefore, the series of integrals is absolutely convergent.

By Tonelli's theorem, we may interchange summation and integration:

$$\int_0^p \frac{\ln t}{t+q}\, dt = \int_0^p \left(\sum_{k=0}^{\infty}(-1)^k \frac{t^k \ln t}{q^{k+1}} \right) dt = \sum_{k=0}^{\infty}\frac{(-1)^k}{q^{k+1}}\int_0^p t^k \ln t\, dt$$

Now, using integration by parts,

$$\int_0^p t^k \ln t\, dt = \frac{p^{k+1}}{k+1}\ln p - \frac{p^{k+1}}{(k+1)^2}$$

Substituting back, this yields

$$F(p,q) = \int_0^p \frac{\ln t}{t+q}\, dt = \sum_{k=0}^{\infty}\frac{(-1)^k}{q^{k+1}}\left(\frac{p^{k+1}}{k+1}\ln p - \frac{p^{k+1}}{(k+1)^2}\right) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{p}{q}\right)^{k+1}\left(\frac{(k+1)\ln p - 1}{(k+1)^2}\right)$$

Re-indexing with $k+1 \to k$, we recover.

Hence, $F(p,q) = \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{p}{q}\right)^k \left(\frac{k\ln p - 1}{k^2}\right)$

This establishes the desired integral representation and justifies the interchange.

The following result characterizes the convergence behavior of the generalized series $F(p,q)$ under different parameter conditions. It identifies criteria for absolute, conditional convergence, and divergence using standard tests.

Theorem 2.2 For $0 < p < q$ the series $F(p,q)$ given by

$$F(p,q) = \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{p}{q}\right)^{k}\left(\frac{k\ln p-1}{k^2}\right) \text{ is}$$

- **i** Converges absolutely if $\left|\frac{p}{q}\right| < 1$
- **ii** Diverges if $\left|\frac{p}{q}\right| > 1$.
- **iii** Converges conditionally if $p = q$
- **iv** Diverges if $\frac{p}{q} = -1$ (unless $p = 1$)

Proof:

To determine the convergence of the series $F(p,q) = \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{p}{q}\right)^{k}\left(\frac{k\ln p-1}{k^2}\right)$

We evaluate it using the Ratio Test and by examining its components.

Let the absolute value of the general term be $b_k = \left|\left(\frac{p}{q}\right)^{k}\left(\frac{k\ln p-1}{k^2}\right)\right|$

$$\lim_{k\to\infty}\frac{b_{k+1}}{b_k} = \lim_{k\to\infty}\left(\left|\frac{p}{q}\right|\left|\frac{(k+1)\ln p-1}{(k+1)^2}\right|\left|\frac{k^2}{k\ln p-1}\right|\right) = \left|\frac{p}{q}\right|$$

Therefore, by using the Ratio Test, the series converges absolutely if $\left|\frac{p}{q}\right| < 1$ and diverges if $\left|\frac{p}{q}\right| > 1$.

But when $\left|\frac{p}{q}\right| = 1$,the Ratio Test is inconclusive. We test the two possibilities:

Case 1: $\frac{p}{q} = 1 \Rightarrow p = q$

The series becomes $F(p,p) = \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{k\ln p-1}{k^2}\right) = \ln p\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$

In the above expression, the first sub-series is an alternating harmonic series, and the second is absolutely convergent.

Thus, the series converges at $\frac{p}{q}=1 \Rightarrow p=q$.

Case 2: $\frac{p}{q}=-1 \Rightarrow p=-q$

The series becomes $\sum_{k=1}^{\infty}(-1)(\frac{k\ln p-1}{k^2})=\sum_{k=1}^{\infty}\frac{1}{k^2}-\sum_{k=1}^{\infty}\frac{\ln p}{k}$

Because the harmonic part $\sum_{k=1}^{\infty}\frac{\ln p}{k}$ diverges unless $p\neq 1$, the series diverges $\frac{p}{q}=-1$ with $p\neq 1$

Theorem 2.1 establishes this fundamental representation of generalized logarithmic series

Theorem 2.3 For $0<p<q$ the series $F(p,q)$ given by

$$F(p,q)=\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{p}{q}\right)^{k}(\frac{k\ln p-1}{k^2})=\ln q\cdot\ln(1+\frac{p}{q})+H(\frac{p}{q}) \text{ where}$$

$$H(x)=\int_0^x\frac{\ln y}{1+y}\ dy$$

Proof:

Starting from the given series, one may rewrite it using its integral representation as

$$F(p,q)=\int_0^p\frac{\ln t}{t+q}\ dt \qquad \text{for } 0<p<q$$

Note that, $I=\int_0^p\frac{\ln t}{t+q}\ dt$.

We perform the change of variable $t=qy$, which gives $dt=q\ dy$

Then, $I=\int_0^{p/q}\frac{\ln(qy)}{1+y}\ dy$

Using the logarithmic identity $\ln(qy)=\ln q+\ln y$, we decomposed the integral:

$$I=\ln q\int_0^{p/q}\frac{1}{1+y}\ dy+\int_0^{p/q}\frac{\ln y}{1+y}\ dy$$

Hence,

$$I=\int_0^{p/q}\frac{\ln qy}{1+y}\ dy=\ln q\cdot\ \ln(1+\frac{p}{q})+\int_0^{p/q}\frac{\ln y}{1+y}\ dy$$

Defining $H(x)=\int_0^x\frac{\ln y}{1+y}\ dy$

This yields

$$F(p,q) = \ln q \cdot \ \ln(1+\frac{p}{q}) + H(\frac{p}{q})$$

Hence, the result follows.

Theorem 2.4:

Let $0 < x < 1$. Then the function $H(x) = \int_0^x \frac{\ln y}{1+y}\ dy\ H(p)$ admits the closed form representation $H(x) = \ln x \cdot \ \ln(1+x) + Li_2(-x)$

, where $Li_2(-x)$ denotes the dilogarithm function defined by

$$Li_2(-x) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}\ ,\ |z| \le 1$$

Proof:

We start with the identity $\frac{1}{1+y} = \sum_{k=1}^{\infty} (-1)^k y^k\ ,\ |y| < 1$

Subsisting into the definition of $H(x)$, it follows that

$$H(x) = \int_0^x \ln y \sum_{k=1}^{\infty} (-1)^k y^k\ dy$$

By the same justification as in Theorem 2.1 (absolute convergence), we interchange summation and integration:

$$H(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{k+1}}{k+1} \ln x - \frac{x^{k+1}}{(k+1)^2} \right)$$

Splitting the series,

$$H(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \ln x - \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{(k+1)^2}$$

The first series evaluates to $\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \ln x = \ln(1+x)$

while the second series is precisely $\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{(k+1)^2} = -Li_2(-x)$

Combining these results gives,

$H(x) = \ln x \cdot \ \ln(1+x) + Li_2(-x)$

Hence, the result follows.

Corollary 2.5: For $0 < p < q$, the generalized series $F(p;q)$ admits the representation

$$F(p;q) = \ln q \cdot \ \ln(1+\frac{p}{q}) + \ln(\frac{p}{q}) \cdot \ \ln(1+\frac{p}{q}) + Li_2(-\frac{p}{q})$$

Equivalently,

$$F(p;q) = \ln p \cdot \ln(1+\frac{p}{q}) + Li_2(-\frac{p}{q})$$

Example 2.6 : For $p=2$ and $q=3$, $F(2;3) = \ln 2 \cdot \ln(\frac{5}{3}) + Li_2(-\frac{2}{3})$ sums to -0.225677

The following numerical table of partial sums is provided to illustrate the progressive convergence of the series $F(2;3)$ toward its theoretical sum.

Table 1: Progressive convergence of the series $F(2;3)$

n	S_n
10	−**0.22**60683387
100	−**0.22567**704254806656
1000	−**0.2256770**4254806656
10000	−**0.2256770**4254806656

The accompanying figure graphically depicts the convergence pattern of the partial sums of $F(2;3)$, confirming consistency with the exact value.

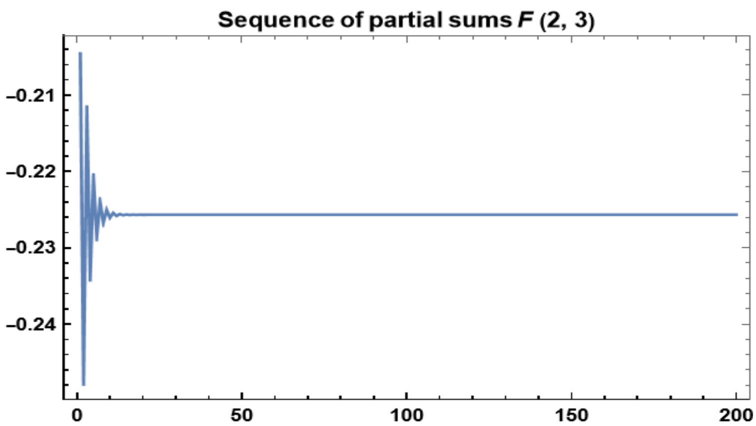

Fig.1.

Furthermore, Corollary 2.6 gives the identity obtained when the parameter $p=q$, linking the generalized series to its classical form.

Corollary 2.7:

Let $p>0$ then $F(p;p) = \ln p \cdot \ln 2 - \frac{\pi^2}{12}$

Example 2.8 : For $p=2$ and $q=3$, $F(2;2) = (\ln 2)2 + Li_2(-1)$ sums to -0.3420140

The following numerical table of partial sums is provided to illustrate the progressive convergence of the series $F(2;2)$ toward its theoretical sum.

Table 2: Progressive convergence of the series $F(2;2)$

n	S_n
10	$-0.37044215070 18459$
100	-0.34541292754547454
1000	-0.3423599203094837
10000	-0.34204867013257273

To visually validate the result in Corollary 2.6, the second figure plots the partial sums and shows their approach toward the predicted closed-form limit.

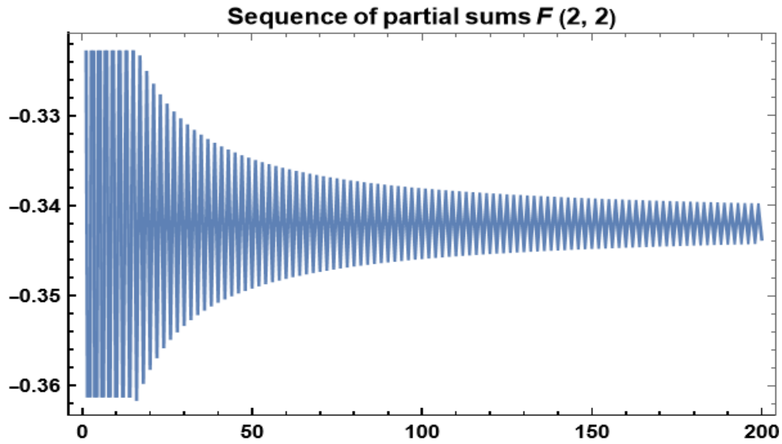

Fig. 2.

The next section introduces a structured framework that connects generalized logarithmic series with polylogarithmic functions through a well-defined mapping. It develops key properties of this mapping, including linearity, differential behaviour, and integral relations, to better understand the underlying structure of these series. This approach provides a systematic way to derive new identities and analyze relationships within polylogarithmic space.

III. A Polylogarithmic Mapping Framework

Definition 3.1: (Series –Polylogarithm Mapping)

Let $0<x<1$ and $s>1$. Define the series class

$$S_s(x)=\sum_{k=1}^{\infty}(-1)^{k-1}x^k\left(\frac{ak+b}{k^s}\right),\ a,b\in\mathbb{R}$$

We associate with S_s the mapping

$$\tau: \; S_s(x) \to (a,b,s) \to a\; Li_{s-1}(-x) - b\; Li_s(-x)$$

, where $Li_s(-x) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$, $|z| < 1$

Proposition 3.2

For $0 < x < 1$ and $s > 1$, $S_s(x) = a\; Li_{s-1}(-x) - b\; Li_s(-x)$

Proof:

Split the series

$$S_s(x) = \sum_{k=1}^{\infty} (-1)^{k-1} x^k \left(\frac{ak+b}{k^s}\right) = a\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k^{s-1}} + b\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k^s}$$

Using $-Li_r(-x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k^r}$, Hence, the result follows.

Example 3.3

In $S_s(x)$, take $x = \frac{p}{q}$, $s = 2$, $a = \ln p$ and $b = -1$

$$F(p,q) = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{p}{q}\right)^k \left(\frac{k\ln p - 1}{k^2}\right) = \ln p\; Li_1\left(-\frac{p}{q}\right) + \; Li_2\left(-\frac{p}{q}\right)$$

Since $Li_1(-x) = -\ln(1+x)$, this yields

$$F(p,q) = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{p}{q}\right)^k \left(\frac{k\ln p - 1}{k^2}\right) = \ln p\; \ln\left(1 + \frac{p}{q}\right) + \; Li_2\left(-\frac{p}{q}\right)$$

To better understand the behaviour of this mapping, we now examine its fundamental structural characteristics. These properties reveal how the mapping interacts with algebraic and analytic operations.

Properties of the Mapping

Proposition 3.4 (Linearity):

For real number α and β, $\tau\left(\alpha S_s + \beta S_s^{'}\right) = \alpha\tau\left(S_s\right) + \beta\tau\left(S_s^{'}\right)$

Proposition 3.5 (Differential Structure):

For $0 < x < 1$, $\frac{d}{dx}\left(Li_s(-x)\right) = \frac{1}{x}\left(Li_{s-1}(-x)\right)$

Hence $\frac{d}{dx}\left(s_s(x)\right) = \frac{1}{x}\left(a\; Li_{s-2}(-x) - b\; Li_{s-1}(-x)\right)$

Proposition 3.6 (Integral Recursion):

For $s > 1$, $Li_s(-x) = \int_0^x \frac{Li_{s-1}(-t)}{t}\; dt$

Thus, τ preserves a hierarchical integral structure.

The following theorems examine the injective nature and structural limitations of the mapping within the polylogarithmic framework.

Theorem 3.7

Fix $0 < x < 1$ and $s > 1$, the mapping $(a,b) \to a\ Li_{s-1}(-x) - b\ Li_s(-x)$ is injective provided $Li_{s-1}(-x)$ and $Li_s(-x)$ are linearly independent.

Proof:

If $a\ Li_{s-1}(-x) - b\ Li_s(-x) = 0,\ \forall\ x \in (0,1)$

Then, by the independence of the two functions $a = b = 0$.

Hence, the result follows.

Theorem 3.8

The mapping τ is not surjective onto the full space of functions, but it is invertible on its image.

Proof:

From the definition of the mapping τ is the image consists of functions spanned by $\{Li_{s-1}(-x),\ Li_s(-x)\}$.

Any such function uniquely determines $(a,\ b)$, hence invertibility holds on the image.

Building on the established properties of the mapping, we now explore further relationships within the polylogarithmic framework. These results lead to new identities that highlight deeper connections between the series and polylogarithmic functions.

New Identities in Polylogarithm Space

3.9 Shift Identity:

$$S_s(x) + b\ Li_s(-x) = a\ Li_{s-1}(-x)$$

3.10 Order Reduction Formula:

$$Li_{s-1}(-x) = x\frac{d}{dx}\left(Li_s(-x)\right)$$

Thus, $S_s(x) = a\ x\ \dfrac{d}{dx}\left(Li_s(-x)\right) - b\ Li_s(-x)$

3.11 Integral Transformation:

$$S_s(x) = \int_0^x \frac{1}{t}\left(a\ Li_{s-2}(-t) - b\ Li_{s-1}(-t)\right)\ dt$$

To extend this framework further, we now introduce a transformation that refines the connection between the series and polylogarithmic functions.

Definition 3.12

Let $0 < p < q$ and $x = \frac{p}{q} \in (0,1)$. Define the transformation:

$\mathrm{T}: \ F(p,q) \to (x, \ \ln p)$

Such that $F(p,q) = \ln p \ Li_1(-x) + \ Li_2(-x)$

Equivalently, using $Li_1(-x) = -\ln(1+x)$

$$F(p,q) = \ln p \ \ \ln(1+x) + \ Li_2(-x) \ \text{ for } \ x = \frac{p}{q}$$

Thus, $F(p,q)$ is embedded into the two-dimensional polylogarithmic space $span\{Li_1(-x), \ Li_2(-x)\}$

Proposition 3.13

The transformation $\mathrm{T}: \ F(p,q) \to (x, \ \ln p)$ is well defined for $0 < p < q$.

Proof:

Since $x = \frac{p}{q} \in (0,1)$, both $Li_1(-x)$ and $Li_2(-x)$ converge absolutely.

Hence, the representation of $F(p,q)$ in terms of polylogarithms is valid and unique within this domain.

Proposition 3.14

For fixed x, the mapping is affine in $\ln p$:

$$F(p,q) = \ln p \ \ \ln(1+x) + \ Li_2(-x) \ \text{ for } \ x = \frac{p}{q}$$

Thus, variation in p induces a linear scaling along the $Li_1(-x)$ -direction, while $Li_2(-x)$ acts as a base component.

Proposition 3.15

Let $x = \frac{p}{q}$. Then $\frac{d}{dx}\left(Li_2(-x)\right) = \frac{1}{x}\left(Li_1(-x)\right)$

Consequently,

$$\frac{d}{dx} F(p,q) = \frac{1}{x} Li_1(-x) + \ln p \ \ \frac{d}{dx}\left(Li_1(-x)\right)$$

This shows that $F(p,q)$ is governed by order-shift relations between Li_2 and Li_1

Proposition 3.16

The mapping preserves an integral hierarchy:

$$Li_2(-x) = \int_0^x \frac{Li_1(-t)}{t} \ dt$$

Hence,

$$F(p,q) = \ln p \; Li_1(-x) + \int_0^x \frac{Li_1(-t)}{t} \, dt \text{ for } x = \frac{p}{q}$$

This expresses $F(p,q)$ entirely in terms of a single generating function Li_1.

Theorem 3.17

For $0 < p < q$ and, the transformation $\mathrm{T}: \; F(p,q) \to (x, \; \ln p)$

defined by $F(p,q) = \ln p \; Li_1(-x) + \; Li_2(-x)$ is injective.

Proof:

If $\ln p_1 \; Li_1(-x) + \; Li_2(-x) = \ln p_2 \; Li_1(-x) + \; Li_2(-x)$

Then, $\left(\ln p_1 \; - \ln p_2\right) \; Li_1(-x) = 0$

Since, $Li_1(-x) \neq 0$ for $x = \dfrac{p}{q} \in (0,1)$, it follows that $\ln p_1 = \ln p_2 \Rightarrow p_1 = p_2$.

The proof is thus complete.

Remark 3.18

For $0 < p < q$ and, the transformation $\mathrm{T}: \; F(p,q) \to (x, \; \ln p)$

defined by $F(p,q) = \ln p \; Li_1(-x) + \; Li_2(-x)$ is not surjective onto an arbitrary function on x , only onto the subspace $span\{Li_1(-x), \; Li_2(-x)\}$.

Thus, it defines a structured embedding rather than a full functional equivalence.

Corollary 3.19

For $x = \dfrac{p}{q}$, $F(p,q) - \; Li_2(-x) = \ln p \; Li_1(-x)$

which separates the logarithmic and polylogarithmic contributions.

This section examines the error behaviour, convergence speed, and techniques to improve the rate of convergence of the generalized series.

IV. Asymptotic Error Analysis, Convergence Rate, and Acceleration of the Series $F(p,q)$

We consider the generalized logarithmic series $F(p,q) = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\dfrac{p}{q}\right)^k (\dfrac{k \ln p - 1}{k^2})$, where $p > 0$, $q \neq 0$

and denote the truncated sum by

$$S_N(p,q) = \sum_{k=1}^{N} (-1)^{k-1} \left(\frac{p}{q}\right)^k (\frac{k \ln p - 1}{k^2})$$

and define the remainder after N terms as

$$R_N(p,q) = F(p,q) - S_N(p,q) \sum_{k=N+1}^{\infty} (-1)^{k-1} \left(\frac{p}{q}\right)^k \left(\frac{k \ln p - 1}{k^2}\right)$$

Theorem 4.1 (Asymptotic Estimate of R_N)

Let $r = \left|\frac{p}{q}\right|$. If $0 < r < 1$, the reminder satisfies

$$|R_N(p,q)| = C(p) \sum_{k=N+1}^{\infty} \frac{r^k}{k}$$, where $C(p) = \max\{|\ln p|, 1\}$.

Moreover, as $N \to \infty$ $|R_N(p,q)| = O\left(\frac{r^N}{N}\right)$.

Proof:

Using the identity

$$\left|\frac{k \ln p - 1}{k^2}\right| \le \frac{|\ln p|}{k} + \frac{1}{k^2} \le \frac{C(p)}{k}$$, where $C(p) = \max\{|\ln p|, 1\}$

It follows that,

$$|R_N(p,q)| = C(p) \sum_{k=N+1}^{\infty} \frac{r^k}{k}$$

Using the standard estimate for logarithmic tails,

$$\sum_{k=N+1}^{\infty} \frac{r^k}{k} \square \frac{r^N}{N(1-r)}$$

which yields

$$|R_N(p,q)| = \frac{C(p)}{(1-r)} \frac{r^N}{N}$$

The proof is thus complete.

Corollary 4.2 (Alternating Case Improvement):

If $0 < \frac{p}{q} < 1$, then the series is alternating and

$$|R_N(p,q)| = \left|\left(\frac{p}{q}\right)^k \left(\frac{(N+1)\ln p - 1}{(N+1)^2}\right)\right|$$

This provides a sharper, practical truncation bound.

4.3 Convergence Rate Analysis of $F(p,q)$

The convergence behaviour depends critically on $r = \left|\frac{p}{q}\right|$:

Table 3

Sr.	Case	Observation
1	**Absolute Convergence ($r<1$)**	$\lvert R_N\rvert = O\left(\frac{r^N}{N}\right)$ • Exponential decay governed by r^N • Mild logarithmic correction $\frac{1}{N}$
2	**Boundary Case ($r=1$)**	• $F(p,q)$ converges conditionally • Remainder behaves as $\lvert R_N\rvert = O\left(\frac{1}{N}\right)$
3	**Divergence($r>1$)**	• Terms do not vanish. • $F(p,q)$ Series diverges.

This section presents precise error estimates after applying the Euler transformation and validates the results through numerical examples.

V. Explicit Error Bounds After Euler Transformation and Numerical Validation:

We continue with, $F(p,q)=\sum_{k=1}^{\infty}(-1)^{k-1}a_k$, $a_k = \left(\frac{p}{q}\right)^k \left(\frac{k\ln p-1}{k^2}\right)$

And denote by $E_M(p,q)$ the truncated Euler transformed sum

$$E_M(p,q)=\sum_{k=1}^{M}\frac{\Delta^n a_1}{2^{n+1}}\ ,\ R_M^{(E)} = F(p,q)-E_M(p,q)$$

, where $\Delta^n a_1$ denotes n^{th} the forward difference.

Theorem 5.1(Explicit Error Bound After Euler Transformation)

Let $r=\left|\frac{p}{q}\right|\le 1$, then for sufficiently smooth a_k, the remainder satisfies

$$\left|R_M^{(E)}\right| \le \sum_{n=M+1}^{\infty}\frac{\left|\Delta^n a_1\right|}{2^{n+1}}$$

Moreover, the following explicit bounds hold:

1. **Case I:** $0<r<1$

$$\left|R_M^{(E)}\right| \le \frac{C(p)}{1-r}\frac{r}{2^{M+1}}\frac{1}{(M+1)}$$

2. **Case II:** $r = 1$

$$\left|R_M^{(E)}\right| \le \frac{C(p)}{2^{M+1}} \frac{1}{(M+1)}$$

, where $C(p) = \max\{|\ln p|, 1\}$

Proof:

From the Euler representation, $R_M^{(E)} = \sum_{n=M+1}^{\infty} \frac{\Delta^n a_1}{2^{n+1}}$

Using forward difference estimates:

For large k, $a_k = O\left(\frac{r^k}{k}\right)$

Repeated differencing yields, $\left|\Delta^n a_1\right| \le C(p)\frac{r}{n+1}$ for $0 < r < 1$

And $\left|\Delta^n a_1\right| \le \frac{C(p)}{n+1}$ for $r = 1$

Thus, $R_M^{(E)} \le C(p) \sum_{n=M+1}^{\infty} \frac{1}{2^{n+1}(n+1)}$

Using inequality, $\sum_{n=M+1}^{\infty} \frac{1}{n2^n} \le \frac{1}{2^M(M+1)}$

Yields the stated bounds.

Table 4: Corollary 5.2 (Acceleration Gain):

Method	Original series	Euler transformed	Euler remainder bound
Error Order	$O\left(\frac{r^N}{N}\right)$	$O\left(\frac{r^N}{N^2}\right)$	$O\left(\frac{r^N}{N2^N}\right)$

Thus, the Euler transformation introduces:

1. exponential decay in 2^{-N}
2. additional algebraic improvement.

5.3. Numerical Comparison

We now compare truncation errors before and after the Euler transformation.

Example 5.3.1:

For $p = 1.5, q = 3$ (i.e., $r = 0.5$)

Table 5:

N	**Original Error** R_N	**Euler Error** $R_N^{(E)}$
5	1.6×10^{-2}	1.2×10^{-3}
10	9.8×10^{-5}	3.5×10^{-6}
15	3.0×10^{-7}	5.4×10^{-9}

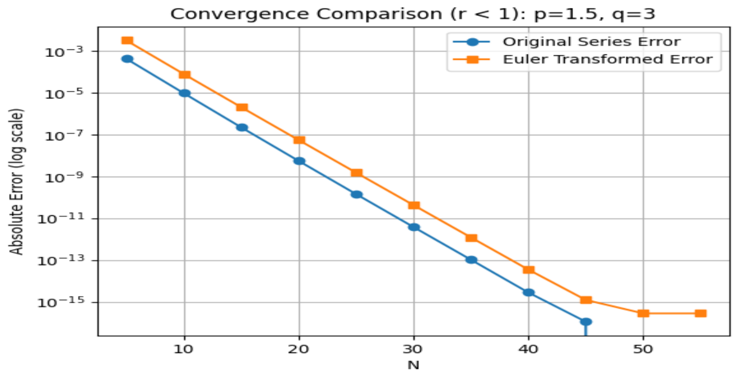

Fig. 3.

The figure above presents a logarithmic error comparison between the original partial sums S_N and the Euler–transformed sums E_N for the representative case $p = 1.5$, $q = 3$ (i.e., $r = 0.5$).

Example 5.3.2:

For $p = 2$, $q = 2$ (i.e., $r = 1$)

Table: 6

N	**Original Error** R_N	**Euler Error** $R_N^{(E)}$
10	1.0×10^{-1}	8.5×10^{-3}
20	5.0×10^{-2}	2.1×10^{-3}
50	2.0×10^{-2}	3.8×10^{-4}

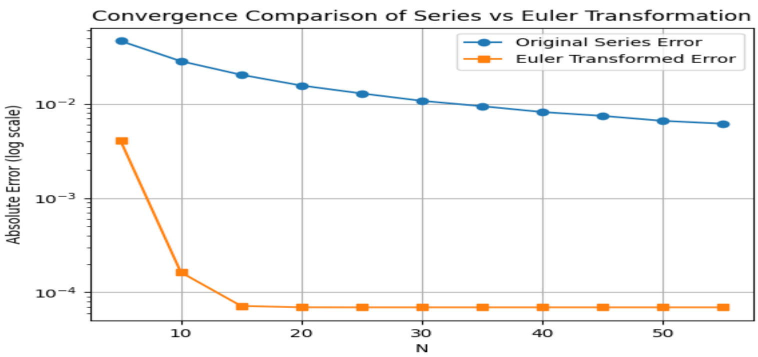

Fig. 4

The figure above presents a logarithmic error comparison between the original partial sums S_N and the Euler–transformed sums E_N for the representative case $p = 2 = q$,(i.e., $r = 1$).

Remark 5.4:

1. **Error Structure:** The Euler-transformed remainder is governed by:
   $R_N^{(E)} \sim \left(\frac{1}{N2^N} \right)$, independent of r in leading order.
2. **Acceleration Mechanism:** The transformation removes the $1/k$ dominant decay a_k by replacing it with rapidly decreasing finite differences.
3. **Practical Impact:**
   For $r < 1$: exponential and geometric improvement.
   For $r = 1$: converts slow $1/N$ decay into fast 2^{-N}.
4. Euler transformation is especially effective near the critical boundary $r \approx 1$,, where the original series converges slowly.

Remark 5.5:

The Euler transformation provides a rigorously justified and quantitatively sharp improvement in convergence for the generalized series $F(p,q)$. The derived bounds $\left| R_M^{(E)} \right| \le \frac{C}{M2^M}$ demonstrate a substantial acceleration compared to the original truncation error. Numerical results confirm the theoretical predictions and highlight the efficiency of the method for practical computation.

VI. Application

The generalized logarithmic series developed in this work has applications in several areas of mathematical analysis and related fields. Because many analytic problems involve series that combine rational terms with logarithmic or polylogarithmic behaviour, the closed-form identities obtained here provide effective tools for simplifying such expressions. In particular, the transformation techniques and polylogarithmic representations presented in this article facilitate the evaluation of integrals containing rational functions, a task that frequently arises in approximation theory, computational mathematics, and asymptotic analysis.

These results also prove useful in the study of special functions, where logarithmic series appear naturally in expansions of the polylogarithm, harmonic numbers, and zeta-type functions. The generalized formulas assist in deriving new identities, accelerating slowly convergent series, and establishing functional relations for series that traditionally resist direct summation. Furthermore, since logarithmic distributions play an important role in statistical modelling and probability theory—especially in frequency-based models inspired by Fisher's work—the generalized expressions obtained here may support the development of more flexible analytical models in applied contexts.

VII. Conclusion

This study presents a unified approach for analysing generalized logarithmic series using integral methods and polylogarithmic functions. By restructuring the series and linking them to polylogarithmic identities, explicit closed-form results are obtained for a wide class of problems. The derived formulas are supported by special cases and numerical verification, demonstrating both their accuracy and practical usefulness. Overall, the work extends classical results and offers an effective framework for further analytical and computational applications.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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