



NEW SEMICIRCULAR DISTRIBUTION WITH APPLICATION

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Abstract

This article introduces a new semicircular distribution known as the stereographic semicircular new Weibull-Pareto distribution. We explore its various properties and provide explicit expressions for trigonometric moments. We delve into its essential mathematical properties and execute a simulation study to estimate its parameter values. Furthermore, we showcase the suggested distribution's modeling capacity in the context of real-world phenomena. An illustrative example is conducted using an authentic data set within the geological domain.

Keywords: Axial data, semicircular model, inverse stereographic projection, trigonometric moments, simulation.

I. Introduction

Angular or circular data is prevalent across a wide range of fields, encompassing disciplines such as biology, geology, meteorology, earth science, political science, economics, and computer science, particularly in data science. Textbooks authored by luminaries such as Fisher (1993) [VII], Jammalamadaka and Sen Gupta (2001) [XVI], and Mardia and Jupp. (2000)[XI] commonly advocate full circular models to analyze such data. However, intriguingly, certain instances exist where the adoption of a complete circular model may prove excessive in the context of modeling angular data. Notable observations related to this phenomenon have been recorded in the research of Guardiola (2004) [X], Jones (1968) [XII], Byoung et al. (2008) [V], and Phani et al. (2013) [XX]. As an illustrative example, when analyzing the behavior of sea turtles as they emerge from the ocean to identify a nesting site on dry land, employing a random variable with values distributed along a semicircle effectively encapsulates the dynamics of this data. Similarly, in scenarios like the loss of an aircraft, where departure and initial headings are known, a semicircular random variable offers an apt framework for handling angular data. Further exemplars of semicircular data are expounded upon in Ugai et al. (1977) [XXIX]. Guardiola (2004) [XIII] innovatively introduced the concept of the semicircular normal distribution via a straightforward projection technique. Byoung et al. (2008) [V] further extended this concept by formulating a family of semicircular Laplace distributions, all tailored to model semicircular data through a parsimonious projection methodology. Phani et al. (2013) [XX], (2013a) [XXI], (2020) [XXII], Ali H. A., (2018) [I], Ayesha Iftikhar et al. (2022) [II], N.A. Oleiwi et al. (2022) [XV], Salah H. A. et al. (2023) [XIX] contributed significantly to this realm by ingeniously crafting semicircular distributions via an inverse stereographic projection.

In this paper, we embark on a novel exploration by deriving a semicircular new Weibull-Pareto distribution, a distinctive endeavor achieved through the application of inverse stereographic projection onto the linear new Weibull-Pareto distribution meticulously tailored to grapple with the intricacies of axial data. To better understand the proposed model, we calculate its first two trigonometric moments.

The structure of this paper unfolds as follows: Section 2 rigorously defines and elaborates on the proposed semicircular distribution, furnishing illuminating graphical representations of density, distribution, survival, and hazard rate functions for various values of its parameters, thereby deepening insights into its practical utility. In Section 3, We explore the mathematical rigor involved in deriving the initial two trigonometric moments of the proposed model, critical elements for discerning population characteristics. Section 5 represents a pioneering extension of the proposed model to accommodate and elucidate the complexities of axial data, enhancing the breadth of its applicability and contributions to the field.

II. Derivation of the Proposed Model

A random variate X residing on the real number line is characterized as having a new Weibull-Pareto distribution (Suleman et al. (2015) [XXVII]) with

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parameters α, β (shape parameters) and λ (scale parameter), when its probability density and cumulative distribution functions are explicitly defined as follows:

$$f(x; \alpha, \beta, \lambda) = \frac{\alpha\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1} \exp\left(-\alpha\left(\frac{x}{\lambda}\right)^\beta\right), \quad (1)$$

and

$$G(x; \alpha, \beta, \lambda) = 1 - \exp\left(-\alpha\left(\frac{x}{\lambda}\right)^\beta\right), \quad (2)$$

where $0 < x < \infty, \alpha, \beta, \lambda > 0$

Stereographic Semicircular New Weibull-Pareto Distribution

Expanding upon the methodology outlined in Phani et al. (2013)[XX], as previously mentioned), we have introduced the semicircular version of the new Weibull-Pareto distribution.

Definition: Let θ be a random variate that follows the stereographic semicircular new Weibull-Pareto distribution with shape parameters α, β and scale parameter λ , written as $\theta \square \text{SSCNW-P}(\alpha, \beta, \lambda)$, the probability density function of θ is given by

$$g(\theta; \alpha, \beta, \lambda) = \frac{\alpha\beta}{\lambda(1+\cos(\theta))} \left(\frac{\tan\left(\frac{\theta}{2}\right)}{\lambda}\right)^{\beta-1} \exp\left(-\alpha\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\lambda}\right)^\beta\right), \quad (3)$$

and the cumulative distribution function of θ is

$$G(\theta; \alpha, \beta, \lambda) = 1 - \exp\left(-\alpha\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\lambda}\right)^\beta\right), \quad (4)$$

where $0 \leq \theta < \pi, \alpha, \beta, \lambda > 0$

Survival and hazard functions: The survival function and hazard function of the SSCNW-P($\theta; \alpha, \beta, \lambda$) distribution are given, respectively, by

$$S(\theta; \alpha, \beta, \lambda) = \exp\left(-\alpha\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\lambda}\right)^\beta\right) \quad (5)$$

and

$$h(\theta; \alpha, \beta, \lambda) = \frac{\alpha\beta}{\lambda(1+\cos(\theta))} \left(\frac{\tan\left(\frac{\theta}{2}\right)}{\lambda}\right)^{\beta-1} \quad (6)$$

Quantile function: The quantile function of the SSCNW-P($\theta; \alpha, \beta, \lambda$) distribution is obtained by solving $G(\theta; \alpha, \beta, \lambda) - \tau = 0$ for $0 < \tau < 1$ and is given by

$$Q(\tau) = 2 \tan^{-1} \left(\lambda \left(\frac{1}{\alpha} \log \left(\frac{1}{1-\tau} \right) \right)^{1/\beta} \right) \quad (7)$$

Median: The second quantile (i.e., $\tau = 1/2$) yields the median, which is given as

$$\text{Median} = Q(0.5) = 2 \tan^{-1} \left(\lambda \left(\frac{1}{\alpha} \log(2) \right)^{1/\beta} \right) \quad (8)$$

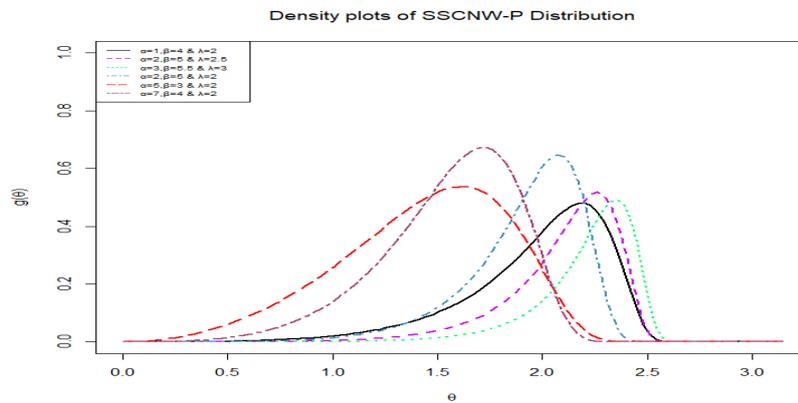


Fig. 1. Plots of pdf of the SSCNW-P for various values of parameters.

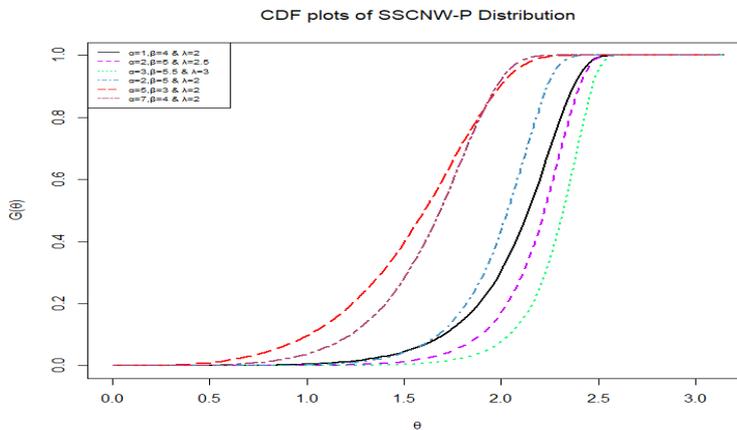


Fig. 2. Plots of cdf of the SSCNW-P for various values of parameters.

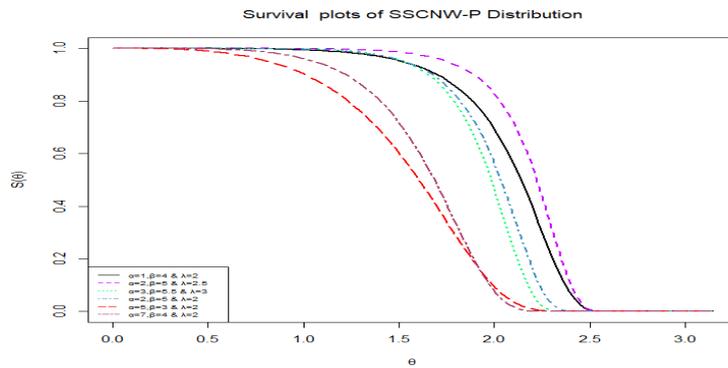


Fig. 3. Plots of survival function of the SSCNW-P for various values of parameters.

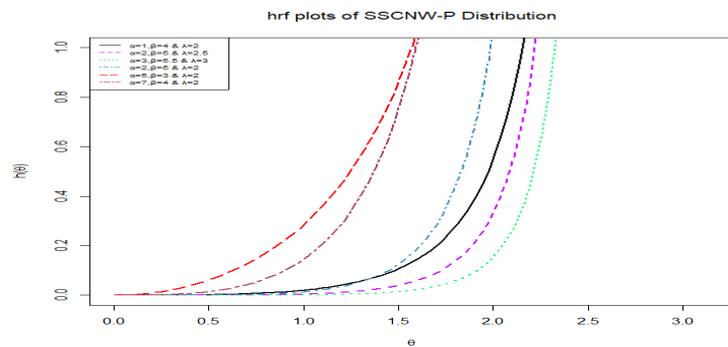


Fig. 4. Plots of hrf of the SSCNW-P for various values of parameters.

III. Characteristic Function

The characteristic function of a semicircular random variable θ can be given

$$\text{by } \varphi_p = E(e^{ip\theta}) = \int_0^\pi e^{ip\theta} g(\theta) d\theta, \quad p \in Z.$$

$$\varphi_p = \frac{\alpha\beta}{2\lambda^\beta} \int_0^\pi e^{ip\theta} \frac{\left(\tan\left(\frac{\theta}{2}\right)\right)^{\beta-1}}{(1+\cos(\theta))} \times e^{-\alpha\left(\frac{\tan(\theta/2)}{\lambda}\right)^\beta} d\theta$$

As this integral lacks a closed-form solution, we proceed with the computation of the first two trigonometric moments, which are crucial and adequate for the examination of population characteristics.

Trigonometric moments: The trigonometric moments of the distribution are given by $\varphi_p = \alpha_p + i\beta_p, p \in Z.$

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$$\alpha_1 = E(\cos(\theta)) = \int_0^\pi \cos(\theta) g(\theta) d\theta \quad \alpha_1 = 1 - 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{\alpha}{\lambda^\beta}\right)^{-\left(\frac{2n+2}{\beta}\right)} \Gamma\left(\frac{2n+\beta+2}{\beta}\right) \quad (9)$$

$$\beta_1 = E(\sin(\theta)) = \int_0^\pi \sin(\theta) g(\theta) d\theta$$

$$\beta_1 = 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{\alpha}{\lambda^\beta}\right)^{-\left(\frac{2n+1}{\beta}\right)} \Gamma\left(\frac{2n+\beta+1}{\beta}\right) \quad (10)$$

$$\alpha_2 = E(\cos(2\theta)) = \int_0^\pi \cos(2\theta) g(\theta) d\theta$$

$$\alpha_2 = 1 + 8 \sum_{n=1}^{\infty} (-1)^n \left((n+1) \left(\frac{\alpha}{\lambda^\beta}\right)^{-\left(\frac{2n+4}{\beta}\right)} \Gamma\left(\frac{2n+\beta+4}{\beta}\right) \right)$$

$$- 8 \sum_{n=1}^{\infty} (-1)^n \left(\left(\frac{\alpha}{\lambda^\beta}\right)^{-\left(\frac{2n+2}{\beta}\right)} \Gamma\left(\frac{2n+\beta+2}{\beta}\right) \right) \quad (11)$$

$$\beta_2 = E(\sin(2\theta)) = \int_0^\pi \sin(2\theta) g(\theta) d\theta$$

$$\beta_2 = 4 \sum_{n=1}^{\infty} (-1)^n \left(\left(\frac{\alpha}{\lambda^\beta}\right)^{-\left(\frac{2n+1}{\beta}\right)} \Gamma\left(\frac{2n+\beta+1}{\beta}\right) \right)$$

$$- 8 \sum_{n=1}^{\infty} (-1)^n (n+1) \left(\left(\frac{\alpha}{\lambda^\beta}\right)^{-\left(\frac{2n+2+3}{\beta}\right)} \Gamma\left(\frac{2n+\beta+3}{\beta}\right) \right) \quad (12)$$

These moments are utilized for computing mean direction, resultant length, circular variance, circular standard deviation, circular skewness, and circular kurtosis.

IV. Maximum Likelihood Estimation

This part discusses statistical inference for the SSCNW-P($\theta; \alpha, \beta, \lambda$) distribution using estimation method namely the maximum likelihood (ML) method. Let $\bar{\theta} = (\theta_1, \theta_2, \theta_3, \dots, \theta_k)$ be a random sample from SSCNW-P(α, β, λ). the likelihood

function is given by $L(\bar{\theta}; \alpha, \beta, \lambda) = \prod_{i=1}^k g(\theta_i; \alpha, \beta, \lambda)$

$$L = \prod_{i=1}^n \left(\frac{\alpha\beta}{\lambda(1+\cos(\theta_i))} \right) \times \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right)^{\beta-1} \exp\left(-\alpha \left(\tan\left(\frac{\theta_i}{2}\right) \right)^\beta\right) \quad (13)$$

The log-likelihood function is

$$\ell = \log(L) = n \log(\alpha\beta) - n \log(\lambda) - \sum_{i=1}^n \log(1 + \cos(\theta_i))$$

$$+(\beta-1)\sum_{i=1}^n \log\left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right) - \alpha \sum_{i=1}^n \log\left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right)^\beta \quad (14)$$

Differentiating equation (14) w.r.to α, β , and λ respectively and equating to zero, we get the following normal equations.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right)^\beta = 0 \quad (15)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n \left(1 - \alpha \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right)^\beta\right) \log\left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right) = 0 \quad (16)$$

$$\frac{\partial \ell}{\partial \lambda} = -\frac{n}{\lambda} - \frac{\alpha\beta}{\lambda(1+\beta)} \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right)^\beta + \frac{n(1-\beta)}{\lambda} = 0 \quad (17)$$

The maximum likelihood estimates (mle) of the parameters are calculated by solving the normal equations (15), (16), and (17) simultaneously. Since these normal equations are non-linear, it is difficult to find closed form expression, so they can be solved by applying a suitable numerical technique.

We now derive the observed information matrix to conduct asymptotic inference for the parameter vector $\mathcal{G} = (\alpha, \beta, \lambda)^T$. The 3×3 observed information matrix $J(\mathcal{G})$ is

$$J(\mathcal{G}) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} \\ \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} & \frac{\partial^2 \ell}{\partial \beta \partial \lambda} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix}, \text{ whose members are as follows}$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2}, \quad \frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} - \alpha \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right)^\beta \left(\log\left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda}\right)\right)^2,$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n(1-\beta)}{\lambda^2} - \frac{\alpha\beta(1+\beta)}{\lambda} \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right)^\beta, \quad \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = \frac{\beta}{\lambda} \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right)^\beta,$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \frac{\partial^2 \ell}{\partial \beta \partial \alpha} = -\sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right)^\beta \log \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right),$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \beta} = \frac{\partial^2 \ell}{\partial \beta \partial \lambda} = -\frac{n}{\lambda} + \frac{\alpha}{\lambda} \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right)^\beta + \frac{\alpha\beta}{\lambda} \frac{\alpha}{\lambda} \sum_{i=1}^n \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right)^\beta \log \left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{\lambda} \right).$$

Under some regular conditions for unknown parameters in the parameter space, we have $\sqrt{n}(\hat{\mathcal{G}} - \mathcal{G}) \xrightarrow{d} N(0, I^{-1}(\mathcal{G}))$, where \xrightarrow{d} denotes convergence in distribution and $I(\mathcal{G})$ is the expected Fisher information matrix of \mathcal{G} .

V. Simulation Study

In this section, a Monte Carlo simulation study is performed to examine the performance, accuracy, and consistency of the maximum likelihood estimates of the stereographic semicircular new Weibull-Pareto distribution. In each simulation, 1000 trials of sizes $n=50, 70, 100, 300$ and 500 were generated for various values of the parameters α, β , and λ using the quantile function(7) of SSCNW-P($\theta; \alpha, \beta, \lambda$). For each sample, the MLE's are obtained and these values are utilized for computing the following quantities with the help of R software.

- Absolute average bias = $\frac{1}{1000} \sum_{i=1}^{1000} |\theta - \hat{\theta}|$
- Mean square error(mse) = $\frac{1}{1000} \sum_{i=1}^{1000} (\theta - \hat{\theta})^2$
- Mean relative error(mre) = $\frac{1}{1000} \sum_{i=1}^{1000} \frac{|\theta - \hat{\theta}|}{\theta}$

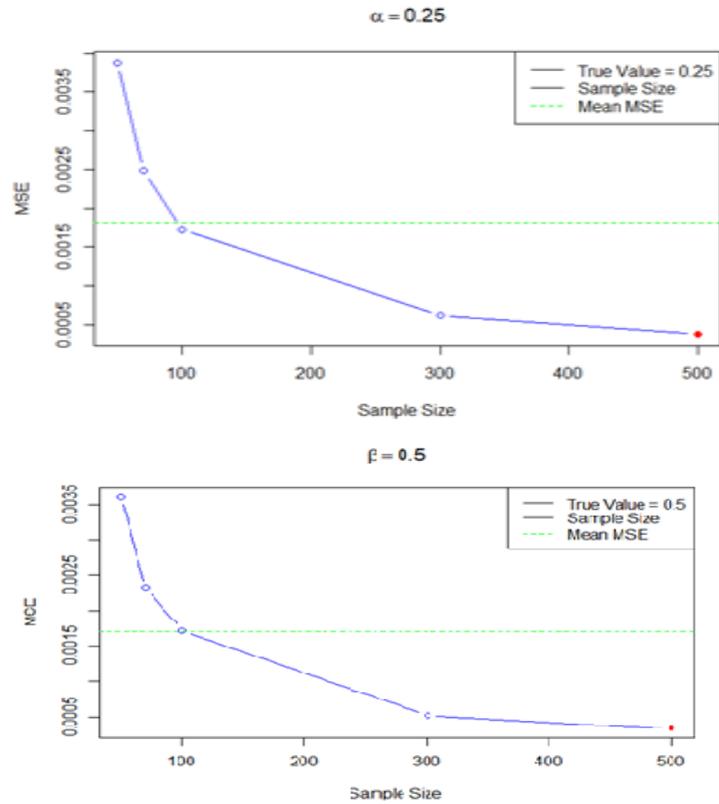


Fig. 5. Sample size versus MSE. Table 1. Average mle, absolute bias, mse, and mre of the simulated estimates for α , β , and λ .

n	$\alpha = 0.25$				$\beta = 0.5$				$\lambda = 0.75$			
	mle	bias	mse	mre	mle	bias	mse	mre	mle	bias	mse	mre
50	0.248	0.047	0.003	0.191	0.515	0.047	0.003	0.095	0.7532	0.011	0.000	0.014
70	06	88	59	51	00	51	79	02	5	19	36	93
100	0.250	0.040	0.002	0.163	0.509	0.039	0.002	0.078	0.7518	0.008	0.000	0.011
200	24	78	63	11	45	16	51	32	8	76	22	68
300	0.249	0.033	0.001	0.135	0.506	0.032	0.001	0.064	0.7510	0.006	0.000	0.008
400	53	77	80	08	59	05	66	09	7	50	11	67
500	0.249	0.019	0.000	0.076	0.502	0.017	0.000	0.035	0.7504	0.003	0.000	0.004
700	78	20	58	81	48	88	51	77	0	46	03	62
1000	0.250	0.015	0.000	0.060	0.501	0.013	0.000	0.027	0.7502	0.002	0.000	0.003
1500	03	01	35	06	36	97	31	95	5	72	02	63
n	$\alpha = 0.5$				$\beta = 0.75$				$\lambda = 1.25$			
	mle	bias	mse	mre	mle	mle	bias	mse	mle	bias	mse	mre
70	0.502	0.069	0.007	0.138	0.770	0.069	0.008	0.092	1.2575	0.027	0.001	0.021
100	11	18	61	36	64	45	09	60	1	40	64	92
200	0.503	0.057	0.005	0.115	0.764	0.058	0.005	0.077	1.2569	0.022	0.001	0.018
300	07	90	34	80	01	35	55	80	6	65	12	12
400	0.502	0.049	0.003	0.098	0.760	0.048	0.003	0.064	1.2555	0.018	0.000	0.014
500	25	15	82	29	65	52	82	70	4	30	75	64
700	0.500	0.028	0.001	0.056	0.753	0.027	0.001	0.036	1.2513	0.009	0.000	0.007
1000	93	16	26	32	11	44	19	59	0	07	16	26
1500	0.500	0.021	0.000	0.043	0.751	0.020	0.000	0.027	1.2506	0.006	0.000	0.005
2000	34	52	73	04	62	82	69	76	1	75	08	40

n	$\alpha = 1.5$				$\beta = 2.75$				$\lambda = 3.25$			
	mle	bias	mse	mre	mle	bias	mse	mre	mle	bias	mse	mre
50	1.517	0.062	0.006	0.041	2.828	0.257	0.111	0.093	3.2459	0.089	0.013	0.027
70	42	31	65	54	75	84	07	76	2	73	29	61
10	1.513	0.050	0.004	0.033	2.803	0.215	0.076	0.078	3.2493	0.074	0.009	0.022
0	88	94	39	96	80	13	17	23	5	63	38	96
30	1.511	0.042	0.003	0.028	2.789	0.177	0.051	0.064	3.2528	0.062	0.006	0.019
0	34	63	02	42	56	54	13	56	7	84	70	34
50	1.506	0.026	0.001	0.017	2.762	0.101	0.016	0.036	3.2530	0.035	0.002	0.010
0	61	97	22	98	77	02	13	73	4	58	19	95
	1.504	0.020	0.000	0.013	2.757	0.076	0.009	0.027	3.2518	0.026	0.001	0.008
	74	97	76	98	81	79	34	93	4	54	19	17
n	$\alpha = 5$				$\beta = 4.5$				$\lambda = 7$			
	mle	bias	mse	mre	mle	bias	mse	mre	mle	bias	mse	mre
50	5.018	0.078	0.011	0.015	4.632	0.424	0.299	0.094	6.9581	0.219	0.075	0.031
70	83	24	61	65	68	90	71	42	1	54	70	36
10	5.012	0.062	0.007	0.012	4.588	0.347	0.198	0.077	6.9715	0.182	0.052	0.026
0	05	92	42	58	37	04	11	12	3	64	09	09
30	5.008	0.050	0.004	0.010	4.566	0.288	0.135	0.064	6.9778	0.154	0.037	0.022
0	35	65	58	13	20	49	20	11	9	85	50	12
50	5.003	0.027	0.001	0.005	4.524	0.163	0.042	0.036	6.9917	0.089	0.012	0.012
0	36	79	22	56	30	92	98	43	2	40	66	77
	5.001	0.021	0.000	0.004	4.511	0.127	0.025	0.028	6.9958	0.069	0.007	0.009
	86	53	75	31	80	62	67	36	3	08	53	87
n	$\alpha = 9$				$\beta = 4.5$				$\lambda = 12$			
	mle	bias	mse	mre	mle	bias	mse	mre	mle	bias	mse	mre
50	9.018	0.151	0.037	0.016	4.629	0.425	0.302	0.094	11.906	0.476	0.357	0.039
70	11	49	70	83	19	18	05	48	58	77	30	73
10	9.015	0.126	0.025	0.014	4.594	0.353	0.205	0.078	11.922	0.400	0.249	0.033
0	52	11	38	01	11	86	12	64	12	90	76	41
30	9.011	0.104	0.017	0.011	4.570	0.287	0.135	0.063	11.942	0.333	0.175	0.027
0	34	10	69	57	66	76	19	95	42	00	06	75
50	9.001	0.058	0.005	0.006	4.519	0.163	0.043	0.036	11.983	0.193	0.059	0.016
0	88	76	61	53	21	94	24	43	91	88	42	16
	9.001	0.045	0.003	0.005	4.515	0.127	0.025	0.028	11.986	0.150	0.035	0.012
	93	18	25	02	26	28	57	28	62	95	54	58

Observing Table 1, it becomes evident that as the sample size increases, the bias, mse, and mre of the estimates for α , β and λ tend to approach zero. This indicates that the parameter estimates are becoming more accurate, precise, and consequently, consistent. In Figure 5, it is further confirmed that as the sample size increases, the mean squared error (MSE) decreases.

VI. Application

In this section, we conduct an extensive analysis of a real-world axial dataset to assess the suitability of the proposed distribution, specifically the stereographic semicircular new Weibull-Pareto distribution. The primary data set under investigation concerns measurements related to the long-axis orientation of feldspar laths within basalt. These measurements were originally sourced from Smith (1988, set 24-6-5 co.prn) and are comprehensively detailed in Fisher (1993[VII]), particularly in Appendix B5. Upon constructing a histogram, we observe a distinct right-skewed pattern within the data set, which lends itself effectively to representation by the proposed distribution. This analytical effort is consistent with

the research conducted by several scholars, including those who have explored the wrapped two-parameter Lindley distribution (Sahana Bhattacharjee, et al. (2021) [XXVIII]) and the wrapped xgamma distribution (Al-Mofleh, H. and Sen, S., et al. (2019) [IX]). Furthermore, our study extends its comparative analysis to include other models, such as the wrapped Lindley distribution (Joshi et al. (2018) [XVIII]), wrapped exponential distribution (Jammalamadaka et al. (2007) [XVII]), wrapped modified Lindley distribution (Christophe Chesneau, et al. (2022) [VI]), and stereographic semicircular exponential distribution (Phani et al. (2013) [XX]).

To evaluate goodness of fit, we utilize statistical metrics, namely log likelihood (LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and p-values. Lower AIC and BIC values indicate a superior fit for the data set. Table 2 presents the Maximum Likelihood Estimates (MLEs) and their corresponding standard errors (S.E) for the parameters of the proposed distribution and the distributions used for comparative analysis.

Simultaneously, Table 3 provides a comprehensive comparison of these models, demonstrating that the proposed distribution outperforms the others in various metrics, including LL, AIC, BIC, KS statistic, and p-values.

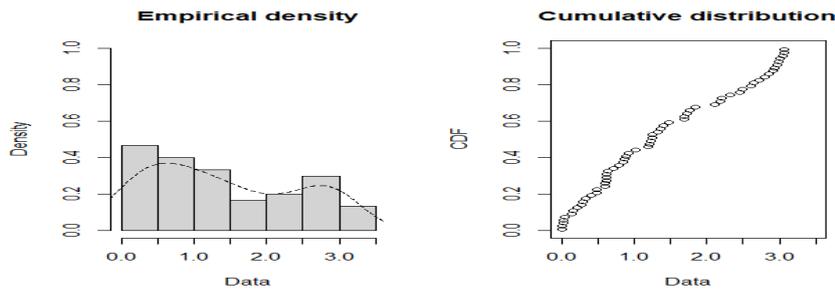


Figure 6. Histogram and cdf plots of an empirical distribution for the Fisher B5 data set.

Table 2: MLEs and their standard errors for the feldspar lathsdata set.

Distribution	$\theta_1 (S.E)$	$\theta_2 (S.E)$	$\theta_3 (S.E)$
SSCNW-P(α, β, λ)	2.311841(9.69370)	0.580375 (0.055969)	7.7869355(6.135079)
SSCEx. (λ)	0.3239258(0.041817)	-	-
WTPLD(α, λ)	1.54454(1.3448139)	1.100696(0.1777949)	-
WLD(λ)	1.030762(0.1095776)	-	-
WED(λ)	0.8182344(0.080313)	-	-
WRXG(λ)	1.320312(0.130022)	-	-
WML(λ)	0.7716361(0.088165)	-	-

Table 3: Summary of statistics.

Distribution	2L	AIC	BIC	KS(p-value)
SSCNW-P (α, β, λ)	-134.3461	140.3461	146.6291	0.12412(0.3137)
SSCEX. (λ)	-177.2558	179.2558	181.075	0.34534(0.0000)
WTPLD(α, λ)	-156.6354	160.6353	164.8240	1.8763(2.2e-16)
WLD(λ)	-156.8536	158.8536	160.9480	0.1112(0.4482)
WED(λ)	-159.4301	161.4301	163.5244	0.1165(0.3900)
WRXG(λ)	-156.0570	158.1260	160.1514	0.1042(0.5323)
WML(λ)	-158.8228	160.8228	162.9172	0.09787(0.6144)

Table 3 reveals that the stereographic semicircular new Weibull-Pareto distribution offers the best fit for the fisher data B5[VII]. It attains the highest log-likelihood value, while the statistical metrics AIC and BIC reach their minimum values, indicating its superior performance in fitting the data.

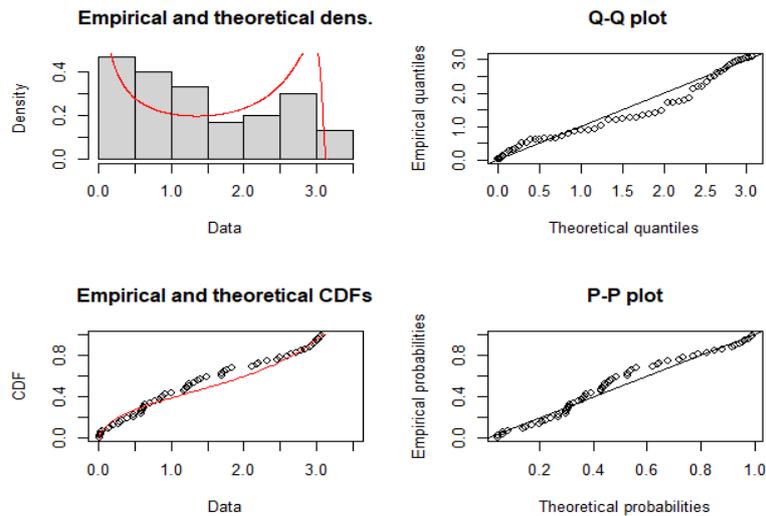


Figure 7. The empirical probability density function (PDF) is shown in the top left panel, while the empirical cumulative distribution function (CDF) is displayed in the bottom left panel. The top right panel illustrates the quantile-quantile (Q-Q) plot, and the bottom right panel presents the probability-probability (P-P) plot for the data set of long axis orientations of feldspar laths (Fisher B5[VII]).

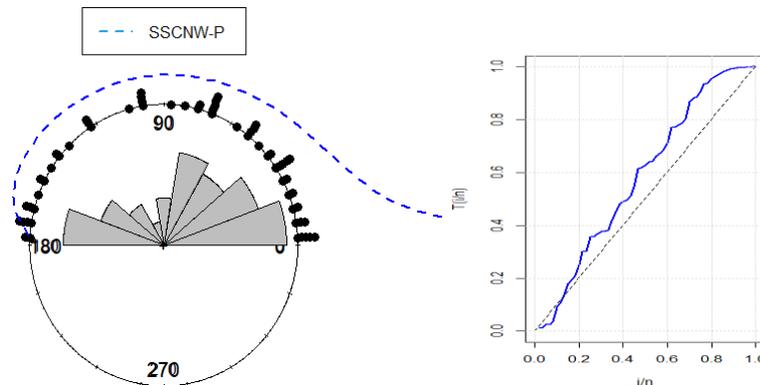


Figure 8. Visualization includes a plot for the data set of long axis orientations of feldspar laths (Fisher B5[VII]), comprising a rose diagram, data plot, and a fitted semicircular probability density function (PDF) on the left. Additionally, a TTT plot for the Fisher B5 data set is displayed.

VII. Conclusions

To sum up, our paper introduces the stereographic semicircular new Weibull-Pareto distribution. We have conducted a thorough examination of its mathematical properties, offering clear expressions for trigonometric moments, and carried out a rigorous simulation study to assess the precision of parameter estimation. Furthermore, our practical application of this distribution to authentic geological data illustrates its efficacy in modeling real-world phenomena.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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