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GENERALIZED JORDAN (σ,τ)-DERIVATIONS IN SEMIPRIME RINGS

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Abstract

Let R be a 2-torsion-free semiprime ring, $F: R \to R$ be a generalized Jordan (σ,τ)-derivation associated with Jordan (σ,τ)-derivation d, and $H: R \to R$ be a left σ-centralizer. If (i) $F(x^2) \mp H(x^2) = 0$ (ii) $F(x^2) \mp H(x^2) \in C_{\sigma,\tau}$; for all $x, y \in R$.

Keywords: Semiprime ring, Derivation, Jordan derivation, Generalized derivation, (σ,τ) -derivation, Generalized (σ,τ) -derivation, Jordan (σ,τ) -derivation, and Generalized Jordan (σ,τ)- derivation.

I. Introduction

In 1969, Herstein in [VII], [VIII] proved a result that every Jordan derivation on a 2-torsion-free prime ring is a derivation. In 1988, Bresar and Vukman in [III], [IV] proved some results on Jordan derivations on semi-prime rings and prime rings. Further, Awtar in [II] generalized this result on lie ideals. In 1991, Bresar and Vukman in [V] proved some results in Jordan (θ,φ) -derivations. In 2003, Mohammad Ashraf, Nadeem-UR-Rehman, and Shakir Ali in [I] studied on lie ideals, and Jordan generalized derivation of prime rings. In 2016, Jaya Subba Reddy et al. in [IX, X] proved some results on prime near rings with generalized Jordan derivations and centralizing with generalized (σ,τ)-derivations in semiprime rings. In 2017, Didem K. Camci and Neset Aydin in [VI] studied on multiplicative (generalized) derivation in semiprime rings. In a classical result, Bresar proved that Jordan derivation on 2-Torsion-free semiprime rings is a derivation. Similar 2-Torsion-free assumptions appear in the works of Herstein and Bresar-Vukman. Compared to Bresar's hypothesis, In this paper, Lemma 2 uses the same strength assumption as 2-

Torsion-free and semiprime. In this paper, we extended some results on generalized Jordan (σ, τ)-derivation in semiprime rings.

II. Preliminaries

Throughout this paper, R denotes an associative ring with center Z. Recall that a ring R is semiprime if $xRx = \{0\}$ implies $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. A ring R is 2-torsion-free if $2x = 0$ implies $x = 0$, for all $x \in R$.Example : $(Z_{11}, +, .)$ is a 2-torsion-free semiprime ring. The (σ, τ)-centre of R denoted by $C_{\sigma,\tau}$ and defined by $C_{\sigma,\tau} = \{c \in R: c\sigma(r) = \tau(r)c$, for all $r \in R\}$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $d: R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$. An additive mapping $d: R \to R$ is called a (σ, τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$. An additive mapping $d: R \to R$ is called a Jordan (σ, τ) -derivation if $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$. An additive mapping $F: R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized Jordan derivation if there exists a Jordan derivation $d: R \to R$ such that $F(x^2) = F(x)x + xd(x)$, for all $x \in R$. An additive mapping $F: R \to R$ is said to be a generalized (σ, τ)-derivation of R, if there exists a (σ, τ)-derivation $d: R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized Jordan (σ, τ)-derivation of R, if there exists a Jordan (σ, τ)-derivation $d: R \to R$ such that $F(x^2) = F(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$. An additive mapping $H: R \to R$ is called a left σ-centralizer if $H(xy) = H(x)\sigma(y)$, for all $x, y \in R$, where σ and τ are automorphisms of R. Throughout this paper, we shall make use of the basic commutator identities: $[x, yz] = y[x, z] + [x, y]z$; $[xy, z] = [x, z]y + x[y, z]$; $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y$.

Lemma 1: Let R be a semiprime ring. If $F: R \to R$ is a generalized Jordan (σ, τ)-derivation associated with Jordan (σ, τ)-derivation d, then d is a Jordan (σ, τ)-derivation, that is, $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$.

Proof: We have $F(x(xx)) = F(x)\sigma(xx) + \tau(x)d(xx)$

$$= F(x)\sigma(xx) + \tau(x)d(x^2), \text{ for all } x \in R \tag{1}$$

On the otherhand, we have $F\big((xx)x\big) = F(xx)\sigma(x) + \tau(xx)d(x)$

$$= \big(F(x)\sigma(x) + \tau(x)d(x)\big)\sigma(x) + \tau(xx)d(x)$$

$$= F(x)\sigma(x)\sigma(x) + \tau(x)d(x)\sigma(x) + \tau(xx)d(x), \text{ for all } x \in R. \tag{2}$$

Equating equations (1) and (2), we get

$$\tau(x)d(x^2) = \tau(x)d(x)\sigma(x) + \tau(xx)d(x)$$

$$\tau(x)\big(d(x^2) - d(x)\sigma(x) - \tau(x)d(x)\big) = 0, \text{ for all } x \in R.$$

Left multiplying equation (2) by $d(x^2) - d(x)\sigma(x) - \tau(x)d(x)$, we get

$(d(x^2) - d(x)\sigma(x) - \tau(x)d(x))\tau(x)\big(d(x^2) - d(x)\sigma(x) - \tau(x)d(x)\big) = 0$, for all $x \in R$.

Recall that the vanishing expression generates a two-sided annihilating ideal, the above expression, obtained $aRa = 0$, where $a = d(x^2) - d(x)\sigma(x) - \tau(x)d(x)$ from two sided ideal, $I = aRa$ take the arbitrary elements $xy = r_1ar_2r_3ar_4 = r_1a(r_2r_3)ar_4$ since $aRa = 0$, $a(r_2r_3)a = 0$. Hence $xy = 0$. Thus $I^2 = 0$. So $I = RaR$ is two sided annihilating ideal. Since τ is an automorphism, we get

$$\big(d(x^2) - d(x)\sigma(x) - \tau(x)d(x)\big)R\big(d(x^2) - d(x)\sigma(x) - \tau(x)d(x)\big) = 0 \forall x \in R \quad (3)$$

R is a semiprimeness used condition, $I^2 = 0$. So $I = 0$.

Suppose $aRb = 0$, for all $r \in R$. If similarly $bRa = 0$, then for all $r, s \in R$.

$ara = ar(bsa)$, for all $r, s \in R$. Since $bsa \in bRa = 0$, $ara = 0$. Hence, $aRa = 0$.

Since R is semiprime ring, we get $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$.

That is, d is a Jordan (σ, τ)-derivation.

Lemma 2: Let R be a 2-torsion-free semiprime ring, and $F: R \to R$ be a generalized Jordan (σ, τ)-derivation associated with Jordan (σ, τ)-derivation d. If $F(x^2) = 0$, for all $x \in R$, then $F = 0$ and $d = 0$.

Proof: We have

$$F(x^2) = 0, \text{ for all } x \in R. \quad (4)$$

By replacing x by $x + y$ in equation (4), we get

$$F((x + y)^2) = 0$$

$$F(x^2) + F(xy) + F(yx) + F(y^2) = 0, \text{ for all } x, y \in R.$$

Using equation (4) in the above equation, we get

$$F(xy) + F(yx) = 0$$

$$F(x)\sigma(y) + \tau(x)d(y) + F(y)\sigma(x) + \tau(y)d(x) = 0, \text{ for all } x, y \in R.$$

We are replacing y by y^2 in the above equation, we get

$$F(x)\sigma(y^2) + \tau(x)d(y^2) + F(y^2)\sigma(x) + \tau(y^2)d(x) = 0, \text{ for all } x, y \in R.$$

Using equation (4) in the above equation, we get

$$F(x)\sigma(y)\sigma(y) + \tau(x)d(y)\sigma(y) + \tau(x)\tau(y)d(y) + \tau(y)\tau(y)d(x) = 0$$

By replacing y by x in the above equation, we get

$$(F(x)\sigma(x) + \tau(x)d(x))\sigma(x) + 2\tau(x)\tau(x)d(x) = 0$$

$$F(x^2)\sigma(x) + 2\tau(x)\tau(x)d(x) = 0, \text{ for all } x \in R.$$

Using equation (4) and using R is a 2-torsion-free ring in the above equation, we get

$$\tau(x)\tau(x)d(x) = 0, \text{ for all } x \in R. \quad (5)$$

Left multiplying equation (5) by $\tau(x)d(x)$, we get

$\tau(x)d(x)\tau(x)\tau(x)d(x) = 0$, for all $x \in R$.

Since τ is an automorphism, we get

$\tau(x)d(x)R\tau(x)d(x) = 0$, for all $x \in R$. Since R is semiprime ring, we get

$\tau(x)d(x) = 0$, for all $x \in R$. (6)

Left multiplying equation (6) by $d(x)$, we get

$d(x)\tau(x)d(x) = 0$, for all $x \in R$. Since τ is an automorphism of R, we get

$d(x)Rd(x) = 0$, for all $x \in R$. Since R is semiprime ring, we get

$d(x) = 0$, for all $x \in R$. (7)

By the hypothesis $F(x^2) = 0$, for all $x \in R$.

$F(x)\sigma(x) + \tau(x)d(x) = 0$, for all $x \in R$.

Using equation (7) in the above equation, we get

$F(x)\sigma(x) = 0$, for all $x \in R$. (8)

Right multiplying equation (8) by $F(x)$, we get

$F(x)\sigma(x)F(x) = 0$, for all $x \in R$. Since σ is an automorphism of R, we get

$F(x)RF(x) = 0$, for all $x \in R$. Since R is semiprime ring, we get

$F(x) = 0$, for all $x \in R$.

Lemma 3: Let R be a 2-torsion-free semiprime ring, and $F: R \to R$ be a generalized Jordan (σ,τ)-derivation associated with Jordan (σ,τ)-derivation d. If $F(x^2) \in C_{\sigma,\tau}$, for all $x \in R$, then $[d(x), x]_{\sigma,\tau} = 0$, for all $x \in R$.

Proof: We have

$F(x^2) \in C_{\sigma,\tau}$, for all $x \in R$. (9)

By replacing x by $x + y$ in equation (9), we get

$F((x + y)^2) \in C_{\sigma,\tau}$

$F(x^2) + F(xy) + F(yx) + F(y^2) \in C_{\sigma,\tau}$, for all $x, y \in R$.

Using equation (9) in the above equation, we get

$F(xy) + F(yx) \in C_{\sigma,\tau}$,

$F(x)\sigma(y) + \tau(x)d(y) + F(y)\sigma(x) + \tau(y)d(x) \in C_{\sigma,\tau}$, for all $x, y \in R$.

We are replacing y by y^2 in the above equation, we get

$F(x)\sigma(y^2) + \tau(x)d(y^2) + F(y^2)\sigma(x) + \tau(y^2)d(x) \in C_{\sigma,\tau}$, for all $x, y \in R$.

Using equation (9) in the above equation, we get

$$F(x)\sigma(y)\sigma(y) + \tau(x)d(y)\sigma(y) + \tau(x)\tau(y)d(y) + \tau(y)\tau(y)d(x) \in C_{\sigma,\tau}$$

By replacing y by x in the above equation, we get

$$(F(x)\sigma(x) + \tau(x)d(x))\sigma(x) + 2\tau(x)\tau(x)d(x) \in C_{\sigma,\tau}$$

$$F(x^2)\sigma(x) + 2\tau(x)\tau(x)d(x) \in C_{\sigma,\tau}, \text{ for all } x \in R.$$

Using equation (9) in the above equation, we get

$$2\tau(x)\tau(x)d(x) \in C_{\sigma,\tau}, \text{ for all } x \in R.$$

$$[2\tau(x)\tau(x)d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

$$2\tau(x)\tau(x)[d(x), x]_{\sigma,\tau} + [\tau(x)\tau(x), \tau(x)]d(x) = 0$$

$$2\tau(x)\tau(x)[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

Using R is a 2-torsion-free ring in the above equation, we get

$$\tau(x)\tau(x)[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R. \quad (10)$$

Left multiplying equation (10) by $\tau(x)[d(x), x]_{\sigma,\tau}$, we get

$$\tau(x)[d(x), x]_{\sigma,\tau}\tau(x)\tau(x)[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

Since τ is an automorphism, we get

$$\tau(x)[d(x), x]_{\sigma,\tau}R\tau(x)[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

Since R is semiprime, we get

$$\tau(x)[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R. \quad (11)$$

Left multiplying equation (11) by $[d(x), x]_{\sigma,\tau}$, we get

$$[d(x), x]_{\sigma,\tau}\tau(x)[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

Since τ is an automorphism of R, we get

$$[d(x), x]_{\sigma,\tau}R[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

Since R is semiprime ring, we get

$$[d(x), x]_{\sigma,\tau} = 0, \text{ for all } x \in R.$$

Lemma 4: Let R be a 2-torsion-free semiprime ring, $F: R \to R$ be a generalized Jordan (σ, τ)-derivation associated with Jordan (σ, τ)-derivation d, and $H: R \to R$ be a left σ-centralizer. If the map $G: R \to R$ is defined as $G(x) = F(x) \mp H(x)$, for all $x \in R$, then G is a generalized Jordan (σ, τ)-derivation associated with Jordan (σ, τ)-derivation d.

Proof: We suppose that

$$G(x) = F(x) \mp H(x), \text{ for all } x \in R. \quad (12)$$

We are replacing x by x^2 in equation (12), we get

$$G(x^2) = F(x^2) \mp H(x^2)$$

$= F(x)\sigma(x) + \tau(x)d(x) \mp H(x)\sigma(x)$

$= \big(F(x) \mp H(x)\big)\sigma(x) + \tau(x)d(x)$, for all $x \in R$.

Using equation (12) in the above equation, we get

$G(x^2) = G(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$.

Then G is a generalized Jordan (σ,τ)-derivation associated with Jordan (σ,τ)-derivation d.

Theorem 1: Let R be a 2-torsion-free semiprime ring, $F: R \to R$ be a generalized Jordan (σ,τ)-derivation associated with Jordan (σ,τ)-derivation d, and $H: R \to R$ be a left σ-centralizer. If $F(x^2) \mp H(x^2) = 0$, for all $x \in R$, then $d = 0$ and $F = \pm H$.

Proof: By the hypothesis, we have

$$F(x^2) - H(x^2) = 0, \text{ for all } x \in R. \tag{13}$$

Using equation (12) in the above equation, we get

So we have $G(x^2) = 0$, for all $x \in R$.

Using Lemma 2, we get $G = 0$.

So we have $F = H$. (14)

By the hypothesis, we have

$F(x^2) - H(x^2) = 0$, for all $x \in R$.

$F(x)\sigma(x) + \tau(x)d(x) - H(x)\sigma(x) = 0$, for all $x \in R$.

Using equation (14) in the above equation, we get

$$\tau(x)d(x) = 0, \text{ for all } x \in R. \tag{15}$$

Left multiplying equation (15) by $d(x)$, we get

$d(x)\tau(x)d(x) = 0$, for all $x \in R$.

Since τ is an automorphism, we get

$d(x)Rd(x) = 0$, for all $x \in R$.

Since R is semiprime ring, we get

$d(x) = 0$, for all $x \in R$.

Similar proof shows that the same conclusion holds as $F(x^2) - H(x^2) = 0$, for all $x \in R$. In this case, we obtain $F = -H$. Therefore, the proof is completed.

Theorem 2: Let R be a 2-torsion-free semiprime ring, $F: R \to R$ be a generalized Jordan (σ,τ)-derivation associated with Jordan (σ,τ)-derivation d, and $H: R \to R$ be a left σ-centralizer. If $F(x^2) \mp H(x^2) \in C_{\sigma,\tau}$, for all $x \in R$, then $[d(x), x]_{\sigma,\tau} = 0$, for all $x \in R$.

Proof: By the hypothesis, we have

$F(x^2) \mp H(x^2) \in C_{\sigma,\tau}$, for all $x \in R$. (16)

Using equation (12) in the above equation, we get

So we have $G(x^2) \in C_{\sigma,\tau}$, for all $x \in R$.

Using Lemma 3, we get $[d(x), x]_{\sigma,\tau} = 0$, for all $x \in R$. Therefore, the proof is completed.

III. Conclusion

The research shows that let R be a 2-torsion-free semiprime ring, $F: R \to R$ be a generalized Jordan (σ,τ)-derivation associated with Jordan (σ,τ)-derivation d, and $H: R \to R$ be a left σ-centralizer. (i) If $F(x^2) \mp H(x^2) = 0$, for all $x \in R$, then $d = 0$ and $F = \pm H$.(ii) If$F(x^2) \mp H(x^2) \in C_{\sigma,\tau}$, for all $x \in R$, then $[d(x), x]_{\sigma,\tau} = 0$, for all $x \in R$. The same structural conclusions are obtained from the vanishing of derivation terms. But they hold in a more general $\sigma -$ framework. Hence, the result extends classical generated derivation theorems to the $\sigma -$ automorphism setting. Hence, theorems' conclusions are a proper generalization of classical results**.** (1) Explicit example where σ ≠ id gives Non-trivial behavior: Let $R = M_n(F)$. By assumption, $\sigma(x) = \mu x \mu^{-1}$, for a fixed invertible matrix μ. By definition, $d(x) = ax - \sigma(x)a$. This satisfies a $\sigma -$type derivation identity, but is not an arbitrary derivation when σ ≠ id (2). Structural Novelty of the $\sigma -$ framework: The $\sigma -$ framework introduces: (i)Twisted multiplication structure: identity involves $d(x)\sigma(y) + \tau(x)d(y)$ instead of the classical Leibniz rule. (ii) Automorphism interaction: The structure now depends on the ring automorphisms σ and τ, not just flexibility. (iii) Greater flexibility: Results apply to skew polynomial extensions. Rings with automorphism actions. An extent classical semiprime annihilation technique. Theorems 1 and 2 generalize classical generalized derivation results by incorporating automorphisms σ and τ. When σ ≠ id, nontrivial twisted derivations arise (e.g., via inner automorphisms on matrix rings). The structural novelty lies in replacing the ordinary Leibniz rule with a σ-twisted identity, extending classical semiprime annihilation techniques to a broader automorphism-controlled framework.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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