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POSITIVE SOLUTIONS FOR A THREE-COMPONENT ITERATIVE SYSTEM OF NONLINEAR TEMPERED FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, we establish the existence of denumerably many positive solutions for the iterative system of nonlinear four-point tempered fractional-order boundary value problems. By an application of Krasnoselskii's fixed point theorem in a Banach space. An illustrative example is also presented.

Keywords: Tempered FDE; Green's function; iterative; fixed point theorem.

I. Introduction

Initially, L. Liouville [XX] and B. Riemann [XXXIV] introduced fractional calculus, which is an effective tool for studying systems that describe different real memory operations. Because fractional calculus has so many applications in fields such as physics, chemistry, biology, control theory, signal and image processing, economics, geology, and sociology, it has garnered the interest of numerous scholars worldwide in recent years [VII, XIII]. Research on nonlinear fractional systems is expanding quickly due to the desire to apply these systems' special qualities to address practical issues. These systems are found in many aspects of nature, such as the viscoelastic behavior of polymers and the unusual particle diffusion in fluids. Fractional calculus offers an effective foundation for these intricate system modeling and comprehension [XIV, XXIX]. Several other types of fractional derivatives have also been defined in recent decades [XXVIII]. They are more general and more suitable for applications than the Riemann-Liouville fractional derivative [II, IV]. These days, research is focused on finding solutions to equations involving fractional derivatives and computing them using iterative approaches [III, XII], respectively. The existence and uniqueness of fractional iterative differential equations have been studied widely by using the fixed-point theorem [VIII]. Recently, Liu and Jia [XIX] presented a class of iterative functional fractional differential equations on the semi-infinite interval with integral boundary conditions. We study the presence of iterative positive solutions to a linked system of fractional differential equations supplemented with multistrip and multipoint mixed boundary conditions, using the monotone iterative technique.

One of the generalized forms of fractional derivatives is the tempered fractional derivative. The tempered fractional derivative is obtained by multiplying the classical fractional derivative by an exponential factor. The conventional Riemann-Liouville and Caputo fractional derivatives are obtained in particular situations for $\lambda = 0$. This novel fractional operator depends on a parameter λ. Because of its applications in physics, groundwater hydrology, poroelasticity, geophysical flow, finance, and other fields, the tempered fractional derivative has gained popularity as a research topic in recent years [IX, XX, XXXVII, V, XI, XXV, XXVI, XXXIII, XVII]. There are numerous other works on fractional differential equations in the sense of tempered derivatives. However, the majority of them only address classical tempered fractional derivatives [XXXIX, XXXII, XVIII, XXVII, XV, XXIV].

In [I], Almeida et al. expanded the concept of tempered fractional derivatives within the Riemann-Liouville and Caputo frameworks, introducing a novel class of functional operators. In [XXXVIII], Zaky studied the well-posedness of the solution and derived and analyzed a Jacobi spectral-collocation method for the numerical solution to the two-point nonlinear tempered fractional boundary value problem. In Pandey et al. [XXXI], the properties of eigenvalues for the regular tempered fractional Strum-Liouville problem are studied using a fractional variational approach. Li et al. [XXI] examined various properties of tempered fractional derivatives and explored the existence, uniqueness, and stability of some tempered fractional ordinary differential equations. In Khuddush [XVI] examined the existence of positive solutions to the iterative system of non-linear two-point tempered fractional order boundary value problem.

Motivated by the works above, in order to discuss the proposed new iterative system results for fractional calculus and the theory of fractional differential equations in the tempered sense, in this paper, we utilize Krasnoselskii's fixed point theorem to establish the existence of a positive solution within the iterative system of a nonlinear four-point tempered fractional boundary value problem:

$$\begin{cases} {}_{0}^{R}\mathcal{D}_{\tau}^{\alpha_1,p} x_i(\tau) + \lambda(\tau) f_i\big(y_{i+1}(\tau)\big) = 0, \quad 0 < \tau < 1, \quad 1 \le i \le n, \\ {}_{0}^{R}\mathcal{D}_{\tau}^{\alpha_2,p} y_i(\tau) + \mu(\tau) g_i\big(z_{i+1}(\tau)\big) = 0, \\ {}_{0}^{R}\mathcal{D}_{\tau}^{\alpha_3,p} z_i(\tau) + \sigma(\tau) h_i\big(x_{i+1}(\tau)\big) = 0, \\ x_{n+1}(\tau) = x_1(\tau),\ y_{n+1}(\tau) = y_1(\tau),\ z_{n+1}(\tau) = z_1(\tau) \end{cases} \tag{1.1}$$

subject to the boundary conditions,

$$\begin{cases} x_i(0), \ \frac{d}{d\tau}[e^{p\tau}x_i(\tau)]|_{\tau=0} = 0, \ \frac{d^2}{d\tau^2}[e^{p\tau}x_i(\tau)]|_{\tau=0} = 0, \\ {}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_i(\tau)]|_{\tau=1} - k_1 {}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_i(\tau)]|_{\tau=\eta_1} = \delta_1 {}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_i(\tau)]|_{\tau=\varsigma_1}, \\ y_i(0), \ \frac{d}{d\tau}[e^{p\tau}y_i(\tau)]|_{\tau=0} = 0, \ \frac{d^2}{d\tau^2}[e^{p\tau}y_i(\tau)]|_{\tau=0} = 0, \\ {}_0^R\mathcal{D}_\tau^{\gamma_2,p}[y_i(\tau)]|_{\tau=1} - k_2 {}_0^R\mathcal{D}_\tau^{\gamma_2,p}[y_i(\tau)]|_{\tau=\eta_2} = \delta_2 {}_0^R\mathcal{D}_\tau^{\gamma_2,p}[y_i(\tau)]|_{\tau=\varsigma_2}, \\ z_i(0), \ \frac{d}{d\tau}[e^{p\tau}z_i(\tau)]|_{\tau=0} = 0, \ \frac{d^2}{d\tau^2}[e^{p\tau}z_i(\tau)]|_{\tau=0} = 0, \\ {}_0^R\mathcal{D}_\tau^{\gamma_3,p}[z_i(\tau)]|_{\tau=1} - k_3 {}_0^R\mathcal{D}_\tau^{\gamma_3,p}[z_i(\tau)]|_{\tau=\eta_3} = \delta_3 {}_0^R\mathcal{D}_\tau^{\gamma_3,p}[z_i(\tau)]|_{\tau=\varsigma_3}, \end{cases} \tag{1.2}$$

where $3 < \alpha_r \leq 4$; $1 < \gamma_r \leq 2$, $0 < \eta_r < \zeta_r < 1$, η_r, ζ_r are positive constants, $r = 1,2,3$. $\lambda(\tau) = \prod_{k=1}^{n} \lambda_k(\tau), \mu(\tau) = \prod_{k=1}^{n} \mu_k(\tau)$, $\sigma(\tau) = \prod_{k=1}^{n} \sigma_k(\tau)$ and each $\lambda_k(\tau), \mu_k(\tau), \sigma_k(\tau) \in L^{p_i}[0,1], (1 \leq p_i \leq +\infty)$ has n-singularities in the interval $(0,1)$.

Assume the following conditions hold throughout this paper:

$(T1)$ $f_r, g_r, h_r: [0,\infty) \to [0,\infty)$ are continuous,
$(T2)$ $3 < \alpha_r \leq 4$, $1 < \gamma_r \leq 2$, $\alpha_r - \gamma_r - 1 \leq \alpha_r - 2$; $k_r, \delta_r \geq 0$; $0 < \eta_r < \zeta_r < 1$ and $\Lambda_r = \Gamma(\alpha_r)\left[e^{-p} - \kappa_r e^{-p\eta_r}\eta_r^{\alpha_r-\gamma_r-1} - \delta_r e^{-p\zeta_r}\zeta_r^{\alpha_r-\gamma_r-1}\right]$ for $r = 1,2,3$,
$(T3)$ $\lim_{\tau\to\tau_k} \lambda_k(\tau) = \infty$, $\lim_{\tau\to\tau_k} \mu_k(\tau) = \infty$, $\lim_{\tau\to\tau_k} \sigma_k(\tau) = \infty$,
where $0 < \tau_n < \tau_{n-1} < \cdots < \tau_1 < 1$,
$(T4)$ $\exists\, \psi_i, \phi_i, \nu_i > 0$ such that $\lambda_i(\tau) > \psi_i, \mu_i(\tau) > \phi_i, \sigma_i(\tau) > \nu_i$ respectively for $\tau \in [0,1], i = 1,2,\cdots,n$.

The rest of the paper is organised as follows. In Section 2, we present some definitions and background results, also construct the Green functions for the homogeneous boundary value problem corresponding to (1.1)-(1.2) and estimate their bounds. In section 3, for the sake of convenience, we state Krasnoselskii's fixed point theorem and also provide criteria for the presence of denumerably many positive solutions to the boundary value problem (1.1)-(1.2). Finally, as an application, we provide an example to demonstrate our results.

II. Mathematical Model

This section goes over the definitions, properties and lemmas of tempered fractional calculus [XXIII, XIV, XXX], which are important for the discussion that follows:

Definition 2.1 *Suppose that the real function $x(\tau)$ is piecewise continuous on (a,b) and $x(\tau) \in L([a,b])$ $\sigma > 0, \lambda \geq 0$. The Riemann-Liouville tempered fractional integral of order σ is defined as*

$$^{\mathbb{RL}}{}_a\mathbb{I}_\tau^{\sigma,\lambda}x(\tau) = e^{-\lambda\tau}\,{}^{\mathbb{RL}}{}_a\mathbb{I}_\tau^{\sigma}(e^{\lambda\tau}x(\tau)) = \frac{1}{\Gamma(\sigma)}\int_a^\tau e^{-\lambda(\tau-s)}(\tau-s)^{\sigma-1}x(s)ds$$

where $^{\mathbb{RL}}{}_a\mathbb{D}_\tau^{\sigma}$ denotes the Riemann-Liouville fractional integral

$$^{\mathbb{RL}}{}_a\mathbb{I}_\tau^{\sigma}x(\tau) = \frac{1}{\Gamma(\sigma)}\int_a^\tau (\tau-s)^{\sigma-1}x(s)ds$$

If $\lambda = 0$ then the tempered fractional integral reduces to the Riemann-Liouville fractional integral.

Definition 2.2 *For $n-1<\alpha<n; n\in N^{+}, \lambda\geq 0$. The Riemann-Liouville tempered fractional derivative is defined by*

$${}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma,\lambda}x(t)=e^{-\lambda\tau}\,{}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma}(e^{\lambda\tau}x(\tau))=\frac{e^{-\lambda\tau}}{\Gamma(n-\sigma)}\frac{d^{n}}{d\tau^{n}}\int_{a}^{\tau}\frac{e^{\lambda s}x(s)}{(\tau-s)^{\sigma-n+1}}ds$$

where ${}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma}$ denotes the Riemann-Liouville fractional derivative

$${}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma}x(\tau)=\frac{1}{\Gamma(n-\sigma)}\frac{d^{n}}{d\tau^{n}}\int_{a}^{\tau}\frac{x(s)}{(\tau-s)^{\sigma-n+1}}ds$$

Lemma 2.1 *Let $x(\tau)\in L([a,b])$ and $\mathbb{I}^{n-\alpha,\lambda}x(\tau)\in AC^{n}[a,b]$. Then the Riemann-Liouville tempered fractional derivative and integral have the composite property:*

$${}^{\mathbb{RL}}{}_{a}\mathbb{I}_{\tau}^{\sigma,\lambda}[{}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma,\lambda}x(\tau)]=x(\tau)-\sum_{k=0}^{n-1}\frac{e^{-\lambda\tau}(\tau-a)^{\sigma-k-1}}{\Gamma(\sigma-k)}[{}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma-k-1}(e^{\lambda\tau}x(\tau))|_{\tau=a}]$$

Remark:(i) ${}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma,\lambda}[{}_{a}I_{\tau}^{\sigma,\lambda}x(\tau)]=x(\tau)$ (ii) ${}^{\mathbb{RL}}{}_{a}\mathbb{D}_{\tau}^{\sigma}\tau^{\eta-1}=\frac{\Gamma(\eta)}{\Gamma(\eta-\sigma)}\tau^{\eta-\sigma-1}$.

In what follows, we calculate the Green's functions associated with (1.1)-(1.2). Consider the homogeneous boundary value problem:

$$-{}_{0}^{R}\mathcal{D}_{\tau}^{\alpha_1,p}x_1(\tau)=0,\ 0<\tau<1,\ 3<\alpha_1\leq 4, \tag{2.1}$$

$$\begin{gathered}x_1(0)=0,\ \frac{d}{d\tau}[e^{p\tau}x_1(\tau)]|_{\tau=0}=0,\ \frac{d^2}{d\tau^2}[e^{p\tau}x_1(\tau)]|_{\tau=0}=0,\\ {}_{0}^{R}\mathcal{D}_{\tau}^{\gamma_1,p}[x_1(\tau)]|_{\tau=1}-k_1\,{}_{0}^{R}\mathcal{D}_{\tau}^{\gamma_1,p}[x_1(\tau)]|_{\tau=\eta_1}=\delta_1\,{}_{0}^{R}\mathcal{D}_{\tau}^{\gamma_1,p}[x_1(\tau)]|_{\tau=\varsigma_1},\end{gathered} \tag{2.2}$$

Lemma 2.2 *Let $\Lambda_1=\Gamma(\alpha_1)\vartheta\neq 0; \vartheta=e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}$. If $\Upsilon(\tau)\in C[0,1]$ and $3<\alpha_1\leq 4$, then the boundary value problem:*

$${}_{0}^{R}\mathcal{D}_{\tau}^{\alpha_1,p}x_1(\tau)+\Upsilon(\tau)=0,\ 0<\tau<1, \tag{2.3}$$

satisfying the boundary condition (2.2), has a unique solution

$$x_1(\tau)=\int_0^1\mathfrak{I}_1(\tau,\ell)e^{-p(\tau-\ell)}\Upsilon(\ell)d\ell,\ \tau\in[0,1],$$

where $\mathfrak{I}_1(\tau,\ell)$ is the Green's function for the BVP (2.3)-(2.2) and is given by

$$\mathfrak{I}_1(\tau,\ell)=\begin{cases}\mathfrak{I}_{11}(\tau,\ell); & 0\leq\ell\leq\min\{\tau,\eta_1\}\leq 1,\\ \mathfrak{I}_{12}(\tau,\ell); & 0\leq\tau\leq\ell\leq\eta_1\leq\zeta_1<1,\\ \mathfrak{I}_{13}(\tau,\ell); & 0\leq\eta_1\leq\ell\leq\min\{\zeta_1,\tau\}<1,\\ \mathfrak{I}_{14}(\tau,\ell); & 0<\max\{\tau,\eta_1\}\leq\ell\leq\zeta_1\leq 1,\\ \mathfrak{I}_{15}(\tau,\ell); & 0\leq\eta_1\leq\zeta_1\leq\ell\leq\tau<1,\\ \mathfrak{I}_{16}(\tau,\ell); & 0<\max\{\zeta_1,\tau\}\leq\ell\leq 1,\end{cases} \tag{2.4}$$

where

$$\begin{aligned}\mathfrak{I}_{11}(\tau,\ell)&=\frac{1}{\Lambda_1}[(e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}-\kappa_1 e^{-p\eta_1}(\eta_1-\ell)^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\\&\quad\times(\zeta_1-x)^{\alpha_1-\gamma_1-1})\tau^{\alpha_1-1}-\vartheta(\tau-\ell)^{\alpha_1-1}],\\ \mathfrak{I}_{12}(\tau,\ell)&=\frac{1}{\Lambda_1}[e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}-\kappa_1 e^{-p\eta_1}(\eta_1-\ell)^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\\&\quad\times(\zeta_1-\ell)^{\alpha_1-\gamma_1-1}]\tau^{\alpha_1-1},\\ \mathfrak{I}_{13}(\tau,\ell)&=\frac{1}{\Lambda_1}[(e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}(\zeta_1-\ell)^{\alpha_1-\gamma_1-1})\tau^{\alpha_1-1}\end{aligned}$$

$$-\vartheta(\tau-\ell)^{\alpha_1-1}],$$

$$\Im_{14}(\tau,\ell)=\frac{1}{\Lambda_1}[(e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}(\zeta_1-\ell)^{\alpha_1-\gamma_1-1})\tau^{\alpha_1-1}],$$

$$\Im_{15}(\tau,\ell)=\frac{1}{\Lambda_1}[(e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}\tau^{\alpha_1-1}-\vartheta(\tau-\ell)^{\alpha_1-1}],$$

$$\Im_{16}(\tau,\ell)=\frac{1}{\Lambda_1}[e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}\tau^{\alpha_1-1}],$$

Proof: Applying the Riemann-Liouville tempered fractional integral operator ${}_0^R\mathcal{D}_\tau^{\alpha_1,p}$ on both sides of the equation and using the composite property, we get

$$x_1(\tau)=c_1e^{-p\tau}\tau^{\alpha_1-1}+c_2e^{-p\tau}\tau^{\alpha_1-2}+c_3e^{-p\tau}\tau^{\alpha_1-3}$$
$$+c_4e^{-p\tau}\tau^{\alpha_1-4}-\frac{1}{\Gamma(\alpha_1)}\int_0^\tau e^{-p(\tau-\ell)}(\tau-\ell)^{\alpha_1-1}\Upsilon(\ell)d\ell.$$

Using boundary conditions, we have $c_4=c_3=c_2=0$. Hence

$$x_1(\tau)=c_1e^{-p\tau}\tau^{\alpha_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_0^\tau e^{-p(\tau-\ell)}(\tau-\ell)^{\alpha_1-1}\Upsilon(\ell)d\ell$$

moreover, we have

$${}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_1(\tau)]=c_1e^{-p\tau}\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)}\tau^{\alpha_1-\gamma_1-1}$$
$$-e^{-p\tau}\frac{1}{\Gamma(\alpha_1-\gamma_1)}\int_0^\tau(\tau-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\gamma(\ell)d\ell$$

From the other boundary condition

$${}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_1(\tau)]|_{\tau=1}-\kappa_1\,{}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_1(\tau)]|_{\tau=\eta_1}=\delta_1\,{}_0^R\mathcal{D}_\tau^{\gamma_1,p}[x_1(\tau)]|_{\tau=\zeta_1}$$

we have

$$c_1\left[e^{-p}\Gamma(\alpha_1)-\kappa_1e^{-p\eta_1}\Gamma(\alpha_1)\eta_1^{\alpha_1-\gamma_1-1}-\delta_1e^{-p\zeta_1}\Gamma(\alpha_1)\zeta_1^{\alpha_1-\gamma_1-1}\right]$$
$$=e^{-p}\int_0^1(1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)dx-\kappa_1e^{-p\eta_1}\int_0^{\eta_1}(\eta_1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell$$
$$-\delta_1e^{-p\zeta_1}\int_0^{\zeta_1}(\zeta_1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell,$$

$$c_1=\frac{1}{\Lambda_1}[e^{-p}\int_0^1(1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell-\kappa_1e^{-p\eta_1}$$
$$\times\int_0^{\eta_1}(\eta_1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell-\delta_1e^{-p\zeta_1}\int_0^{\zeta_1}(\zeta_1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell].$$

As a result,

$$x_1(\tau)=\frac{-1}{\Gamma(\alpha_1)}\int_0^\tau e^{-p(\tau-\ell)}(\tau-\ell)^{\alpha_1-1}\Upsilon(\ell)d\ell+\frac{e^{-p\tau}\tau^{\alpha_1-1}}{\Lambda_1}$$
$$\times[e^{-p}\int_0^1(1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell-\kappa_1e^{-p\eta_1}\int_0^{\eta_1}(\eta_1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell$$

$$-\delta_1 e^{-p\zeta_1}\int_0^{\zeta_1}(\zeta_1-\ell)^{\alpha_1-\gamma_1-1}e^{p\ell}\Upsilon(\ell)d\ell]$$

$$=\frac{1}{\Lambda_1}[e^{-p}\tau^{\alpha_1-1}\int_0^1(1-\ell)^{\alpha_1-\gamma_1-1}e^{-p(\tau-\ell)}\Upsilon(\ell)d\ell-\kappa_1 e^{-p\eta_1}\tau^{\alpha_1-1}$$

$$\times\int_0^{\eta_1}(\eta_1-\ell)^{\alpha_1-\gamma_1-1}e^{-p(\tau-\ell)}\Upsilon(\ell)d\ell-\delta_1 e^{-p\zeta_1}\tau^{\alpha_1-1}$$

$$\times\int_0^{\zeta_1}(\zeta_1-\ell)^{\alpha_1-\gamma_1-1}e^{-p(\tau-\ell)}\Upsilon(\ell)d\ell-\vartheta\int_0^{\tau}(\tau-\ell)^{\alpha_1-1}e^{-p(\tau-\ell)}\Upsilon(\ell)d\ell]$$

$$=\int_0^1\mathfrak{I}_1(\tau,\ell)\,e^{-p(\tau-\ell)}\Upsilon(\ell)d\ell$$

Lemma 2.3: Assume that condition (T2) is satisfied. Then the Green's function $\mathfrak{I}_1(\tau,\ell)$ is given by (2.4) satisfies the following properties:

- **(i)** $\mathfrak{I}_1(\tau,\ell)$ is nonnegative and continuous on $[0,1]\times[0,1]$.
- **(ii)** $\mathfrak{I}_1(\tau,\ell)\le\mathfrak{I}_1(1,\ell)$, for all $(\tau,\ell)\in[0,1]\times[0,1]$.
- **(iii)** There is some $\flat\in(0,\frac{1}{2})$ such that $\mathfrak{I}_1(\tau,\ell)\ge\flat^{\alpha_1-1}\mathfrak{I}_1(1,\ell)$, for all $(\tau,\ell)\in[\flat,1-\flat]\times(0,1)$.

Proof: Consider the Green's function $\mathfrak{I}_1(\tau,\ell)$ given by (2.4)

(i) Let $0\le\ell\le\min\{\tau,\eta_1\}\le 1$. Then

$$\mathfrak{I}_{11}(\tau,\ell)\ge\frac{\tau^{\alpha_1-1}}{\Lambda_1}[(e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1})(1-\ell)^{-\gamma_1}-\vartheta]\times(1-\ell)^{\alpha_1-1},$$

$$=\frac{\tau^{\alpha_1-1}}{\Lambda_1}[\vartheta(1-\ell)^{-\gamma_1}-\vartheta](1-\ell)^{\alpha_1-1},$$

$$=\frac{\tau^{\alpha_1-1}}{\Lambda_1}[\vartheta(\gamma_1\ell+\frac{\gamma_1(\gamma_1+1)}{2}\ell^2+\cdots)](1-\ell)^{\alpha_1-1}\ge 0.$$

Let $0\le\tau\le\ell\le\eta_1\le\zeta_1\le 1$. Then

$$\mathfrak{I}_{12}(\tau,\ell)\ge\frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1},$$

$$=\frac{\tau^{\alpha_1-1}}{\Lambda_1}[\vartheta(1-\ell)^{\alpha_1-\gamma_1-1}]\ge 0.$$

Let $0\le\eta_1\le\ell\le\min\{\zeta_1,\tau\}<1$. Then

$$\mathfrak{I}_{13}(\tau,\ell)\ge\frac{\tau^{\alpha_1-1}}{\Lambda_1}[(e^{-p}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1})(1-\ell)^{-\gamma_1}-\vartheta](1-\ell)^{\alpha_1-1},$$

$$=\frac{\tau^{\alpha_1-1}}{\Lambda_1}[(\vartheta+\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1})(1-\ell)^{-\gamma_1}-\vartheta](1-\ell)^{\alpha_1-1}\ge 0.$$

Let $0<\max\{\tau,\eta_1\}\le\ell\le\zeta_1\le 1$. Then

$$\mathfrak{I}_{14}(\tau,\ell) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1}$$
$$= \frac{\tau^{\alpha_1-1}}{\Lambda_1}[\vartheta-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1} \geq 0.$$

Let $0 < \max\{\tau,\eta_1\} \leq \ell \leq \zeta_1 \leq 1$. Then

$$\mathfrak{I}_{15}(\tau,x) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[(e^{-p}(1-\ell)^{-\gamma_1}-\vartheta](1-x)^{\alpha_1-1},$$
$$= \frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}(\gamma_1\ell+\frac{\gamma_1(\gamma_1+1)}{2}\ell^2+\cdots)$$
$$+\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}+\delta_1\zeta_1^{\alpha_1-\gamma_1-1}](1-x)^{\alpha_1-1} \geq 0.$$

Let $0 < \max\{\eta_1,\tau\} \leq \ell \leq 1$. Then

$$\mathfrak{I}_{16}(\tau,\ell) = \frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}] \geq 0.$$

Which implies that $\mathfrak{I}_1(\tau,\ell) \geq 0$ for all $\tau,\ell \in [0,1]$.

(ii) Let $0 \leq \ell \leq \min\{\tau,\eta_1\} \leq 1$. Then

$$\frac{\partial\mathfrak{I}_{11}(\tau,\ell)}{\partial\tau} \geq \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[(e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1})$$
$$\times(1-\ell)^{\alpha_1-\gamma_1-1}-\vartheta(1-\ell)^{\alpha_1-2}],$$
$$= \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[\vartheta(1-\ell)^{\alpha_1-\gamma_1-1}-\vartheta(1-\ell)^{\alpha_1-2}] \geq 0.$$

Let $0 \leq \tau \leq \ell \leq \eta_1 \leq \zeta_1 \leq 1$. Then

$$\frac{\partial\mathfrak{I}_{12}(\tau,x)}{\partial\tau} \geq \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}](1$$
$$-\ell)^{\alpha_1-\gamma_1-1},$$
$$= \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[\vartheta(1-\ell)^{\alpha_1-\gamma_1-1}] \geq 0.$$

Let $1 \leq \eta_1 \leq \ell \leq \min\{\zeta_1,\tau\}$. Then

$$\frac{\partial\mathfrak{I}_{13}(\tau,x)}{\partial\tau} \geq \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[(e^{-p}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1})(1-\ell)^{\alpha_1-\gamma_1-1}-\vartheta(1$$
$$-\ell)^{\alpha_1-2}],$$

$$= \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[(\vartheta+\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1})(1-\ell)^{\alpha_1-\gamma_1-1}-\vartheta(1-\ell)^{\alpha_1-2}]$$
$$\geq 0$$

Let $0 < \max\{\tau,\eta_1\} \leq \ell \leq \zeta_1 \leq 1$. Then

$$\frac{\partial \Im_{14}(\tau,\ell)}{\partial \tau} \geq \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[e^{-p}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1}$$
$$= \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[\vartheta+\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1} \geq 0$$

Let $0 < \max\{\tau,\eta_1\} \leq \ell \leq \zeta_1 \leq 1$. Then

$$\frac{\partial \Im_{15}(\tau,\ell)}{\partial \tau} \geq \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}-\vartheta(1-\ell)^{\alpha_1-2}] \geq 0$$

Let $0 < \max\{\eta_1,\tau\} \leq \ell \leq 1$. Then

$$\frac{\partial \Im_{16}(\tau,\ell)}{\partial \tau} = \frac{(\alpha_1-1)\tau^{\alpha_1-2}}{\Lambda_1}[e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}] \geq 0.$$

Which implies that $\Im_1(\tau,\ell)$ is a monotone non-decreasing function, so we have $\Im_1(\tau,\varsigma) \leq \Im_1(1,\varsigma)$ for all $(\tau,\ell) \in [0,1]\times[0,1]$.

Hence the inequality (ii) is proved.

(iii) Let $0 \leq \ell \leq \min\{\tau,\eta_1\} \leq 1$. Then

$$\Im_{11}(\tau,\ell) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[(e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1})(1-\ell)^{-\gamma_1} - \vartheta]\ (1-\ell)^{\alpha_1-1}$$
$$= \tau^{\alpha_1-1}\Im_{11}(1,\ell) \geq \flat^{\alpha_1-1}\Im_{11}(1,\ell)$$

Let $0 \leq \tau \leq \ell \leq \eta_1 \leq \zeta_1 \leq 1$. Then

$$\Im_{12}(\tau,\ell) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}-\kappa_1 e^{-p\eta_1}\eta_1^{\alpha_1-\gamma_1-1}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1},$$
$$= \tau^{\alpha_1-1}\Im_{12}(1,\ell) \geq \flat^{\alpha_1-1}\Im_{12}(1,\ell)$$

Let $1 \leq \eta_1 \leq \ell \leq \min\{\zeta_1,\tau\}$. Then

$$\Im_{13}(\tau,\ell) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[(e^{-p}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1})(1-x)^{-\gamma_1}-\vartheta](1-\ell)^{\alpha_1-1},$$
$$= \tau^{\alpha_1-1}\Im_{13}(1,\ell) \geq \flat^{\alpha_1-1}\Im_{13}(1,\ell)$$

Let $0 < \max\{\tau,\eta_1\} \leq \ell \leq \zeta_1 \leq 1$. Then

$$\Im_{14}(\tau,\ell) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}-\delta_1 e^{-p\zeta_1}\zeta_1^{\alpha_1-\gamma_1-1}](1-\ell)^{\alpha_1-\gamma_1-1}$$
$$= \tau^{\alpha_1-1}\Im_{14}(1,\ell) \geq \flat^{\alpha_1-1}\Im_{14}(1,\ell)$$

Let $0 < \max\{\tau,\eta_1\} \leq \ell \leq \zeta_1 \leq 1$. Then

$$\Im_{15}(\tau,\ell) \geq \frac{\tau^{\alpha_1-1}}{\Lambda_1}[(e^{-p}(1-\ell)^{-\gamma_1}-\vartheta](1-\ell)^{\alpha_1-1},$$
$$= \tau^{\alpha_1-1}\Im_{15}(1,\ell) \geq \flat^{\alpha_1-1}\Im_{15}(1,\ell)$$

Let $0 < \max\{\eta_1, \tau\} \leq \ell \leq 1$. Then

$$\mathfrak{J}_{16}(\tau,\ell) = \frac{\tau^{\alpha_1-1}}{\Lambda_1}[e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1}] = \tau^{\alpha_1-1}\mathfrak{J}_{16}(1,\ell) \geq \flat^{\alpha_1-1}\mathfrak{J}_{16}(1,\ell).$$

Therefore, we have $\mathfrak{J}_1(1,\ell) \geq \flat^{\alpha_1-1}\mathfrak{J}_1(1,\ell)$, for all $(\tau,\ell) \in [\flat, 1-\flat] \times (0,1)$. Hence the inequality (iii) is proved.

We can also formulate similar results as Lemmas 2.2-2.3 for the tempered fractional order boundary value problems.

$$\begin{cases} {}_0^R\mathcal{D}_\tau^{\alpha_2,p} y_1(\tau) + k_1(\tau) = 0, \;\; 0 < \tau < 1, \;\; 3 < \alpha_2 \leq 4, \\ y_1(0), \;\; \frac{d}{d\tau}[e^{p\tau}y_1(\tau)]|_{\tau=0} = 0, \;\; \frac{d^2}{d\tau^2}[e^{p\tau}y_1(\tau)]|_{\tau=0} = 0, \\ {}_0^R\mathcal{D}_\tau^{\gamma_2,p}[y_1(\tau)]|_{\tau=1} - k_2\,{}_0^R\mathcal{D}_\tau^{\gamma_2,p}[y_1(\tau)]|_{\tau=\eta_2} = \delta_2\,{}_0^R\mathcal{D}_\tau^{\gamma_2,p}[y_1(\tau)]|_{\tau=\varsigma_2}, \end{cases} \tag{2.5}$$

Lemma 2.4 *Let $\Lambda_2 > 0$. Then the Green's function $\mathfrak{J}_2(\tau,\ell)$ has the following inequalities:*

(i) $\mathfrak{J}_2(\tau,\ell) > 0$, for all $\tau, \ell \in (0,1)$.

(ii) $\mathfrak{J}_2(\tau,\ell) \leq \mathfrak{J}_2(1,\ell)$, for all $(\tau,\ell) \in [0,1] \times [0,1]$.

(iii) There is some $\wp \in (0,\frac{1}{2})$ such that $\mathfrak{J}_2(\tau,\ell) \geq \wp^{\alpha_2-1}\mathfrak{J}_2(1,\ell)$, for all $(\tau,\ell) \in [\wp, 1-\wp] \times (0,1)$.

Similarly, we also formulate results as Lemmas 2.2-2.3 for the tempered fractional order boundary value problems

$$\begin{cases} {}_0^R\mathcal{D}_\tau^{\alpha_3,p} z_1(\tau) + \mathrm{q}(\tau) = 0, \;\; 0 < \tau < 1, \;\; 3 < \alpha_3 \leq 4, \\ z_1(0), \;\; \frac{d}{d\tau}[e^{p\tau}z_1(\tau)]|_{\tau=0} = 0, \;\; \frac{d^2}{d\tau^2}[e^{p\tau}z_1(\tau)]|_{\tau=0} = 0, \\ {}_0^R\mathcal{D}_\tau^{\gamma_3,p}[z_1(\tau)]|_{\tau=1} - k_3\,{}_0^R\mathcal{D}_\tau^{\gamma_3,p}[z_1(\tau)]|_{\tau=\eta_3} = \delta_3\,{}_0^R\mathcal{D}_\tau^{\gamma_3,p}[z_1(\tau)]|_{\tau=\varsigma_3}, \end{cases} \tag{2.6}$$

Lemma 2.5 *Let $\Lambda_3 > 0$. Then the Green's function $\mathfrak{J}_3(\tau,\ell)$ has the following inequalities:*

(i) $\mathfrak{J}_3(\tau,\ell) > 0$, for all $\tau, \ell \in (0,1)$.

(ii) $\mathfrak{J}_3(\tau,\ell) \leq \mathfrak{J}_3(1,\ell)$, for all $(\tau,\ell) \in [0,1] \times [0,1]$.

(iii) There is some $\aleph \in (0,\frac{1}{2})$ such that $\mathfrak{J}_3(\tau,\ell) \geq \aleph^{\alpha_3-1}\mathfrak{J}_3(1,\ell)$, for all $(\tau,\ell) \in [\aleph, 1-\aleph] \times (0,1)$.

We note that an $3n$-tuple

$$(x_1(\tau), x_2(\tau), \cdots x_n(\tau), y_1(\tau), y_2(\tau), \cdots, y_n(\tau), z_1(\tau), z_2(\tau), \cdots, z_n(\tau))$$

is a solution of the boundary value problem (1.1)-(1.2) if and only if

$$x_i(\tau) = \int_0^1 \mathfrak{J}_1(\tau,\ell)e^{-p(\tau-\ell)}\lambda(\ell)\, f_i(y_{i+1}(\ell))\, d\ell$$

$$y_i(\tau) = \int_0^1 \mathfrak{J}_2(\tau,\ell)e^{-p(\tau-\ell)}\mu(\ell)\, g_i(z_{i+1}(\ell))\, d\ell$$

$$z_i(\tau) = \int_0^1 \Im_3(\tau,\ell)e^{-p(\tau-\ell)}\sigma(\ell)\, h_i(x_{i+1}(\ell))\, d\ell, \qquad \tau \in (0,1), \qquad 1 \le i \le n,$$

$$x_{n+1}(\tau) = x_1(\tau), \qquad y_{n+1}(\tau) = y_1(\tau), \qquad z_{n+1}(\tau) = z_1(\tau), \qquad \tau \in (0,1).$$

Equivalently, if n is a multiple of $3m, m \in N$ then

$$x_1(\tau) = \int_0^1 \Im_1(\tau,\ell_1)e^{-p(\tau-\ell_1)}\lambda(\ell_1)\, f_1(\int_0^1 \Im_2(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\mu(\ell_2)$$
$$\times\, g_2(\int_0^1 \Im_3(\ell_2,\ell_3)e^{-p(\ell_2-\ell_3)}\sigma(\ell_3)h_3 \cdots$$
$$\times\, g_{n-1}(\int_0^1 \Im_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\sigma(\ell_n)\, h_n(x_1(\ell_n))dx_n) \cdots d\ell_3)d\ell_2)d\ell_1,$$

And

$$y_1(\tau) = \int_0^1 \Im_2(\tau,\ell_1)e^{-p(\tau-\ell_1)}\mu(\ell_1)\, g_1(\int_0^1 \Im_3(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\sigma(\ell_2)$$
$$\times\, h_2(\int_0^1 \Im_1(\ell_2,\ell_3)e^{-p(\ell_2-\ell_3)}\mu(\ell_3)f_3 \cdots$$
$$\times\, h_{n-1}(\int_0^1 \Im_1(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\lambda(\ell_n)\, f_n(y_1(\ell_n))d\ell_n) \cdots d\ell_3)d\ell_2)d\ell_1,$$

$$z_1(\tau) = \int_0^1 \Im_3(\tau,\ell_1)\, e^{-p(\tau-\ell_1)}\sigma(\ell_1)h_1(\int_0^1 \Im_1(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\lambda(\ell_2)$$
$$\times\, f_2(\int_0^1 \Im_2(\ell_2,\ell_3)e^{-p(\ell_3-\ell_2)}\mu(\ell_3)\, g_3 \cdots$$
$$\times\, f_{n-1}(\int_0^1 \Im_2(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\mu(\ell_n)\, g_n(z_1(\ell_n))d\ell_n) \cdots d\ell_3)d\ell_2)d\ell_1,$$

If n is a multiple of $3m-1$, then

$$x_1(\tau) = \int_0^1 \Im_1(\tau,\ell_1)e^{-p(\tau-\ell_1)}\lambda(\ell_1)\, f_1(\int_0^1 \Im_2(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\mu(\ell_2)$$
$$\times\, g_2(\int_0^1 \Im_3(\ell_2,\ell_3)e^{-p(\ell_2-\ell_3)}\sigma(\ell_3)\, h_3 \cdots$$
$$\times\, h_{n-1}(\int_0^1 \Im_1(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\lambda(\ell_n)\, f_n(y_1(\ell_n))d\ell_n) \cdots d\ell_3)d\ell_2)d\ell_1,$$

and

$$y_1(\tau) = \int_0^1 \Im_2(\tau,\ell_1)e^{-p(\tau-\ell_1)}\mu(\ell_1)\, g_1(\int_0^1 \Im_3(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\sigma(\ell_2)h_2(\int_0^1 \Im_1(\ell_2,\ell_3)e^{-p(\ell_2-\ell_3)}$$
$$\times\, \lambda(\ell_3)f_3 \cdots f_{n-1}(\int_0^1 \Im_2(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\mu(\ell_n)\, g_n(z_1(\ell_n))d\ell_n) \cdots d\ell_3)d\ell_2)d\ell_1,$$

$$z_1(\tau)) = \int_0^1 \Im_3(\tau,\ell_1)e^{-p(\tau-\ell_1)}\sigma(\ell_1)\, h_1(\int_0^1 \Im_1(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\lambda(\ell_2)f_2(\int_0^1 \Im_2(\ell_2,\ell_3)e^{-p(\ell_2-\ell_3)}$$
$$\times\, \mu(\ell_3)g_3 \cdots g_{n-1}(\int_0^1 \Im_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\sigma(\ell_n)\, h_n(x_1(\ell_n))d\ell_n) \cdots d\ell_3)d\ell_2)d\ell_1$$

Theorem 2.6 (Unified positivity of the Green Kernels). Assume that condition (T2) holds and that $\Lambda_r > 0$ for $r = 1,2,3$. Let $\mathfrak{I}_r(\tau,\ell)$ $(r = 1,2,3)$ denotes the Green's functions associated respectively with the boundary value problems (2.3), (2.5), and (2.6).

Then the following properties hold:

(i) (Global negativity) $\mathfrak{I}_r(\tau,\ell) \geq 0$ for all $(\tau,\ell) \in [0,1] \times [0,1]$.

(ii) *(Interior strict positivity) For every $\varepsilon \in \left(0,\frac{1}{2}\right)$, there exists a constant $c_{r,\varepsilon} > 0$ such that* $\mathfrak{I}_r(\tau,\ell) \geq c_{r,\varepsilon}$ for all $(\tau,\ell) \in [\varepsilon, 1-\varepsilon] \times [\varepsilon, 1-\varepsilon]$.

(iii) (Uniform integral lower bound) *For every $\varepsilon \in \left(0,\frac{1}{2}\right)$, there exists a constant $c_{r,\varepsilon} > 0$ such that* $\inf_{\tau\in[\varepsilon,1-\varepsilon]} \int_{\varepsilon}^{1-\varepsilon} \mathfrak{I}_r(\tau,\ell)d\ell \geq c_{r,\varepsilon} > 0$.

Proof: We give the proof for $\mathfrak{I}_1$; the arguments for $\mathfrak{I}_2$ and $\mathfrak{I}_3$ are analogous.

From Lemma 2.2, the Green function $\mathfrak{I}_r(\tau,\ell)$ admits the unified structural representation

$$\mathfrak{I}_1(\tau,\ell) = \frac{1}{\Lambda_1}\left[A(\ell)\tau^{\alpha_1-1} - \vartheta(\tau-\ell)_+^{\alpha_1}\right],$$

Where $(\tau-\ell)_+ = max\{\tau-\ell, 0\}$ and

$$A(\ell) = e^{-p}(1-\ell)^{\alpha_1-\gamma_1-1} - \kappa_1 e^{-p\eta_1}(\eta_1-\ell)_+^{\alpha_1-\gamma_1-1} - \delta_1 e^{-p\zeta_1}(\zeta_1-\ell)_+^{\alpha_1-\gamma_1-1}$$

Under the condition (T_2) we have $\Lambda_1 > 0$ and $\vartheta > 0$. Moreover, since $0 \leq (\tau-\ell)_+ \leq \tau$, it follows that,

$$(\tau-\ell)_+^{\alpha_1-1} \leq \tau^{\alpha_1-1}.$$

Therefore,

$$\mathfrak{I}_1(\tau,\ell) = \frac{\tau^{\alpha_1-1}}{\Lambda_1}[A(\ell)-\vartheta].$$

By the definition of ϑ and the ordering $0 < \eta_1 < \zeta_1 < 1$, one varies that $A(\ell) \geq \vartheta$ for all $\ell \in [0,1]$. Hence $\mathfrak{I}_1(\tau,\ell) \geq 0$ on $[0,1] \times [0,1]$, providing (i).

To prove (ii), fix $\varepsilon \in \left(0,\frac{1}{2}\right)$. On the compact rectangle $[\varepsilon, 1-\varepsilon]^2$ we have,

$$\tau^{\alpha_1-1} \geq \varepsilon^{\alpha_1-1} > 0.$$

All functions involved in the definition of $\mathfrak{I}_1$ are continuous on this compact set. Since $\mathfrak{I}_1$ is nonnegative and not identically zero, continuity implies that,

$$\min_{(\tau,\ell)\in[\varepsilon,1-\varepsilon]^2} \mathfrak{I}_1(\tau,\ell) = c_{1,\varepsilon} > 0.$$

This prove interion strict positivity.

For (iii), using (ii) we obtain,

$$\int_{\varepsilon}^{1-\varepsilon} \mathfrak{I}_1(\tau,\ell)d\ell \geq \int_{\varepsilon}^{1-\varepsilon} c_{1,\varepsilon}d\ell = (1-2\varepsilon)c_{1,\varepsilon}.$$

Taking the infimum over $\tau \in [\varepsilon, 1-\varepsilon]$ gives

$$C_{1,\varepsilon} := (1-2\varepsilon)c_{1,\varepsilon} > 0.$$

The proof for $\mathfrak{I}_2$ and $\mathfrak{I}_3$ are identical.

III. Existence of Positive Solutions

Now we establish the existence of positive solutions for the iterative system (1.1) by using Hölder's inequality and the theorem developed by Krasnoselskii's fixed-point theorem.

Let $\mathcal{B} = \mathcal{E} \times \mathcal{E} \times \mathcal{E}$ be a Banach space $C([0,1], R)$ with norm $\| (x, y, z) \| = \| x \| + \| y \| + \| z \|$ for $(x, y, z) \in \mathcal{B}$, where $\mathcal{E} = \{x : x \in C([0,1], R)\}$ and $\| x \| = \max_{\tau \in [0,1]} |x(\tau)|$.

For $\zeta \in (0, \frac{1}{2})$, let $\mathcal{S}(\zeta) = \min\{\flat^{\alpha_1 - 1}, \wp^{\alpha_2 - 1}, \aleph^{\alpha_3 - 1}\}\}$. We define the cone $\mathcal{K}_\zeta \subset \mathcal{B}$ as

$$\mathcal{K}_\zeta \quad = \{(x, y, z) \in \mathcal{B} : x(\tau) \geq 0;\ y(\tau) \geq 0;\ z(\tau) \geq 0;$$
$$c\{x(\tau) + y(\tau) + z(\tau)\} \geq \mathcal{S}(\zeta) \| (x, y, z) \|\}.$$

For any $(x_1, y_1, z_1) \in \mathcal{K}_\zeta$, define an operator $\varpi : \mathcal{K}_\zeta \to \mathcal{B}$ by $\varpi(x_1, y_1, z_1) = (\varpi_1 x_1, \varpi_2 y_1, \varpi_3 z_1)$ where $\varpi_1 : \mathcal{P}_\flat \to \mathcal{E}, \varpi_2 : \mathcal{P}_\wp \to \mathcal{E}, \varpi_3 : \mathcal{P}_\aleph \to \mathcal{E}$ and

c

$$\mathcal{P}_\flat \quad = \{x \in \mathcal{E} : x(\tau) \geq 0 \ and \ \min_{\tau \in [\flat, 1-\flat]} x(\tau) \geq \flat^{\alpha_1 - 1} \| x(\tau) \|\},$$
$$\mathcal{P}_\wp \quad = \{y \in \mathcal{E} : y(\tau) \geq 0 \ and \ \min_{\tau \in [\wp, 1-\wp]} y(\tau) \geq \wp^{\alpha_2 - 1} \| y(\tau) \|\},$$
$$\mathcal{P}_\aleph \quad = \{z \in \mathcal{E} : z(\tau) \geq 0 \ and \ \min_{\tau \in [\aleph, 1-\aleph]} z(\tau) \geq \aleph^{\alpha_3 - 1} \| z(\tau) \|\}.$$

Lemma 3.1 (Positivity of Operator). Assume that the hypotheses of Theorem 2.6 hold. *For* $r = 1, 2, 3$, *define the nonlinear operators*

$$\varpi_1 x(\tau) = \int_0^1 \Im_1(\tau, \ell) e^{-p(\tau - \ell)} \lambda(\ell) f(\ell, y(\ell)) d\ell,$$
$$\varpi_2 y(\tau) = \int_0^1 \Im_2(\tau, \ell) e^{-p(\tau - \ell)} \mu(\ell) g(\ell, z(\ell)) d\ell,$$
$$\varpi_3 z(\tau) = \int_0^1 \Im_3(\tau, \ell) e^{-p(\tau - \ell)} \sigma(\ell) h(\ell, x(\ell)) d\ell,$$

Assume furthermore that whenever $s > 0$.

$$\lambda(\ell), \mu(\ell), \sigma(\ell) > 0 \ i.e.\ on\ (0,1),$$

And that

$$f(\ell, s), g(\ell, s), h(\ell, s) \geq 0 \ for\ s \geq 0,$$

Then for each $r = 1, 2, 3$, *the operator* ϖ_r *is strongly positive in the sense that,*

$$x \geq 0, x \not\equiv 0 \Longrightarrow \varpi_r x(\tau) > 0, for\ all\ \tau \in (0,1).$$

Proof: We prove the result for ϖ_1; the arguments for ϖ_2 and ϖ_3 are analogous.

Let $y \in C[0,1]$ satisfy $y \geq 0$ and $y \not\equiv 0$. Then there exists a subinterval $[a, b] \subset (0,1)$ of positive length such that

$$y(\ell) > 0 \text{ for } \ell \in [a, b].$$

Since $f(\ell, s) > 0$ whenever $s > 0$, it follows that,

$$f(\ell, y(\ell)) > 0 \text{ for } \ell \in [a, b].$$

Fix $\varepsilon > 0$ such that $[a, b] \subset [\varepsilon, 1 - \varepsilon]$. By the Theorem 2.6(ii), there exists $c_{1,\varepsilon} > 0$ such that

$$\Im_1(\tau, \ell) \geq c_{1,\varepsilon}, \text{ for all } (\tau, \ell) \in [\varepsilon, 1 - \varepsilon]^2.$$

Let $\tau \in [\varepsilon, 1 - \varepsilon]$. Then

$$\varpi_1 x(\tau) = \int_0^1 \mathfrak{J}_1(\tau,\ell)e^{-p(\tau-\ell)}\lambda(\ell)f\big(\ell,y(\ell)\big)d\ell$$

$$\geq \int_a^b \mathfrak{J}_1(\tau,\ell)e^{-p(\tau-\ell)}\lambda(\ell)f\big(\ell,y(\ell)\big)d\ell,$$

$$\geq c_{1,\varepsilon}e^{-p}\left(\inf_{\ell\in[a,b]}\lambda(\ell)\right)\int_a^b f\big(\ell,y(\ell)\big)d\ell.$$

Since $\lambda(\ell) > 0$ almost everywhere and $f\big(\ell,y(\ell)\big) > 0$ on $[a,b]$, the right hand side is strictly positive. Thus

$$\varpi_1 x(\tau) > 0 \text{ for } \tau \in [\varepsilon, 1-\varepsilon].$$

By continuity, it follows that,

$$\varpi_1 x(\tau) > 0 \text{ for } \tau \in (0,1).$$

Therefore ϖ_1 is strongly positive. The proofs for ϖ_2 and ϖ_3 are identical.

Lemma 3.2. *Assume that (T1)(T4)hold that the nonlinearities f,g,h are continuous and nonnegative on* $[0,1]\times[0,\infty]$. *Then for each* $\zeta \in \left(0,\frac{1}{2}\right)$,

$$\varpi: \mathcal{K}_\zeta \longrightarrow \mathcal{K}_\zeta$$

is completely continuous.

Proof: Positivity of Components. Let $(x,y,z) \in \mathcal{K}_\zeta$. Since $\mathfrak{J}_r(\tau,\ell) \geq 0$ (Theorem 2.6), $e^{-p(\tau-\ell)} > 0$, and the weight functions are nonnegative, and since $f,g,h \geq 0$ for nonnegative arguments, we obtain,

$$\varpi_1 x(\tau) \geq 0,\ \varpi_2 y(\tau) \geq 0,\ \varpi_3 z(\tau) \geq 0 \text{ for all } \tau \in [0,1].$$

Uniform Boundedness. Let $(x,y,z) \in \mathcal{K}_\zeta$ with $\|(x,y,z)\| \leq R$. By continuity of f,g,h on a bounded set, there exists $M_r > 0$ such that,

$$f(\ell,y(\ell)), g(\ell,z(\ell)), h(\ell,x(\ell)) \geq M_r \text{ for all } \ell \in [0,1].$$

Since the kernels are bounded on $[0,1]^2$, there exists $C > 0$ such that,

$$\|\varpi_r u\| \leq CM_r \quad (r = 1,2,3).$$

Hence ϖ maps bounded sets into bounded sets.

Equicontinuity. Let $\tau_1,\tau_2 \in [0,1]$. For ϖ_1 we estimate

$$|\varpi_1 x(\tau_1) - \varpi_1 x(\tau_2)|$$
$$\leq \int_0^1 \left|\mathfrak{J}_1(\tau_1,\ell)e^{-p(\tau_1-\ell)} - \mathfrak{J}_1(\tau_2,\ell)e^{-p(\tau_2-\ell)}\right|\lambda(\ell)f\big(\ell,y(\ell)\big)d\ell.$$

Since the kernel

$$K_1(\tau,\ell) = \mathfrak{J}_1(\tau,\ell)e^{-p(\tau-\ell)}\lambda(\ell)$$

Is continuous on $[0,1]^2$, it is uniformly continuous. Thus,

$$\sup_{\ell\in[0,1]} |K_1(\tau_1,\ell) - K_1(\tau_2,\ell)| \to 0 \text{ as } \tau_1 \to \tau_2$$

Using the boundedness of f on bounded sets, we obtain

$$|\varpi_1 x(\tau_1) - \varpi_1 x(\tau_2)| \leq M_R \sup_{\ell\in[0,1]} |K_1(\tau_1,\ell) - K_1(\tau_2,\ell)|.$$

Hence ϖ_1 maps bounded sets into equicontinuous sets. The same holds for ϖ_2 and ϖ_3.

Therefore, by ArzelAscoli, ϖ is completely continuous.

Cone invariance. Fix $(x, y, z) \in \mathcal{K}_\zeta$. By the interior strict positivity of the kernels (Theorem 2.6), there exists $c_\zeta > 0$ such that,

$$\mathfrak{I}_r(\tau, \ell) \geq c_\zeta \text{ for } (\tau, \ell) \in [\zeta, 1-\zeta]^2.$$

For $\tau \in [\zeta, 1-\zeta]$,

$$\varpi_1 x(\tau) \geq c_\zeta \int_\zeta^{1-\zeta} e^{-p(\tau-\ell)} \lambda(\ell) f(\ell, y(\ell)) d\ell.$$

Using the boundedness of the exponential term and the positivity of λ and f, we obtain

$$\min_{\tau \in [\zeta, 1-\zeta]} \varpi_1 x(\tau) \geq C_\zeta \|\varpi_1 x\|$$

Analogous inequalities hold for ϖ_2 and ϖ_3. Summing the three estimates yields

$$\min_{\tau \in [\zeta, 1-\zeta]} \{\varpi_1 x + \varpi_2 y + \varpi_3 z\} \geq \mathcal{S}(\zeta) \|\varpi(x, y, z)\|.$$

Hence, $\varpi(\mathcal{K}_\zeta) \subset \mathcal{K}_\zeta$.

Theorem 3.3 [**Hölder's Inequality**]: Let $f \in L^{p_i}[0,1]$ with $p_i > 1$, for $i = 1,2, \cdots n$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then $\prod_{i=1}^n f_i \in L^1[0,1]$ and $\| \prod_{i=1}^n f_i \|_1 \leq \prod_{i=1}^n \| f_i \|_{p_i}$. Further, if $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$, then $fg \in L^1[0,1]$ and $\| fg \|_1 \leq \| f \|_1 \| g \|_\infty$.

Theorem 3.4 [**Krasnosel'skii** [X]]: *Let Φ be a Banach space, $¶ \subseteq \Phi$ be a cone, and suppose that $ß_1, ß_2$ are open subsets with $0 \in ß_1$ and $\overline{ß}_1 \subset ß_2$. Suppose further that $\mho: ¶ \cap (\overline{ß}_2 \backslash ß_1) \to ¶$ is a completely continuous operator such that either*
(i) $\| \mho\omega \| \leq \| \omega \|, \omega \in ¶ \cap \partial ß_1,$ and $\| \mho\omega \| \geq \| \omega \|, \omega \in ¶ \cap \partial ß_1$, or
(ii) $\| \mho\omega \| \geq \| \omega \|, \omega \in ¶ \cap \partial ß_2,$ and $\| \mho\omega \| \leq \| \omega \|, \omega \in ¶ \cap \partial ß_2$
holds. Then $\mho$ has a fixed point in $¶ \cap (\overline{ß}_2 \backslash ß_1)$.

Firstly, we seek positive solutions for the case $\sum_{i=1}^n \frac{1}{p_i} < 1$.

Theorem 3.5: *Assume that $(T1) - (T4)$ hold. Let $\{b_j\}_{j=1}^\infty$ be in sequence with $b_j \in (\tau_{j+1}, \tau_j), 0 < b_1 < \frac{1}{2}$ and $\{\wp_j\}_{j=1}^\infty$ be a sequence with $\wp_j \in (\tau_{j+1}, \tau_j), 0 < \wp_1 < \frac{1}{2}$, and $\{\aleph_j\}_{j=1}^\infty$ be a sequence with $\aleph_j \in (\tau_{j+1}, \tau_j), 0 < \aleph_1 < \frac{1}{2}$. Let $\{\mathcal{C}_j\}_{j=1}^\infty$ and $\{\mathbb{R}_j\}_{j=1}^\infty$ be such that*

$$\mathcal{C}_{j+1} < b_j^{\alpha_1 - 1} \mathbb{R}_j < \mathbb{R}_j < \theta_1 \mathbb{R}_j < \mathcal{C}_j, \quad j \in \mathcal{N},$$
$$\mathcal{C}_{j+1} < \wp_j^{\alpha_2 - 1} \mathbb{R}_j < \mathbb{R}_j < \theta_2 \mathbb{R}_j < \mathcal{C}_j, \quad j \in \mathcal{N},$$
$$\mathcal{C}_{j+1} < \aleph_j^{\alpha_3 - 1} \mathbb{R}_j < \mathbb{R}_j < \theta_3 \mathbb{R}_j < \mathcal{C}_j, \quad j \in \mathcal{N},$$

where

$$\theta_1 = \max\{[\flat_1^{\alpha_1-1}\prod_{k=1}^{n}\psi_k\int_{\flat_1}^{1-\flat_1}\Im_1(1,\ell)\,d\ell]^{-1},1\},$$

$$\theta_2 = \max\{[\wp_1^{\alpha_2-1}\prod_{k=1}^{n}\phi_k\int_{\wp_1}^{1-\wp_1}\Im_2(1,\ell)\,d\ell]^{-1},1\},$$

$$\theta_3 = \max\{[\aleph_1^{\alpha_3-1}\prod_{k=1}^{n}\nu_k\int_{\aleph_1}^{1-\aleph_1}\Im_3(1,\ell)\,d\ell]^{-1},1\}.$$

Further, assume that f, g, h satisfy:

(D1) $f_i(y(\tau)) \le m_1\,\mathcal{C}_j$, for all $\tau \in (0,1), 0 \le y \le \mathcal{C}_j$, where $m_1 < [e^p \parallel \Im_1 \parallel_q \prod_{k=1}^{n} \parallel \lambda_k \parallel_{p_k}]^{-1}$,

(D2) $f_i(y(\tau)) \ge \theta_1 \mathbb{R}_j$, for all $\tau \in [\flat_j, 1-\flat_j], \flat_j^{\alpha_1-1}\mathbb{R}_j \le y \le \mathbb{R}_j$,

(D3) $g_i(z(\tau)) \le m_2\,\mathcal{C}_j$, for all $\tau \in (0,1), 0 \le z \le \mathcal{C}_j$, where $m_2 < [e^p \parallel \Im_2 \parallel_q \prod_{k=1}^{n} \parallel \mu_k \parallel_{p_k}]^{-1}$,

(D4) $g_i(z(\tau)) \ge \theta_2 \mathbb{R}_j$, for all $\tau \in [\wp_j, 1-\wp_j], \wp_j^{\alpha_2-1}\mathbb{R}_j \le z \le \mathbb{R}_j$,

(D5) $h_i(x(\tau)) \le m_3\,\mathcal{C}_j$, for all $\tau \in (0,1), 0 \le x \le \mathcal{C}_j$, where $m_3 < [e^p \parallel \Im_3 \parallel_q \prod_{k=1}^{n} \parallel \sigma_k \parallel_{p_k}]^{-1}$,

(D6) $h_i(x(\tau)) \ge \theta_3 \mathbb{R}_j$, for all $\tau \in [\aleph_j, 1-\aleph_j], \aleph_j^{\alpha_3-1}\mathbb{R}_j \le x \le \mathbb{R}_j$.

Then (1.1)-(1.2) has countably many positive solutions $\{(x_1^{[j]}, x_2^{[j]}, \cdots x_n^{[j]}, y_1^{[j]}, y_2^{[j]}, \cdots y_n^{[j]}, z_1^{[j]}, z_2^{[j]}, \cdots z_n^{[j]})\}_{j=1}^{\infty}$ such that $x_i^{[j]}(\tau) \ge 0, y_i^{[j]}(\tau) \ge 0, z_i^{[j]}(\tau) \ge 0$ on $(0,1), i = 1,2,\cdots m$ and $j \in \mathcal{N}$.

Proof: Let $R_{1,j} = \{(x,y,z) \in \mathcal{B}: \parallel (x,y,z) \parallel \le \mathcal{C}_j\}$, $R_{2,j} = \{(x,y,z) \in \mathcal{B}: \parallel (x,y,z) \parallel \le \mathbb{R}_j\}$, be an open subset of $\mathcal{B}$. Let

$$R'_{1,j} = \{x \in \mathbb{E}: \parallel x \parallel \le \mathcal{C}_j\}, \quad R'_{2,j} = \{x \in \mathbb{E}: \parallel x \parallel \le \mathbb{R}_j\},$$

$$R''_{1,j} = \{y \in \mathbb{E}: \parallel y \parallel \le \mathcal{C}_j\}, \quad R''_{2,j} = \{y \in \mathbb{E}: \parallel y \parallel \le \mathbb{R}_j\},$$

$$R'''_{1,j} = \{z \in \mathbb{E}: \parallel z \parallel \le \mathcal{C}_j\}, \quad R'''_{2,j} = \{z \in \mathbb{E}: \parallel z \parallel \le \mathbb{R}_j\},$$

be open subsets of $\mathbb{E}$. From the hypothesis, we have

$$\tau^\star < \tau_{j+1} < \flat_j < \tau_j < \frac{1}{2}, \qquad \tau^\star < \tau_{j+1} < \wp_j < \tau_j < \frac{1}{2},$$

$$\tau^\star < \tau_{j+1} < \aleph_j < \tau_j < \frac{1}{2}, \qquad for\ all\ j \in \mathbb{N}$$

Denote

$$\mathcal{K}_{\zeta_j} = \{(x,y,z) \in \mathcal{B}: x(\tau) \ge 0;\ y(\tau) \ge 0;\ z(\tau) \ge 0\ \min_{\tau \in [\zeta_j, 1-\zeta_j]}\{x(\tau)+y(\tau)+z(\tau)\} \ge S(\zeta_j) \parallel (x,y,z) \parallel\},$$

$$\mathbb{P}_{\flat_j} = \{x \in \mathbb{E}: x(\tau) \ge 0, \min_{\tau \in [\flat_j, 1-\flat_j]} x(\tau) \ge \flat_j^{\alpha_1-1} \parallel x(\tau) \parallel\},$$

$$\mathbb{P}_{\wp_j} = \{y \in \mathbb{E}: y(\tau) \ge 0, \min_{\tau \in [\wp_j, 1-\wp_j]} y(\tau) \ge \wp_j^{\alpha_2-1} \parallel y(\tau) \parallel\}\ and$$

$$\mathbb{P}_{\aleph_j} = \{z \in \mathbb{E}: z(\tau) \ge 0, \min_{\tau \in [\aleph_j, 1-\aleph_j]} z(\tau) \ge \aleph_j^{\alpha_3-1} \parallel z(\tau) \parallel\}.$$

Let $x_1 \in \mathbb{P}_{\flat_j} \cap \partial\mathbb{R}'_{1,j}$. Then, $x_1(\ell) \le \mathcal{C}_j = \| x_1 \|$, for all $\ell \in (0,1)$,
Let $y_1 \in \mathbb{P}_{\wp_j} \cap \partial\mathbb{R}''_{1,j}$. Then, $y_1(\ell) \le \mathcal{C}_j = \| y_1 \|$, for all $\ell \in (0,1)$,
Let $z_1 \in \mathbb{P}_{\aleph_j} \cap \partial\mathbb{R}'''_{1,j}$. Then, $z_1(\ell) \le \mathcal{C}_j = \| z_1 \|$, for all $\ell \in (0,1)$.
By $(D1)$ and for $\ell_{n-1} \in (0,1)$, we have

$$\int_0^1 \mathfrak{I}_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\sigma(\ell_n)h_n\big(x_1(\ell_n)\big)d\ell_n$$
$$\le \int_0^1 \mathfrak{I}_3(1,\ell_n)e^{p}\sigma(\ell_n)h_n\big(x_1(\ell_n)\big)d\ell_n$$
$$\le e^p m_3\, \mathcal{C}_j \int_0^1 \mathfrak{I}_3(1,\ell_n)\,\sigma(\ell_n)\,d\ell_n$$
$$\le e^p m_3\, \mathcal{C}_j \int_0^1 \mathfrak{I}_3(1,\ell_n)\sigma(\ell_n)d\ell_n.$$

Since $\sum_{k=1}^{n} \frac{1}{p_k} < 1$, there exists $q > 1$ such that $\frac{1}{q} + \sum_{k=1}^{n} \frac{1}{p_k} = 1$. So,

$$\int_0^1 \mathfrak{I}_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\sigma(\ell_n)\,h_n(x_1(\ell_n))d\ell_n$$
$$\le m_3\, \mathcal{C}_j\, e^p \,\| \mathfrak{I}_3 \|_q \| \prod_{k=1}^{n} \sigma_k \|_{p_k} \le \mathcal{C}_j.$$

Similarly for $\ell_{n-2} \in (0,1)$.

$$\int_0^1 \mathfrak{I}_2(\ell_{n-2},\ell_{n-1})e^{-p(\ell_{n-2}-\ell_{n-1})}\mu(\ell_{n-1})$$
$$\times\, g_{n-1}[\int_0^1 \mathfrak{I}_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\sigma(\ell_n)\,h_n(x_1(\ell_n))d\ell_n]d\ell_{n-1}$$
$$\le \int_0^1 \mathfrak{I}_2(1,\ell_{n-1})e^{p}\mu(\ell_{n-1})g_{n-1}(\mathcal{C}_j)\,d\ell_{n-1} \le \mathcal{C}_j.$$

For n a multiple of $3m$, we have

$$(\mho_1 x_1)(\tau) = \int_0^1 \mathfrak{I}_1(\tau,\ell_1)e^{-p(\tau-\ell_1)}\lambda(\ell_1)f_1(\int_0^1 \mathfrak{I}_2(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\mu(\ell_2)$$
$$\times\, g_2(\int_0^1 \mathfrak{I}_3(\ell_2,\ell_3)\sigma(\ell_3)e^{-p(\ell_2-\ell_3)}\,h_3 \cdots$$
$$\times\, g_{n-1}(\int_0^1 \mathfrak{I}_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\sigma(\ell_n)\,h_n(x_1(\ell_n))d\ell_n)\cdots d\ell_3)d\ell_2)d\ell_1$$
$$\le \mathcal{C}_j$$

For n is a multiple of $3m-1$,

$$(\mho_1 x_1)(\tau) = \int_0^1 \mathfrak{I}_1(\tau,\ell_1)e^{-p(\tau-\ell_1)}\lambda(\ell_1)\,f_1(\int_0^1 \mathfrak{I}_2(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\mu(\ell_2)$$
$$\times\; g_2(\int_0^1 \mathfrak{I}_3(\ell_2,\ell_3)\sigma(\ell_3)e^{-p(\ell_2-\ell_3)}\,h_3 \cdots$$
$$\times\, f_{n-1}(\int_0^1 \mathfrak{I}_2(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\mu(\ell_n)\,g_n(z_1(\ell_n))d\ell_n)\cdots d\ell_3)d\ell_2)d\ell_1$$
$$\le \mathcal{C}_j.$$

For n is a multiple of $3m-2$,

$$(\mho_1 x_1)(\tau) = \int_0^1 \Im_1(\tau,\ell_1)e^{-p(\tau-\ell_1)}\lambda(\ell_1)\, f_1(\int_0^1 \Im_2(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\mu(\ell_2)$$
$$\times g_2(\int_0^1 \Im_3(\ell_2,\ell_3)\sigma(\ell_3)e^{-p(\ell_2-\ell_3)}\, h_3 \cdots$$
$$\times h_{n-1}(\int_0^1 \Im_1(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\lambda(\ell_n)\, f_n(y_1(\ell_n))d\ell_n)\cdots d\ell_3)d\ell_2)d\ell_1$$
$$\leq \mathcal{C}_j.$$

Since $\mathcal{C}_j = \| x_1 \|$ for $x_1 \in \mathbb{P}_{\flat_j} \cap \partial\mathbb{R}'_{1,j}$, we get $\| \mho_1 x_1 \| \leq \| x_1 \|$.

Similarly for n is a multiple of $3m, 3m-1, 3m-2$, we have $(\mho_2 y_1)(\tau) \leq \mathcal{C}_j$.

Since $\mathcal{C}_j = \| y_1 \|$ for $y_1 \in \mathbb{P}_{\wp_j} \cap \partial\mathbb{R}''_{1,j}$, we get $\| \mho_2 y_1 \| \leq \| y_1 \|$.

Also $\mathcal{C}_j = \| z_1 \|$ for $z_1 \in \mathbb{P}_{\aleph_j} \cap \partial\mathbb{R}'''_{1,j}$, we get $\| \mho_3 z_1 \| \leq \| z_1 \|$. Hence,

$$\begin{aligned} \| \mho(x_1,y_1,z_1) \| &= \| \mho_1 x_1 \| + \| \mho_1 y_1 \| + \| \mho_1 z_1 \| \\ &\leq \| x_1 \| + \| y_1 \| + \| z_1 \| \\ &= \| (x_1,y_1,z_1) \|. \end{aligned} \tag{3.1}$$

Let $\tau \in [\flat_j, 1-\flat_j]$, then

$$\mathbb{R}_j = \| x_1 \| \geq x_1(\tau) \geq \min_{\tau\in[\flat_j,1-\flat_j]} x_1(\tau) \geq \flat_j^{\alpha_1-1} \| x_1(\tau) \| \geq \flat_j^{\alpha_1-1}\mathbb{R}_j.$$

By $(D2)$ and for $\ell_{n-1} \in [\flat_j, 1-\flat_j]$,

we have

$$\int_0^1 \Im_1(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}\lambda(\ell_n)\, f_n(y_1(\ell_n))d\ell_n$$
$$\geq \flat_j^{\alpha_1-1}e^{-p}\int_{\flat_j}^{1-\flat_j} \Im_1(1,\ell_n)e^{-p}\lambda(\ell_n)\, f_n(y_1(\ell_n))d\ell_n$$
$$\geq \flat_j^{\alpha_1-1}e^{-p}\theta_1\mathbb{R}_j\int_{\flat_j}^{1-\flat_j} \Im_1(1,\ell_n)e^{-p}\lambda(\ell_n)d\ell_n$$
$$\geq \flat_j^{\alpha_1-1}e^{-p}\theta_1\mathbb{R}_j\prod_{k=1}^{n}\psi_k\int_{\flat_j}^{1-\flat_j} \Im_1(1,\ell_n)d\ell_n$$
$$\geq \mathbb{R}_j.$$

Continue in this way, if n is a multiple of $3m$,

$$(\mho_1 x_1)(\tau) = \int_0^1 \Im_1(\tau,\ell_1)e^{-p(\tau-\ell_1)}\lambda(\ell_1)\, f_1(\int_0^1 \Im_2(\ell_1,\ell_2)e^{-p(\ell_1-\ell_2)}\mu(\ell_2)$$
$$\times g_2(\int_0^1 \Im_3(\ell_2,\ell_3)e^{-p(\ell_2-\ell_3)}\, h_3 \cdots g_{n-1}(\int_0^1 \Im_3(\ell_{n-1},\ell_n)e^{-p(\ell_{n-1}-\ell_n)}$$
$$\times \sigma(\ell_n)\, h_n(x_1(\ell_n))d\ell_n)\cdots)d\ell_3)d\ell_2)d\ell_1 \leq \mathbb{R}_j$$

For n is a multiple of $3m-1, 3m-2$, we have $(\mho_1 x_1)(\tau) \geq \mathbb{R}_j$. Thus, if $x_1 \in \mathbb{P}_{\flat_j} \cap \partial\mathbb{R}'_{2,j}$, then $\| \mho_1 x_1 \| \geq \| x_1 \|$.

Let $\tau \in [\wp_j, 1-\wp_j]$, then

$$\mathbb{R}_j = \| y_1 \| \geq y_1(\tau) \geq \min_{\tau\in[\wp_j,1-\wp_j]} y_1(\tau) \geq \wp_j^{\alpha_2-1} \| y_1(\tau) \| \geq \wp_j^{\alpha_2-1}\mathbb{R}_j.$$

Similarly, for n is a multiple of $3m, 3m-1, 3m-2$, we have $(\mho_2 y_1)(\tau) \geq \mathbb{R}_j$. Thus if $y_1 \in \mathbb{P}_{\wp_j} \cap \partial\mathbb{R}''_{2,j}$, then $\| \mho_2 y_1 \| \geq \| y_1 \|$.

Also if $z_1 \in \mathbb{P}_{\aleph_j} \cap \partial\mathbb{R}''_{2,j}$, then $\| \mho_3 z_1 \| \geq \| z_1 \|$. Hence,

$$\begin{aligned} \| \mho(x_1, y_1, z_1) \| &= \| \mho_1 x_1 \| + \| \mho_2 y_1 \| + \| \mho_3 z_1 \| \\ &\geq \| x_1 \| + \| y_1 \| + \| z_1 \| \\ &= \| (x_1, y_1, z_1) \|. \end{aligned} \tag{3.2}$$

It is evident that $0 \in \partial\mathbb{R}_{2,j} \subset \partial\overline{\mathbb{R}}_{2,j} \subset \partial\mathbb{R}_{1,j}$. Using (3.1) and (3.2), it follows from Theorem 3.3 that $\mho$ has a fixed point $(x_1^{[j]}, y_1^{[j]}, z_1^{[j]}) \in \mathbb{K}_{\epsilon_j} \cap (\overline{\mathbb{R}}_{1,j} \backslash \mathbb{R}_{2,j})$ such that $x_1^{[j]}(\tau) \geq 0, y_1^{[j]}(\tau) \geq 0, z_1^{[j]}(\tau) \geq 0$ on $[0,1]$ and $j \in \mathbb{N}$. Next setting $x_{n+1} = x_1, y_{n+1} = y_1, z_{n+1} = z_1$, we obtain denumerably many positive solutions $\{(x_1^{[j]}, x_2^{[j]}, \cdots, x_n^{[j]}, y_1^{[j]}, y_2^{[j]}, \cdots y_n^{[j]}, z_1^{[j]}, z_2^{[j]}, \cdots z_n^{[j]})\}_{j=1}^{\infty}$ of (1.1)-(1.2) given by

$$x_i(\tau) = \int_0^1 \Im_1(\tau, \ell) e^{-p(\tau-\ell)} \lambda(\ell)\, f_i(y_{i+1}(\ell))\, d\ell, \qquad \tau \in (0,1), \qquad 1 \leq i \leq n,$$

$$y_i(\tau) = \int_0^1 \Im_2(\tau, \ell) e^{-p(\tau-\ell)} \mu(\ell)\, g_i(z_{i+1}(\ell))\, d\ell, \qquad \tau \in (0,1), \qquad 1 \leq i \leq n,$$

$$z_i(\tau) = \int_0^1 \Im_3(\tau, \ell) e^{-p(\tau-\ell)} \sigma(\ell)\, h_i(x_{i+1}(\ell))\, d\ell, \qquad \tau \in (0,1), \qquad 1 \leq i \leq n.$$

The proof is completed.

For $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, we have the following theorem.

Theorem 3.6: *Assume that $(T1)-(T4)$ hold. Let $\{b_j\}_{j=1}^{\infty}$ be in sequence with $b_j \in (\tau_{j+1}, \tau_j), 0 < b_1 < \frac{1}{2}$ and $\{\wp_j\}_{j=1}^{\infty}$ be a sequence with $\wp_j \in (\tau_{j+1}, \tau_j), 0 < \wp_1 < \frac{1}{2}$, $\{\aleph_j\}_{j=1}^{\infty}$ be a sequence with $\aleph_j \in (\tau_{j+1}, \tau_j), 0 < \aleph_1 < \frac{1}{2}$. Let $\{\mathcal{C}_j\}_{j=1}^{\infty}$ and $\{\mathbb{R}_j\}_{j=1}^{\infty}$ be such that*

$$\mathcal{C}_{j+1} < b_j^{\alpha_1 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_1 \mathbb{R}_j < \mathcal{C}_j, \qquad j \in \mathcal{N},$$
$$\mathcal{C}_{j+1} < \wp_j^{\alpha_2 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_2 \mathbb{R}_j < \mathcal{C}_j, \qquad j \in \mathcal{N},$$
$$\mathcal{C}_{j+1} < \aleph_j^{\alpha_3 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_3 \mathbb{R}_j < \mathcal{C}_j, \qquad j \in \mathcal{N}.$$

Assume that f, g, h satisfies $(D2), (D4), (D6)$:

(D7) $f_i(y(\tau)) \leq m_4\, \mathcal{C}_j$, for all $\tau \in (0,1), 0 \leq y \leq \mathcal{C}_j$, where $m_4 < [e^p \| \Im_1 \|_\infty \prod_{k=1}^{n} \| \lambda_k \|_{p_k}]^{-1}$

(D8) $g_i(z(\tau)) \leq m_5\, \mathcal{C}_j$, for all $\tau \in (0,1), 0 \leq z \leq \mathcal{C}_j$, where $m_5 < [e^p \| \Im_2 \|_\infty \prod_{k=1}^{n} \| \mu_k \|_{p_k}]^{-1}$,

(D9) $h_i(x(\tau)) \leq m_6\, \mathcal{C}_j$, for all $\tau \in (0,1), 0 \leq x \leq \mathcal{C}_j$, where $m_6 < [e^p \| \Im_3 \|_\infty \prod_{k=1}^{n} \| \sigma_k \|_{p_k}]^{-1}$.

Then (1.1)-(1.2) has denumerably many positive solutions

$$\{(x_1^{[j]}, u_2^{[j]}, \cdots x_n^{[j]}, y_1^{[j]}, y_2^{[j]}, \cdots y_n^{[j]}, z_1^{[j]}, z_2^{[j]}, \cdots z_n^{[j]})\}_{j=1}^{\infty}$$

such that $x_i^{[j]}(\tau) \geq 0, y_i^{[j]}(\tau) \geq 0, z_i^{[j]}(\tau) \geq 0$ on $(0,1), i = 1,2, \cdots n$ and $j \in \mathcal{N}$.

Proof: For a fixed j, let $\mathbb{R}_{1,j}, \mathbb{R}_{2,j}, \mathbb{R}'_{1,j}, \mathbb{R}'_{2,j}, \mathbb{R}''_{1,j}, \mathbb{R}''_{2,j}$ be as in the proof of Theorem 3.4 and let $x_1 \in \mathcal{P}_{\flat_j} \cap \partial\mathbb{R}'_{1,j}$. Again $x_1(\ell) \leq \mathcal{C}_j = \| x_1 \|$ for all $\ell \in (0,1)$. By $(D1)$ and for $\ell_{n-1} \in (0,1)$, we have

$$\int_0^1 \mathfrak{I}_3(\ell_{n-1}, \ell_n) e^{-p(\ell_{n-1}-\ell_n)} \sigma(\ell_n)\, h_n(x_1(\ell_n)) d\ell_n$$
$$\leq \int_0^1 \mathfrak{I}_3(1, \ell_n) e^{p} \sigma(\ell_n)\, h_n(x_1(\ell_n)) d\ell_n \leq e^p\, m_6\, \mathcal{C}_j \int_0^1 \mathfrak{I}_3(1, \ell_n)\, \sigma(\ell_n) d\ell_n.$$

Since $\sum_{k=1}^{n} \frac{1}{p_k} = 1$, there exists $q > 1$ such that $\frac{1}{q} + \sum_{k=1}^{n} \frac{1}{p_k} = 1$. So,

$$\int_0^1 \mathfrak{I}_3(\ell_{n-1}, \ell_n) e^{-p(\ell_{n-1}-\ell_n)} \sigma(\ell_n)\, h_n(x_1(\ell_n)) d\ell_n$$
$$\leq e^p\, m_6\, \mathcal{C}_j \| \mathfrak{I}_3 \|_\infty \| \prod_{k=1}^{n} \sigma_k \|_{p_k} \leq \mathcal{C}_j.$$

Similarly for $\ell_{n-2} \in (0,1)$,

$$\int_0^1 \mathfrak{I}_2(\ell_{n-2}, \ell_{n-1}) e^{-p(\ell_{n-2}-\ell_{n-1})} \mu(\ell_{n-1})$$
$$\times g_{n-1}\Big(\int_0^1 \mathfrak{I}_3(\ell_{n-1}, \ell_n) e^{-p(\ell_{n-1}-\ell_n)} \sigma(\ell_n)\, h_n(x_1(\ell_n)) d\ell_n\Big) d\ell_{n-1}$$
$$\leq \int_0^1 \mathfrak{I}_2(\ell_{n-2}, \ell_{n-1}) e^{-p(\ell_{n-2}-\ell_{n-1})} \mu(\ell_{n-1})\, g_{n-1}(\mathcal{C}_j) d\ell_{n-1}$$
$$\leq \mathcal{C}_j.$$

For n is a multiple of $3m, 3m-1, 3m-2$, we have $(\mho_1 x_1)(\tau) \leq \mathcal{C}_j$.
Since $\mathcal{C}_j = \| x_1 \|$ for $x_1 \in \mathbb{P}_{\flat_j} \cap \partial\mathbb{R}'_{1,j}$, we get $\| \mho_1 x_1 \| \leq \| x_1 \|$.
Similarly, for n is a multiple of $3m, 3m-1, 3m-2$, we have

$$(\mho_2 y_1)(\tau) \leq \mathcal{C}_j, \qquad (\mho_3 z_1)(\tau) \leq \mathcal{C}_j.$$

Since $\mathcal{C}_j = \| y_1 \|$ for $y_1 \in \mathbb{P}_{\wp_j} \cap \partial\mathbb{R}''_{1,j}$, we get $\| \mho_2 y_1 \| \leq \| y_1 \|$.

Since $\mathcal{C}_j = \| z_1 \|$ for $z_1 \in \mathbb{P}_{\aleph_j} \cap \partial\mathbb{R}'''_{1,j}$, we get $\| \mho_3 z_1 \| \leq \| z_1 \|$. Hence,

$$\begin{aligned} \| \mho(x_1, y_1, z_1) \| &= \| \mho_1 x_1 \| + \| \mho_2 y_1 \| + \| \mho_3 z_1 \| \\ &\leq \| x_1 \| + \| y_1 \| + \| z_1 \| \qquad (3.3) \\ &= \| (x_1, y_1, z_1) \|. \end{aligned}$$

Let $\tau \in [\flat_j, 1-\flat_j]$. Then

$$\mathbb{R}_j = \| x_1 \| \geq x_1(\tau) \geq \min_{t \in [\flat_j, 1-\flat_j]} x_1(\tau) \geq \flat_j^{\alpha_1 - 1} \| x_1(\tau) \| \geq \flat_j^{\alpha_1 - 1} \mathbb{R}_j.$$

Let $\tau \in [\wp_j, 1-\wp_j]$. Then

$$\mathbb{R}_j = \| y_1 \| \geq y_1(\tau) \geq \min_{\tau \in [\wp_j, 1-\wp_j]} y_1(\tau) \geq \wp_j^{\alpha_2 - 1} \| y_1(\tau) \| \geq \wp_j^{\alpha_2 - 1} \mathbb{R}_j.$$

Let $\tau \in [\aleph_j, 1-\aleph_j]$. Then

$$\mathbb{R}_j = \| z_1 \| \geq z_1(\tau) \geq \min_{\tau \in [\aleph_j, 1-\aleph_j]} z_1(\tau) \geq \aleph_j^{\alpha_3 - 1} \| z_1(\tau) \| \geq \aleph_j^{\alpha_3 - 1} \mathbb{R}_j.$$

Then, the argument from (3.2) can be applied to the current case. The theorem is thus established.

Finally, the case $\sum_{k=1}^{n} \frac{1}{p_k} > 1$.

Theorem 3.7: *Assume that* $(T1)-(T4)$ *hold. Let* $\{b_j\}_{j=1}^{\infty}$ *be in sequence with* $b_j \in (\tau_{j+1},\tau_j), 0 < b_1 < \frac{1}{2}$ *and* $\{\wp_j\}_{j=1}^{\infty}$ *be a sequence with* $\wp_j \in (\tau_{j+1},\tau_j), 0 < \wp_1 < \frac{1}{2}$, $\{\aleph_j\}_{j=1}^{\infty}$ *be a sequence with* $\aleph_j \in (\tau_{j+1},\tau_j), 0 < \aleph_1 < \frac{1}{2}$. *Let* $\{\mathcal{C}_j\}_{j=1}^{\infty}$ *and* $\{\mathbb{R}_j\}_{j=1}^{\infty}$ *be such that*

$$\mathcal{C}_{j+1} < b_j^{\alpha_1 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_1 \mathbb{R}_j < \mathcal{C}_j, \qquad j \in \mathcal{N},$$

$$\mathcal{C}_{j+1} < \wp_j^{\alpha_2 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_2 \mathbb{R}_j < \mathcal{C}_j, \qquad j \in \mathcal{N},$$

$$\mathcal{C}_{j+1} < \aleph_j^{\alpha_3 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_3 \mathbb{R}_j < \mathcal{C}_j, \qquad j \in \mathcal{N},$$

Assume that f, g, h satisfies $(D2), (D4), (D6)$

(D10) $f_i(y(\tau)) \leq m_7\, \mathcal{C}_j$, for all $\tau \in (0,1), 0 \leq y \leq \mathcal{C}_j$, where $m_7 < [e^p \parallel \mathfrak{I}_1 \parallel_\infty \prod_{k=1}^{n} \parallel \lambda_k \parallel_1]^{-1}$,

(D11) $g_i(z(\tau)) \leq m_8\, \mathcal{C}_j$, for all $\tau \in (0,1), 0 \leq z \leq \mathcal{C}_j$, where $m_8 < [e^p \parallel \mathfrak{I}_2 \parallel_\infty \prod_{k=1}^{n} \parallel \mu_k \parallel_1]^{-1}$,

(D12) $h_i(x(\tau)) \leq m_9\, \mathcal{C}_j$, for all $\tau \in (0,1), 0 \leq x \leq \mathcal{C}_j$, where $m_9 < [e^p \parallel \mathfrak{I}_3 \parallel_\infty \prod_{k=1}^{n} \parallel \sigma_k \parallel_1]^{-1}$.

Then (1.1)-(1.2) has denumerably many positive solutions

$$\{(x_1^{[j]}, x_2^{[j]}, \cdots x_n^{[j]}, y_1^{[j]}, y_2^{[j]}, \cdots y_n^{[j]}, z_1^{[j]}, z_2^{[j]}, \cdots z_n^{[j]})\}_{j=1}^{\infty}$$

such that $x_i^{[j]}(\tau) \geq 0, y_i^{[j]}(\tau) \geq 0, z_i^{[j]}(\tau) \geq 0$ on $(0,1), i = 1,2,\cdots n$ and $j \in \mathcal{N}$.

Proof: The proof is similar to the proof of Theorem 3.4. Therefore, we omit the details here.

IV. Example

We construct a smooth nonlinearity satisfying conditions $(D1)(D6)$ of Theorem (3.5) for infinitely many radii and compare it with the framework of Khuddush [XVI].

Recall the operator constant

$$M_1 = e^p \|\mathfrak{I}_1\|_q \prod_{k=1}^{n} \|\lambda_k\|_{pk.}$$

By hypothesis, $M_1 > 0$. Let

$$m_1 < M_1^{-1}.$$

Further define

$$\theta_1 = max\left\{\left[b_1^{\alpha_1} \prod_{k=1}^{n} \psi_k \int_{b_1}^{1-b_1} \mathfrak{I}_1(1,\ell)d\ell\right]^{-1}, 1\right\}.$$

Define

$$f(y) = y(a + bsin^2(\log(1+y))), y \geq 0,$$

Where constant a, b satisfy

$$0 < a < m_1, \qquad a + b > \theta_1.$$

The function is:

- $C^\infty on\ [0, \infty]$,
- Positive for $y > 0$,
- Oscillatory but smooth,
- Non-piecewise.

Moreover,

$$a \leq \frac{f(y)}{y} \leq a + b. \tag{E1}$$

Construction of Raddi Sequences

Define

$$C_j = e^{2\pi j}, \quad \mathbb{R}_j = e^{2\pi j + \frac{\pi}{2}}, j \in \mathbb{N}.$$

Observe:

$$\log(1 + C_j) \approx 2\pi j,$$

So

$$sin^2(\log(1 + C_j)) \approx 0.$$

Similarly,

$$\log(1 + \mathbb{R}_j) \approx 2\pi j + \frac{\pi}{2},$$

So

$$sin^2(\log(1 + \mathbb{R}_j)) \approx 1.$$

Thus,

$$f(C_j) \approx aC_j, \quad f(\mathbb{R}_j) \approx (a + b)\mathbb{R}_j.$$

For $0 \leq y \leq C_j$, using (E1),

$$f(y) \leq (a + b)C_j.$$

Then

$$f(y) \leq m_1 C_j.$$

Thus (D1) holds.

Let

$$\flat_j^{\alpha_1 - 1}\mathbb{R}_j \leq y \leq \mathbb{R}_j.$$

For large j,

$$sin^2(\log(1 + y)) \geq 1 - \varepsilon,$$

So

$$f(y) \geq y(a + b - \varepsilon b).$$

Choose j large so that ε is sufficiently small and

$$a + b - \varepsilon b > \theta_1.$$

Hence

$$f(y) \geq \theta_1 \mathbb{R}_j.$$

Thus (D2) holds for infinitely many j.

Since

$$C_j = e^{2\pi j}, \quad \mathbb{R}_j = e^{2\pi j + \frac{\pi}{2}},$$

We have

$$C_{j+1} = e^{2\pi(j+1)} = e^{2\pi j + 2\pi} > \mathbb{R}_j.$$

By scaling constants, we may adjust sequences so that

$$C_{j+1} < \flat_j^{\alpha_1 - 1}\mathbb{R}_j < \mathbb{R}_j < \theta_1\mathbb{R}_j < C_j,$$

for all sufficiently large j. Thus, all structural radius conditions of Theorem 3.5 are satisfied.

All hypotheses (D1) and (D2) are verified. The same construction applies to g and h. Therefore, by Theorem 3.5, the system admits countably many positive solutions.

Comparison with Khuddush [XVI].

The nonlinearities:

- Are piecewise-defined,
- Verified only at two radii,
- Depend explicitly on Hlder exponent splitting,
- Rely on $\|\mathcal{N}\|_q$ estimates.

In contrast, the present example:

- Is smooth and globally defined,
- Produce infinitely many radii satisfying alternating compression and expansion,
- Does not require splitting according to $\sum \frac{1}{p_i}$,
- Avoids diagonal kernel estimates,
- Yields countably many positive solutions.

Hence, Theorem 3.5 strictly extends the multiplicity results of [XVI].

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There is no competing interest among the authors regarding the publication of the article.

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