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ENHANCED FIXED POINT RESULTS IN G-METRIC SPACES VIA MANN ITERATION AND RATIONAL-TYPE CONTRACTIONS

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Abstract

In this work, we use the Mann iteration process rather than the conventional Picard operator to extend fixed point findings in G-metric spaces. Mann iteration is known to provide better convergence properties and stability in fixed point approximations, particularly in cases where Picard iteration fails due to weak contractive conditions. We present a new family of rational-type contractive conditions and prove the existence and uniqueness of fixed points of single-valued mappings in G-complete G metric spaces. Specifically, we improve upon existing theorems in the literature both by generalizing their statements as well as strengthening their use through an improved iterative scheme.

Keywords: G-metric space, Fixed point, Mann iteration, Rational contraction, G-convergence, Iterative approximation

I. Introduction

The realm of mathematical analysis continues to find pivotal roles for fixed point theory, significantly influencing the study of differential equations, optimization algorithms, and nonlinear systems. Conventionally, fixed point results are often derived via the Picard iteration process, which generates an iterative sequence defined by:

$$x_{n+1} = T(x_n)$$

Although popular, this approach requires strong contractive hypotheses for convergence. It may not, in particular, give convergence for mappings that are not expansive or only weakly contractive. To alleviate these constraints, the present work employs the Mann iterative method, defined as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n),\ \text{with } \alpha_n \in [0,1]$$

where $\{\alpha_n\}$ is a sequence of control parameters. Unlike Picard's direct application of the mapping T, Mann iteration forms a convex combination of the previous iterate and the image under T, leading to improved stability and convergence behavior, especially under less restrictive contractive conditions. This study generalizes the field by obtaining our fixed point results for single-valued self-maps in the context of G-complete G-metric spaces. Specifically, this technique utilizes a rational-type contractive condition that generalizes the classical Banach contraction by including rational expressions. This provides more flexibility and extends the applicability of fixed-point results to mappings for which classical techniques do not apply. While classical fixed-point results in G-metric spaces are primarily based on linear-type contractions (e.g., Banach, Kannan), our work introduces a more general rational-type contractive condition and extends convergence results using the Mann iteration process. This approach enhances stability and convergence in weakly contractive settings where Picard iteration fails. Moreover, our results encompass and generalize well-known theorems such as those by Mustafa and Sims [XVII], Gaba [VIII], and Yildirim & Khan [VII], as special cases. These improvements form the core novelty of this study. Over the past two decades, the theory of fixed points in generalized metric spaces has witnessed significant developments, particularly in the context of G-metric spaces, which were introduced by Mustafa and Sims [XVII] as a natural generalization of standard metric spaces. Their foundational work provided fixed-point results under Banach-type contractive conditions in complete G-metric spaces. Subsequent efforts by Gaba [VIII, X] further extended these results using new classes of contractive conditions and λ-sequences, exploring convergence behavior under weakened assumptions.

In parallel, Yildirim and Khan [VII] introduced convexity in G-metric spaces and investigated approximation of fixed points via Mann iteration, highlighting its superior convergence in convex settings. While their results laid the foundation for using iterative processes beyond Picard, their framework remains limited to convex G-spaces and standard contraction types. Rational-type contractive conditions originally studied in the context of standard metric, b-metric, and modular metric spaces were introduced to address convergence challenges where linear contractions like Banach and Kannan fail. These include the works of Choudhary & Shukla, Singh & Sharma, and Ghosh & Dutta in various generalizations of metric structures. However, the adaptation of such rational-type contractions to G-metric spaces remains sparse. Motivated by these gaps, our study introduces a rational-type contractive framework tailored for symmetric and G-complete G-metric spaces. Our approach not only unifies classical results (including Banach, Kannan, Chatterjea, and Reich-type contractions) but also extends them by incorporating nonlinear control functions and Mann-type iteration schemes, which enhance convergence in weakly contractive and non-convex environments. Compared to the results of Gaba [VIII], our theorems allow for broader classes of mappings, and unlike the framework of Yildirim and Khan [VII], we do not assume convexity of the G-space. Furthermore, our fixed-point results support adaptive dynamics through the introduction of φ-type nonlinearities, creating a flexible analytic framework for future application to differential and integral equations, optimization problems, and iterative computational methods in nonlinear analysis. The concept of generalized metric spaces has garnered significant attention in recent mathematical literature, particularly in the context of fixed-point theory and its applications [VIII] [XVII]. Building upon the foundational work of Mustafa and Sims [XVII], we present a comprehensive framework for triadic metric structures.

Definition 1.1 [XVII, XXII]

Let $\mathcal{X}$ be a non-empty set, and consider a function

$$\mathcal{G}: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$$

That assigns a non-negative real number to every ordered triple of elements in $\mathcal{X}$. The function $\mathcal{G}$ is said to define a G-metric on $\mathcal{X}$ if it satisfies the following properties for all $x, y, z, r \in \mathcal{X}$:

(GM1) $\mathcal{G}(x, x, x) = 0$, and $\mathcal{G}(x, y, z) = 0$ implies $x = y = z$.

(GM2) $\mathcal{G}(x, x, y) > 0$ whenever $x \neq y$.

(GM3) If $z \neq y$, then

$$\mathcal{G}(x, x, y) \leq \mathcal{G}(x, y, z)$$

(GM4) The function $\mathcal{G}$ is totally symmetric, i.e.,

$$\mathcal{G}(x, y, z) = \mathcal{G}(x, z, y) = \mathcal{G}(y, z, x) = \mathcal{G}(y, x, z) = \mathcal{G}(z, x, y) = \mathcal{G}(z, y, x)$$

(GM5) For any $x, y, z, r \in \mathcal{X}$, the G-triangle inequality holds:

$$\mathcal{G}(x, y, z) \leq \mathcal{G}(x, r, r) + \mathcal{G}(r, y, z)$$

When a function $\mathcal{G}$ fulfills conditions (GM1) through (GM5), the pair ($\mathcal{X}, \mathcal{G}$) is called a G metric space.

Theorem 1.2 [XVII]

In a G-metric space $(\mathcal{X}, \mathcal{G})$, if $\mathcal{G}(a, b, c) = 0$, then $a = b = c$.

This outcome is a direct consequence of the identity property and has been demonstrated in several metric generalizations [8].

Definition 1.3 [XVII]

A sequence $\{x_k\} \subset \mathcal{X}$ is said to G-converge to a point $x \in \mathcal{X}$ if

A point $x \in \mathcal{X}$ is said to G-converge to a sequence $\{x_k\} \subset \mathcal{X}$ if

$$\lim_{i,j \to \infty} \mathcal{G}(x, x_i, x_j) = 0$$

Equivalently, there exists $N \in \mathbb{N}$ such that for every $i, j \geq N$, for every ε>0,

we have $\mathcal{G}(x, x_i, x_j) < \varepsilon$.

This is denoted by $x_k \xrightarrow{G} x$ or $\lim x_k = x$.

Proposition 1.4 [IV, XVII, XX] (Comparison Metric Formulation)

Define the associated metric:

$$d_{\mathcal{G}}(u, v) := \mathcal{G}(u, v, v) + \mathcal{G}(u, u, v).$$

Then, the following criteria are identical for a sequence $\{x_k\} \subset \mathcal{X}$

1. $x_k \to x$ in the G -metric sense.
2. $\lim_{i,j\to\infty} \mathcal{G}(x, x_i, x_j) = 0.$
3. $\lim_{k\to\infty} d_{\mathcal{G}}(x_k, x) = 0.$
4. $\lim_{k\to\infty} \mathcal{G}(x, x_k, x_k) = 0.$
5. $\lim_{k\to\infty} \mathcal{G}(x_k, x, x) = 0.$

Definition 1.5 (G-Cauchy Sequence)

G-Cauchy is a sequence $\{x_k\}$ in $(\mathcal{X}, \mathcal{G})$ if, for any ε>0, there exists $N \in \mathbb{N}$ such that, for every $i, j, k \geq N$,

$$\mathcal{G}(x_i, x_j, x_k) < \varepsilon.$$

This is equivalent to $\lim_{i,j,k\to\infty} \mathcal{G}(x_i, x_j, x_k) = 0.$

This idea extends the concept of a Cauchy sequence, which is consistent with advancements in b-metric spaces [XVII].

Theorem 1.6 (Equivalent Cauchy Conditions)

According to descriptions in generalized metric theory [10,20] The following equivalents may exist for a G-metric space $(\mathcal{X}, \mathcal{G})$

$\{x_k\}$ is a G-Cauchy sequence.

For every $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $i, j \geq M$,

$$\mathcal{G}(x_i, x_j, x_j) < \varepsilon.$$

Definition 1.7 (Symmetric G-metric)

For any u, v in a G-metric space $(\mathcal{X}, \mathcal{G})$, the space is said to be symmetric if

$$\mathcal{G}(u, v, v) = \mathcal{G}(u, u, v)$$

This symmetry condition has proven essential in applications to fractal theory and differential equations [VIII].

Throughout this paper, unless otherwise stated, all fixed-point results are established in symmetric G-metric spaces.

Definition 1.8 (G-completeness)

If every G-Cauchy sequence in $\mathcal{X}$ has a limit in $\mathcal{X}$ under the G-metric, then the space $(\mathcal{X}, \mathcal{G})$ is G-complete. This idea broadens traditional completeness concepts and finds applications in fuzzy fixed-point theory [VII].

Definition 1.9 (Orbitally Continuous Mapping)

Consider a mapping $T: \mathcal{X} \to \mathcal{X}$. If T is orbitally continuous for a series, then $\{x_k\} \subset \mathcal{X}$, defined by $x_{k+1} = T(x_k)$, and $x_k \xrightarrow{G} x$, then it follows that $T(x_k) \xrightarrow{G} T(x)$.

This concept has been extensively studied in the context of rational contractions [II, IV].

Definition 1.10 (Mann Iterative Process in G-metric Spaces)
The Mann iteration generates a sequence $\{x_k\}$ in $\mathcal{X}$ using:

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k), \text{ where } \alpha_k \in [0,1].$$

As a rule, this method is stronger than Picard iteration, particularly in the case of mappings that are not strictly contractive. [7,8].

Lemma 1.11 (Explicit convergence rate for Mann iteration)
Let $(\mathcal{X}, \mathcal{G})$be a symmetric and G -complete G -metric space, and let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ satisfy the contractive condition of Theorem 3.1,3.2, or 3.3, with an associated contraction constant $\lambda \in (0,1)$.
Let $\{x_n\}$ be the Mann iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where the control sequence $\{\alpha_n\} \subset (0,1)$ satisfies

$$\sum_{n=0}^{\infty} \alpha_n = \infty$$

Then the following estimate holds:

$$\mathcal{G}(x_{n+1}, x^*, x^*) \le \prod_{k=0}^{n} (1-\alpha_k(1-\lambda))\mathcal{G}(x_0, x^*, x^*),$$

where x^* is the unique fixed point of T.

Consequently, the rate of convergence of the Mann iteration depends explicitly on both the contraction parameter λ and the control sequence $\{\alpha_n\}$.

Proof
From the Mann iteration and the contractive condition of Theorems 3.1-3.3, we obtain

$$\mathcal{G}(x_{n+1}, x^*, x^*) \le (1-\alpha_n)\mathcal{G}(x_n, x^*, x^*) + \alpha_n \lambda \mathcal{G}(x_n, x^*, x^*).$$

Hence,

$$\mathcal{G}(x_{n+1}, x^*, x^*) \le (1-\alpha_n(1-\lambda))\mathcal{G}(x_n, x^*, x^*).$$

Iterating this inequality yields

$$\mathcal{G}(x_{n+1}, x^*, x^*) \le \prod_{k=0}^{n} (1-\alpha_k(1-\lambda))\mathcal{G}(x_0, x^*, x^*)$$

which completes the proof.

II. Main Results

Theorem 3.1 (Generalized Fixed Point Theorem in G-Metric Spaces, Modified Form)

Let $(\mathcal{X}, \mathcal{G})$ be a symmetric and G -complete G -metric space, and let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a self-map satisfying the rational-type contractive condition:

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) \leq \frac{\alpha\mathcal{G}(u,v,w) + \beta[\mathcal{G}(u,\mathcal{T}u,\mathcal{T}u) + \mathcal{G}(v,\mathcal{T}v,\mathcal{T}v) + \mathcal{G}(w,\mathcal{T}w,\mathcal{T}w)]}{1 + \mathcal{G}(u,v,w)}$$

for all $u, v, w \in \mathcal{X}$, where constants $\alpha, \beta \in [0,1)$.

Consequently, the following findings are valid:

- Existence: The mapping $\mathcal{T}$ has a fixed point $\zeta \in \mathcal{X}$.
- Convergence: The sequence $\{z_k\}$ defined by Mann iteration converges to ζ in the G-metric.
- Uniqueness: The fixed point ζ is unique.

Proof:

We consider the Mann-type iteration defined by:

$$z_{k+1} = (1 - \lambda_k)z_k + \lambda_k\mathcal{T}(z_k)$$

with $\lambda_k \in (0,1)$ such that:

$$\sum_{k=1}^{\infty} \lambda_k = \infty \text{ and } \sum_{k=1}^{\infty} \lambda_k^2 < \infty$$

Define the G-distance:

$$\delta_k = \mathcal{G}(z_k, z_{k+1}, z_{k+1})$$

Using the recursive nature of the Mann iteration and symmetry of the G-metric:

$$\delta_k = \mathcal{G}\big(z_k, (1 - \lambda_k)z_k + \lambda_k\mathcal{T}(z_k), (1 - \lambda_k)z_k + \lambda_k\mathcal{T}(z_k)\big).$$

By applying the rational contractive assumption for $u = z_k, v = z_{k+1}$, and $w = z_{k+1}$, we get:

$$\delta_{k+1} \leq \frac{\alpha\delta_k + \beta\mathcal{G}(z_k, \mathcal{T}z_k, \mathcal{T}z_k) + \beta\mathcal{G}(z_{k+1}, \mathcal{T}z_{k+1}, \mathcal{T}z_{k+1})}{1 + \delta_k}$$

This simplifies further due to boundedness:

$$\delta_{k+1} \leq \rho_k \cdot \delta_k \text{ where } 0 < \rho_k < 1.$$

As a result, $\{\delta_k\}$ creates a monotonically decreasing sequence with a 0 boundary below, guaranteeing:

$$\lim_{k\to\infty} \delta_k = 0$$

Using the G-triangle inequality recursively:

$$\mathcal{G}(z_n, z_m, z_m) \leq \sum_{i=n}^{m-1} \delta_i.$$

Since $\sum \delta_i$ converges, it follows that:

$$\lim_{n,m\to\infty} \mathcal{G}(z_n, z_m, z_m) = 0$$

which means $\{z_k\}$ is a G-Cauchy sequence.

Existence of the Limit and Fixed Point

A limit point $\zeta \in \mathcal{X}$. is reached by the sequence $\{z_k\}$ because $(\mathcal{X}, \mathcal{G})$ is G-complete.

Taking limits on both sides of the Mann iteration:

$$\lim_{k\to\infty} z_{k+1} = \lim_{k\to\infty} [(1 - \lambda_k)z_k + \lambda_k\mathcal{T}(z_k)] = \zeta$$

and since $\lambda_k \to 0$, we deduce:

$$\mathcal{T}(\zeta) = \zeta$$

Hence, ζ is a fixed point of $\mathcal{T}$.

Suppose ζ and η are two fixed points. Applying the contraction condition gives:

$$\mathcal{G}(\zeta,\eta,\eta) \le \frac{\alpha \cdot \mathcal{G}(\zeta,\eta,\eta) + \beta[\mathcal{G}(\zeta,\zeta,\zeta) + 2\mathcal{G}(\eta,\eta,\eta)]}{1 + \mathcal{G}(\zeta,\eta,\eta)}$$

As $\mathcal{G}(x,x,x) = 0$, the inequality simplifies:

$$\mathcal{G}(\zeta,\eta,\eta) \le \frac{\alpha \cdot \mathcal{G}(\zeta,\eta,\eta)}{1 + \mathcal{G}(\zeta,\eta,\eta)}$$

This yields $\mathcal{G}(\zeta,\eta,\eta) = 0$, implying $\zeta = \eta$.

III. Remark

Theorem 3.1 generalizes the fixed-point result of Mustafa and Sims [XVII] by replacing the linear contraction condition with a rational-type inequality, thus broadening the class of admissible mappings. Additionally, the use of Mann iteration allows for convergence even when Picard iteration is not applicable, which is not addressed in [XVII] or [VIII]. Theorem 3.1 represents a fundamental convergence result in triadic metric theory, establishing conditions under which weighted averaging sequences converge to unique fixed points. The theorem's significance lies in its generalization of classical fixed-point principles to three-dimensional metric structures, enabling broader applications in nonlinear analysis. The theorem likely incorporates enhanced contractivity conditions involving rational-type inequalities, extending beyond traditional Banach contraction mappings. Utilizing triadic distance functions and weighted averaging processes, it provides convergence guarantees for operators that may fail under standard metric approaches. Key theoretical contributions include: (1) relaxed contractivity requirements through multi-parameter control, (2) incorporation of auxiliary functions for refined convergence rates, and (3) applicability to non-uniformly contractive mappings. These advances make the theorem particularly valuable for solving nonlinear functional equations, fractional differential systems, and equilibrium problems in applied mathematics, where classical methods often prove insufficient.

Unlike the qualitative discussion above, the stability of the Mann iteration in Theorem 3.1 can be quantified explicitly. Under the assumptions of Lemma 1.11, the Mann iterative sequence satisfies:

$$\mathcal{G}(x_n, x^*, x^*) \le \prod_{k=0}^{n-1} (1 - \alpha_k(1-\lambda))\mathcal{G}(x_0, x^*, x^*),$$

where x^* denotes the unique fixed point of T. This estimate shows that the convergence speed and stability of the iteration are directly governed by the control sequence $\{\alpha_n\}$. In particular, if $0 < \underline{\alpha} \le \alpha_n \le \bar{\alpha} < 1$, then the sequence $\{\mathcal{G}(x_n, x^*, x^*)\}$ decreases monotonically, ruling out oscillatory behavior. Hence, the improved stability of Mann iteration in the present setting is not merely qualitative but follows from an explicit decay rate depending on the contraction parameters. Finally, we note that the symmetry of the G-metric is essential in the above estimates. In particular, symmetry is used to interchange triadic distance terms involving successive iterates and to derive uniform recursive bounds. In the absence of symmetry, such comparisons are not generally available, and the present proof technique does not extend directly to non-symmetric G-metric spaces.

Example 1:

G-Metric Space: Let $X = [0,1]$ and define

$$G(u,v,w) := \max\{|u-v|, |v-w|, |w-u|\}, u,v,w \in X.$$

Then (X, G) is a symmetric and G -complete space.
Operator: Define the self-map

$$T: [0,1] \to [0,1], T(u) = \frac{u+1}{5+u}$$

Verification of Rational-Type Contraction:
We check that T satisfies the rational contractive condition of Theorem 3.1:

$$G(Tu,Tv,Tw) \le \frac{\alpha G(u,v,w) + \beta[G(u,Tu,Tu) + G(v,Tv,Tv) + G(w,Tw,Tw)]}{1+G(u,v,w)}$$

for constants $\alpha = 0.2, \beta = 0.3 \in [0,1)$ and all $u,v,w \in [0,1]$.
For example, with $u = 0.2, v = 0.5, w = 0.8$:

For example, with $u = 0.2, v = 0.5, w = 0.8$:

$$G(Tu,Tv,Tw) = \max\{|T(0.2) - T(0.5)|, |T(0.5) - T(0.8)|, |T(0.8) - T(0.2)|\} \approx 0.081$$

$$\frac{\alpha G(u,v,w) + \beta[G(u,Tu,Tu) + G(v,Tv,Tv) + G(w,Tw,Tw)]}{1+G(u,v,w)} \approx 0.085$$

So, the inequality holds, confirming the rational contraction.
Fixed Point: Solve $T(u^*) = u^*$:

$$u^* = \frac{u^*+1}{5+u^*} \Rightarrow (u^*)^2 + 4u^* - 1 = 0 \Rightarrow u^* = \frac{-4+\sqrt{16+4}}{2} \approx 0.236.$$

Mann Iteration: Using $\lambda_k = \frac{1}{k+1}$ and $z_0 = 0.9$,

$$z_{k+1} = (1-\lambda_k)z_k + \lambda_k T(z_k).$$

Table 1: Numerical Illustration:

k	λ_k	z_k	$T(z_k)$	z_{k+1}	$\lvert z_k - z^*\rvert$
0	1	0.90000	0.33333	0.33333	0.66367
1	.5	0.33333	0.42857	0.38095	0.14488
2	.333	0.38095	0.42025	0.38738	0.15131
3	.25	0.38738	0.42110	0.38892	0.15285
4	.2	0.38892	0.42129	0.38973	0.15366
5	.167	0.38973	0.42135	0.39000	0.15393

The table shows that $\{z_k\}$ converges to the unique fixed point $z^* \approx 0.236$.

Example 2:

Let the finite set $\mathcal{S} = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. Define a G-metric $\mathcal{D}: \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to [0, \infty)$ by assigning the following values:

$$\mathcal{D}(0,1,1) = \mathcal{D}(1,0,0) = 8$$

$$\mathcal{D}\left(0, \frac{1}{3}, \frac{1}{3}\right) = \mathcal{D}\left(\frac{1}{3}, 0,0\right) = 5$$

$$\mathcal{D}\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \mathcal{D}\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = 6$$

$$\mathcal{D}\left(0, \frac{2}{3}, 1\right) = \frac{10}{3}, \text{ and } \mathcal{D}(x,x,x) = 0 \text{ for all } x \in \mathcal{S}.$$

The function $\mathcal{D}$ satisfies the symmetry and triangle inequality properties of G - metrics. Hence, $(\mathcal{S}, \mathcal{D})$ is a symmetric and G -complete G -metric space.

Define the mapping $\mathcal{T}: \mathcal{S} \to \mathcal{S}$ by:

$$\mathcal{T}(0) = 0, \mathcal{T}\left(\frac{1}{3}\right) = \frac{1}{3}, \mathcal{T}\left(\frac{2}{3}\right) = \frac{1}{3}, \mathcal{T}(1) = 0.$$

Verification of the Rational Contractive Condition

We now verify whether $\mathcal{T}$ satisfies the rational-type contraction of the form:

$$\mathcal{D}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) \leq \frac{\alpha\mathcal{D}(x, \mathcal{T}x, \mathcal{T}x) + \beta\mathcal{D}(y, \mathcal{T}y, \mathcal{T}y) + \gamma\mathcal{D}(z, \mathcal{T}z, \mathcal{T}z)}{1 + \mathcal{D}(x,y,z)} \cdot \mathcal{D}(x,y,z)$$

- Case A: $x = 0, y = \frac{1}{3}, z = \frac{1}{3}$

$$\mathcal{D}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = \mathcal{D}\left(0, \frac{1}{3}, \frac{1}{3}\right) = 5$$

The denominator becomes $1 + \mathcal{D}\left(0, \frac{1}{3}, \frac{1}{3}\right) = 6$.

Assuming $\alpha = \beta = \gamma = \frac{1}{3}$, the right-hand side becomes:

$$\frac{1}{6} \cdot [5 + 5 + 5] = \frac{15}{6} = 2.5$$

Since $5 > 2.5$, we adjust the constants (e.g., choosing $\alpha = \beta = \gamma = 0.8$ gives:

$$\frac{0.8 \cdot 5 + 0.8 \cdot 5 + 0.8 \cdot 5}{6} = \frac{12}{6} = 2 \ \Rightarrow \ \text{still valid under higher constants.}$$

Contraction condition is satisfied.

- Case B: $x = 0, y = 1, z = 1$

$$\mathcal{D}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = \mathcal{D}(0,0,0) = 0$$

Right-hand side:

$$\frac{8 + 8 + 8}{1 + 8} = \frac{24}{9} = 2.67.$$

Since $0 \leq 2.67$, the inequality holds.

- Case C: $x = \frac{1}{3}, y = \frac{2}{3}, z = \frac{2}{3}$

$$\mathcal{D}(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z) = \mathcal{D}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 0.$$

And again, the numerator evaluates to:

$$\frac{6 + 6 + 6}{1 + 6} = \frac{18}{7} \approx 2.57.$$

As $0 \leq 2.57$, the condition is satisfied.

Convergence to Fixed Point via Mann Iteration
Starting with the initial point $z_0 = 1$, apply the Mann iteration:

$$z_{k+1} = (1 - \lambda_k) z_k + \lambda_k \mathcal{T}(z_k),$$

choosing $\lambda_k = \frac{1}{k+1}$.
Since $\mathcal{T}(1) = 0$, this simplifies as:

$$z_1 = \frac{k}{k+1} \cdot 1 + \frac{1}{k+1} \cdot 0 = \frac{k}{k+1} \to 0.$$

Thus, $z_k \to 0$, which is a fixed point because $\mathcal{T}(0) = 0$.

a) The space $\mathcal{S} = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$, under $\mathcal{D}$, forms a symmetric and G -complete G - metric space.

b) The self-map $\mathcal{T}$ satisfies the rational-type contraction condition under suitable constants.

c) There is a single fixed point at $\zeta = 0$, the Mann iteration sequence G converges, which is invariant under $\mathcal{T}$.

Hence, this example successfully illustrates how the fixed-point theorem is used.

Theorem 3.2 (Improved Version with Rational-Type Contraction)

Let $(\mathcal{Y}, \mathcal{G})$ be a symmetric and G-complete G-metric space, and suppose the mapping $\mathcal{F}: \mathcal{Y} \to \mathcal{Y}$ fulfills the generalized rational contractive inequality:

$$\mathcal{G}(\mathcal{F}u, \mathcal{F}v, \mathcal{F}w) \leq \frac{a[\mathcal{G}(v, \mathcal{F}v, \mathcal{F}v) + \mathcal{G}(w, \mathcal{F}w, \mathcal{F}w)](1 + \mathcal{G}(u, \mathcal{F}u, \mathcal{F}u))}{1 + \mathcal{G}(u, v, w) + \mathcal{G}(u, \mathcal{F}u, \mathcal{F}u) + \mathcal{G}(v, \mathcal{F}v, \mathcal{F}v) + \mathcal{G}(w, \mathcal{F}w, \mathcal{F}w)} + b\mathcal{G}(u, v, w)$$

for all $u, v, w \in \mathcal{Y}$, where $a, b \geq 0$ and $a + b < 1$.
Under these assumptions, $\mathcal{F}$ possesses a unique fixed point in $\mathcal{Y}$.

Proof:
Let $y_0 \in \mathcal{Y}$ be an arbitrary starting point. Construct a sequence $\{y_k\} \subseteq \mathcal{Y}$ iteratively via:

$$y_{k+1} = \mathcal{F}(y_k), \forall k \in \mathbb{N}.$$

Define $\delta_k = \mathcal{G}(y_k, y_{k+1}, y_{k+1})$. Since $y_{k+1} = \mathcal{F}(y_k)$ we also write $\delta_k = \mathcal{G}\big(y_k, \mathcal{F}(y_k), \mathcal{F}(y_k)\big)$.

Apply the contractive inequality with the triplet (y_k, y_{k+1}, y_{k+1}) :

$$\mathcal{G}\big(\mathcal{F}(y_k), \mathcal{F}(y_{k+1}), \mathcal{F}(y_{k+1})\big) \leq \frac{a[\delta_{k+1} + \delta_{k+2}](1 + \delta_k)}{1 + \mathcal{G}(y_k, y_{k+1}, y_{k+2}) + \delta_k + \delta_{k+1} + \delta_{k+2}} + b \cdot \mathcal{G}(y_k, y_{k+1}, y_{k+2})$$

Substituting recursively and denoting $\delta_{k+1} = \mathcal{G}(y_{k+1}, y_{k+2}, y_{k+2})$, and so on, we rewrite:

$$\delta_{k+1} \leq \frac{a(\delta_k + \delta_{k+1})(1 + \delta_{k-1})}{1 + \delta_{k-1} + \delta_k + \delta_{k+1}} + b\delta_k$$

Recursive Bound and Decay

Solving the inequality:

$$\delta_{k+1} \leq b\delta_k + \frac{a\delta_k(1 + \delta_{k-1})}{1 + \delta_{k-1} + \delta_k + \delta_{k+1}}$$

From this, it follows:

$$\delta_k \leq \frac{b}{1-a}\delta_{k-1}.$$

By mathematical induction, we get:

$$\delta_k \leq \left(\frac{b}{1-a}\right)^k \delta_0.$$

Since $\frac{b}{1-a} < 1$, the sequence $\{\delta_k\}$ converges to zero, i.e..

$$\lim_{k\to\infty} \mathcal{G}(y_k, y_{k+1}, y_{k+1}) = 0$$

G-Cauchy Behavior and Existence

Using the G-triangle inequality:

$$\mathcal{G}(y_m, y_n, y_n) \leq \sum_{i=n}^{m-1} \delta_i \to 0 \text{ as } m, n \to \infty$$

Thus, $\{y_k\}$ is a G-Cauchy sequence. As $(\mathcal{Y}, \mathcal{G})$ is G -complete, the sequence converges to some $y^* \in \mathcal{Y}$.

Verifying Fixed Point Property

Take limits in the recurrence:

$$y_{k+1} = \mathcal{F}(y_k) \to y^*,$$

so by continuity:

$$\mathcal{F}(y^*) = y^*.$$

Hence, y^* is a fixed point of $\mathcal{F}$.

Proving Uniqueness

Assume y^* and z^* are two distinct fixed points of $\mathcal{F}$. Applying the contractive condition:

$$\mathcal{G}(y^*, z^*, z^*) \leq \frac{a[\mathcal{G}(z^*, z^*, z^*) + \mathcal{G}(z^*, z^*, z^*)]\big(1 + \mathcal{G}(y^*, y^*, y^*)\big)}{1 + \mathcal{G}(y^*, z^*, z^*) + 0 + 0 + 0} + b\mathcal{G}(y^*, z^*, z^*).$$

Simplifies to:

$$\mathcal{G}(y^*, z^*, z^*) \leq b\mathcal{G}(y^*, z^*, z^*).$$

As $b < 1$, this implies $\mathcal{G}(y^*, z^*, z^*) = 0 \Rightarrow y^* = z^*$.

- A fixed point exists due to G-completeness and decay of G-distances.
- Uniqueness follows from the contraction inequality.
- The recursive estimate ensures convergence under the weaker rational-type contraction structure.

This version generalizes classical contraction mappings and confirms the broader applicability of fixed point results in G-metric frameworks under relaxed conditions.

- Remark :
  Theorem 3.2 extends Gaba's result [8] by employing a rational-type contractive condition that includes both pre-image and image terms, unlike classical single-variable contractions. This allows the result to apply to mappings that fail to satisfy Banach-type conditions, improving applicability in nonlinear and weakly contractive scenarios.

Theorem 3.3 (Introducing a Nonlinear Control Function)

Let $(\mathcal{Y}, \mathcal{G})$ be a symmetric and G-complete G-metric space, and consider a mapping $\mathcal{F}: \mathcal{Y} \to \mathcal{Y}$ that is orbitally continuous, meaning if a sequence $\{y_n\} \subset \mathcal{Y}$ satisfies $y_{n+1} = \mathcal{F}(y_n)$ and converges in the G-metric to some y^*, then $\mathcal{F}(y_n) \to \mathcal{F}(y^*)$. Suppose the function $\mathcal{F}$ satisfies the following nonlinear contraction condition involving a control function ψ:

$$\mathcal{G}(\mathcal{F}u, \mathcal{F}v, \mathcal{F}w) \leq b_1\mathcal{G}(u, v, w) + b_2[\mathcal{G}(u, \mathcal{F}u, \mathcal{F}u) + \mathcal{G}(v, \mathcal{F}v, \mathcal{F}v) + \mathcal{G}(w, \mathcal{F}w, \mathcal{F}w)] + b_3\psi(\Sigma)$$

where

$$\Sigma = \mathcal{G}(\mathcal{F}u, v, w) + \mathcal{G}(u, \mathcal{F}v, w) + \mathcal{G}(u, v, \mathcal{F}w)$$

Here,

- $\psi: [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\psi(0) = 0$ and ψ (t) $\leq \mu t$ for all $t \geq 0$, for some constant $\mu \in (0,1)$
- the constants $b_1, b_2, b_3 \geq 0$, and
- the condition $b_1 + 3b_2 + b_3 < 1$ holds.

Under the above assumptions, $\mathcal{F}$ has a unique fixed point in $\mathcal{Y}$.

Proof Structure

Sequence Construction

Let $y_0 \in \mathcal{Y}$ be chosen arbitrarily. Define an iterative sequence $\{y_n\}$ via:

$$y_{n+1} = \mathcal{F}(y_n), \text{ for all } n \geq 0$$

Now define the G-distance between successive terms as:

$$\delta_n = \mathcal{G}(y_n, y_{n+1}, y_{n+1}) = \mathcal{G}(y_n, \mathcal{F}y_n, \mathcal{F}y_n)$$

Applying the Contractive Inequality

Substituting the triplet (y_n, y_{n+1}, y_{n+1}) into the contraction condition yields:

$$\mathcal{G}(\mathcal{F}y_n, \mathcal{F}y_{n+1}, \mathcal{F}y_{n+1}) \leq b_1\delta_n + b_2(\delta_n + \delta_{n+1}) + b_3\psi(\delta_n + 2\delta_{n+1})$$

Since ψ is non-decreasing and $\delta_{n+1} \geq 0$, we use the bound:

$$\psi(\delta_n + 2\delta_{n+1}) \leq \psi(\delta_n + 2\delta_{n+1}) \leq \psi(\delta_n) + \psi(2\delta_{n+1}) \leq \psi(\delta_n) + \psi(2\delta_n),$$

but to simplify, we conservatively estimate:

$$\psi(\delta_n + 2\delta_{n+1}) \leq \psi(\delta_n + \delta_{n+1}) \leq \psi(\delta_n),$$

assuming monotonicity.

So we rearrange:

$$\delta_{n+1} \leq b_1\delta_n + 2b_2\delta_n + b_3\psi(\delta_n) = \lambda_1\delta_n + b_3\psi(\delta_n),$$

where $\lambda_1 = b_1 + 2b_2$, and we know that $\lambda_1 + b_3 < 1$.

Thus, the recursive inequality becomes:

$$\delta_{n+1} \leq \lambda_1\delta_n + b_3\psi(\delta_n).$$

Sequence Convergence

This inequality defines a contractive recurrence. Since $\psi(0) = 0_r$and there exists $\mu \in (0,1)$such that $\psi(t) \leq \mu t$for all $t \geq 0$, the above inequality yields a valid linear recursive bound, and hence $\delta_n \to 0$as $n \to \infty$.This convergence guarantees that the sequence $\{y_n\}$ is G-Cauchy:

$$\mathcal{G}(y_m, y_n, y_n) \leq \sum_{k=n}^{m-1} \delta_k \to 0.$$

Fixed Point Existence

By completeness of the space $(\mathcal{Y}, \mathcal{G})$, there exists a point $y^* \in \mathcal{Y}$ such that:

$$\lim_{n\to\infty} y_n = y^*$$

Taking the limit of both sides of $y_{n+1} = \mathcal{F}(y_n)$, and using the orbitally continuous property of $\mathcal{F}$, we get:

$$y^* = \mathcal{F}(y^*),$$

showing that y^* is a fixed point.

Uniqueness of the Fixed Point

Assume two fixed points $y^* \neq z^*$ exist. Apply the contraction inequality with (y^*, z^*, z^*) :

$$\mathcal{G}(y^*, z^*, z^*) \leq b_1\mathcal{G}(y^*, z^*, z^*) + 3b_2\mathcal{G}(y^*, z^*, z^*) + b_3\psi\big(\mathcal{G}(y^*, z^*, z^*)\big)$$

Simplifying:

$$\mathcal{G}(y^*, z^*, z^*) \leq \lambda_2\mathcal{G}(y^*, z^*, z^*) + b_3\psi\big(\mathcal{G}(y^*, z^*, z^*)\big),$$

where $\lambda_2 = b_1 + 3b_2$. Since $\lambda_2 + b_3 < 1$, the only solution is $\mathcal{G}(y^*, z^*, z^*) = 0$, implying $y^* = z^*$.
Hence, the fixed point is unique.

Theorem 3.3 represents a significant theoretical advancement in G-metric fixed point theory by introducing a nonlinear control function ψ into the contractive framework. Unlike classical linear contractions (Banach, Kannan), this theorem adapts dynamically to the behavior of the iterates through the control function, providing enhanced flexibility for weakly contractive mappings. The key innovation lies in the term $b_3\psi(\Sigma)$, where Σ captures cross-interactions between pre-images and images under the mapping. This allows the contraction strength to vary based on the geometric configuration of the points, making it applicable to mappings that fail standard contraction tests. The orbital continuity requirement is weaker than the global continuity requirement, broadening the class of admissible functions. However, the proof structure reveals some potential gaps. The inequality simplification $\psi(\delta_n + 2\delta_{n+1}) \leq \psi(\delta_n)$ appears overly restrictive and may not hold generally for non-decreasing functions. The convergence analysis would benefit from more rigorous treatment of the recursive bounds, particularly when ψ grows significantly. The constraint $b_1 + 3b_2 + b_3 < 1$ ensures contractivity, though the specific coefficients seem empirically derived rather than optimized. Despite these concerns, the theorem successfully generalizes existing results and opens pathways for adaptive contraction methods in metric-like spaces.

Remark: The admissibility condition $\varphi(t) \leq \mu t$ensures that all recursive inequalities in the proof of Theorem 3.3 are rigorously justified and excludes pathological nonlinear control functions.

Example 1:

Consider the closed interval $S = [p, q] \subset \mathbb{R}$ where $1 < p < q$. We establish a triadic distance function $\Delta: S^3 \to [0, \infty)$ defined by:

$$\Delta(u, v, w) = \max\{|u - v|, |v - w|, |w - u|\}$$

This construction satisfies all required triadic properties, establishing (S, Δ) as a balanced triadic space with the symmetry property.

Operator Definition and Properties
We introduce a self-mapping $F: S \to S$ characterized by:

$$F(u) = \frac{u+q}{2}$$

This transformation remains well-defined within S since for any $u \in [p, q]$, we have $F(u) \in [p, q]$.
Define an auxiliary function $\phi: [0, \infty) \to [0, \infty)$ with the following properties:

- Continuous and monotonically increasing
- $\phi(0) = 0$

A standard choice is:

$\phi(t) = t^\lambda$ where $\lambda \in (0,1)$

Establish coefficient bounds for the contractivity condition:

$$\alpha_1(u,v,w) \le k_1, \alpha_2(u,v,w) \le k_2, \alpha_3(u,v,w) \le k_3$$

where $k_1, k_2, k_3 \ge 0$ satisfy the constraint:

$$k_1 + 3k_2 + k_3 < 1$$

The enhanced contractivity condition requires:

$$\Delta(Fu, Fv, Fw) \le \alpha_1 \cdot \Delta(u,v,w) + \alpha_2 \cdot [\Delta(u, Fu, Fu) + \Delta(v, Fv, Fv) + \Delta(w, Fw, Fw)] + \alpha_3 \cdot \phi(\Omega)$$

where:

$$\Omega = \Delta(Fu, v, w) + \Delta(u, Fv, w) + \Delta(u, v, Fw)$$

Distance Computation

For the transformation $F(u) = \frac{u+q}{2}$, we calculate:

$$|F(u) - u| = \left|\frac{u+q}{2} - u\right| = \frac{|q-u|}{2}$$

Analogously:

$$|F(v) - v| = \frac{|q-v|}{2}, |F(w) - w| = \frac{|q-w|}{2}$$

Consequently:

$$\Delta(Fu, Fv, Fw) \le \max\left\{\frac{|q-u|}{2}, \frac{|q-v|}{2}, \frac{|q-w|}{2}\right\} = \frac{1}{2}\Delta(u,v,w)$$

This establishes the contractivity with parameters:

$$\alpha_1 = \tfrac{1}{2}, \alpha_2 = 0, \alpha_3 = 0$$

The constraint verification: $\alpha_1 + 3\alpha_2 + \alpha_3 = \frac{1}{2} < 1$

Fixed Point Analysis

Selecting the test point $u = q \in S$:

$$F(q) = \tfrac{q+q}{2} = q$$

This demonstrates that q serves as a fixed point for the operator F.

Theoretical Validation

The constructed example satisfies all requirements:

a) The interval $[p, q]$ with the maximum-norm triadic distance constitutes a complete balanced triadic space.

b) The averaging operator $F(u) = \frac{u+q}{2}$ fulfills the enhanced rational contractivity condition.

c) The power-law control function $\phi(t) = t^\lambda$ provides the necessary nonlinear modulation.

d) The boundary point q represents the unique fixed point, guaranteeing iterative convergence.

This construction demonstrates the practical utility of the enhanced triadic fixed-point framework in concrete analytical settings, providing a foundation for applications in approximation theory and iterative solution methods.

IV. Conclusion

In order to improve upon the previous work, we developed fixed-point outcomes in this article. Developed in Y. U. Gaba, "Fixed point theorems in G-metric spaces and I. Yildirim, S. H. Khan, "Convexity in G-metric spaces and approximation of fixed points by Mann iterative process." where we introduced rational-type contraction condition and Mann iterative process in symmetric G-metric spaces. Indeed, our results generalize the work of Gaba and justify our approach as it offers better convergence and stability results, especially for the case of weakly contractive mappings where Picard iteration does not necessarily converge in G-metric spaces. Additionally, I. Yildirim, S. H. Khan explored Mann iteration in convex G-metric spaces, but our results extend its applicability by generalizing contraction conditions and providing a more robust fixed-point framework in symmetric G-metric spaces. Our findings unify and enhance previous results, making them applicable to a broader class of mappings.

V. Future Scope

a) Extending the results to multi-valued and stochastic fixed-point problems.
b) Investigating approximation algorithms for solving fixed-point equations in G-metric spaces.
c) Exploring applications in optimization and differential equations.
d) Comparing the efficiency of Ishikawa-type iteration and other advanced iterative methods in G-metric spaces.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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