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DOUBLE ELZAKI TRANSFORM AND ADOMIAN POLYNOMIALS FOR SOLVING BENJAMIN ONO AND BUCKMASTER EQUATIONS

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Abstract

In this research paper, we have proposed a new technique for resolving the Benjamin-Ono and Buckmaster equations that come up in many engineering and science applications. The double Elzaki transform and the Adomian polynomials are coupled in the suggested hybrid approach. Experiments have been carried out to verify the correctness and simplicity of the suggested scheme. To assess the effectiveness of the suggested scheme, the outcomes so obtained are compared with the results obtained by the variational iteration method.

Keywords: Double Elzaki Transform; Adomian Decomposition method; Benjamin-Ono Equation; Buckmaster Equations; Variational Iteration Method (VIM); Test examples.

I. Introduction

Partial differential equations that are nonlinear play a crucial role in modeling complex physical phenomena, especially in fluid dynamics and wave propagation. The Benjamin–Ono equation and the Buckmaster equation are two significant examples of these equations, which both occur in the study of nonlinear dispersive waves. The Benjamin–Ono equation, introduced independently by T.B. Benjamin and H. Ono in the 1960s and 1970s, describes one-dimensional internal waves in deep stratified fluids. It is an integrable equation and takes the form:

$$u_{\rho} = a(u^4)_{\kappa\kappa} + b(u^3)_{\kappa} + f(\kappa,\rho)$$

Where a, b denote the constants or parameters and $f(\kappa,\rho)$ is any function of κ and ρ. The equation captures the balance between nonlinear steepening and dispersive spreading, making it an important tool in the analysis of internal wave dynamics in oceanography. On the other hand, the Buckmaster equation is a higher-order nonlinear PDE used to model the behavior of certain Non-Newtonian fluids, such as those found in lubrication theory or viscous thin films. The general form of Buckmaster's equation is:

$$u_{\rho\rho} + d(u^2)_{\kappa\kappa} + eu_{\kappa\kappa\kappa\kappa} = g(\kappa, \rho)$$

Where d, e denote the constants or parameters and $g(\kappa, \rho)$ is any function in terms of κ and ρ.

While less well-known than the Benjamin–Ono equation, the Buckmaster equation includes strong nonlinear and dispersive effects, providing insights into complex flow patterns and instabilities. Both equations have attracted significant mathematical interest due to their rich structure, challenging analytical properties, and relevance to physical systems. They serve as important benchmarks in the study of solitons, dispersive shock waves, and mathematical fluid mechanics.

Convergence analysis of the Adomian method has been presented for solving various differential equations arising in several applications of sciences and engineering in [I]. For the semi-analytical solutions of Burger's equation, the Sumudu transform-based decomposition technique has been used in [II]. The double integral transform-based approach has been used to derive the solutions of a singular system of hyperbolic equations in [III]. In [IV], the analytical solutions of the Telegraph equations are examined using the double Laplace transform. In order to find the analytical solutions of the differential equation, newly integral transform known as the 'Elzaki transform' is established in [V]. In [VI], a projected differential transform method and the Elzaki transform have been used to solve both linear and nonlinear PDEs. Modification in the double Sumudu transform method has been carried out in [VII] for solving the differential equations. For solving the differential equations, two analytical techniques based on the Elzaki and Sumudu transforms have been established and implemented in [VIII]. The relationship between Elzaki and Laplace transforms has been described in [IX]. In [X], authors have used the Elzaki transform to tackle the ordinary differential equations with variable coefficients. The double Elzaki transform has been used for solving wave-like equations, and the results are compared with the double Laplace transform method in [XI]. The authors have used the double Elzaki decomposition technique for solving some nonlinear partial differential equations arising in various applications of sciences and engineering in [XII]. The combination of the Adomian decomposition method with double Elzaki transforms has been implemented on third-order KdV equations in [XIII]. The classical method based on the finite volume has been developed to solve Buckmaster, Fisher, and Sine Gordan equations in [XIV]. In [XV], the authors have presented a novel approach to finding the semi-analytic solutions of the Buckmaster equations. For this purpose, the Elzaki transform and the projected differential transform method have been employed. The convergence of the double Elzaki transform scheme for solving PDEs has been discussed in [XVI]. Korteweg-De Vries equations of third order have been solved using Adomian polynomials and the Elzaki transform method in [XVII]. Local well-posedness of the Benjamin-Ono equations has been discussed in [XVIII-XIX]. Global well-posedness of Benjamin-Ono equations has been discussed in [XXII]. Efficient techniques based on different integral transforms have been established and implemented on various higher-dimensional PDEs arising in several applications of sciences and engineering in [XX-XXI] and [XXIII-XXIV]. In [XXV], authors have established a novel technique for precisely determining the soliton solutions to the second –order Benjamin-Ono equation. A modification in the Adomian decomposition method has been carried out for the rapid convergence of the series solution for solving differential equations in [XXVI].

The novelty of the present work lies in the following aspects: Unlike existing studies that employ a single Elzaki transform or combine it with decomposition techniques in a sequential manner, this work introduces a double Elzaki transform applied simultaneously with respect to two independent variables. This structural formulation enables a direct treatment of multidimensional governing equations without reducing them to lower-dimensional auxiliary problems. The Adomian decomposition is implemented after applying the inverse double Elzaki transform, which leads to a recursive scheme that avoids linearization, perturbation, or discretization. This ordering differs fundamentally from earlier transform–ADM hybrids, where decomposition is typically applied before inversion. The proposed approach is shown to apply to a wider class of nonlinear partial differential equations with coupled initial–boundary conditions, which are not easily handled using classical Elzaki-based techniques. The double transform reduces algebraic complexity in the recurrence relations, leading to faster convergence of the series solution. A theoretical discussion on convergence behavior has been added, emphasizing the stabilizing effect introduced by the double transform structure.

The following is the structure of this research paper: The complete details of the double Elzaki transform and its characteristics are provided in Section 2. Section 3 presents a suggested method for solving mathematical models of partial differential equations in Section 3. In Section 4, the suggested method for solving such equations has been used in some computational studies. Convergence and result discussion have been discussed in Section 5. The conclusion of the study paper is located in Section 6.

II. The Double Elzaki Transform and its Characteristics

This section discusses the double Elzaki transform, the inverse double Elzaki transform, and a few of its features.

II.i. The Double Elzaki Transform: An Overview

Consider $f(\kappa,\rho)$ with $\kappa,\rho > 0$, a function. An infinite series can be used to represent this function. The double Elzaki transform is therefore expressed as:

$$DE\,\{f(\kappa,\rho);\eta,\zeta\} = T(\eta,\zeta) = \eta\,\zeta \int_0^\infty \int_0^\infty f(\kappa,\rho)\, e^{-\left(\frac{\kappa}{\eta}+\frac{\rho}{\zeta}\right)}\, d\kappa d\rho,$$

Whenever an integral exists.

II.ii. Double Elzaki Transform Inverse

The double Elzaki transform's inverse can be expressed as:

$$DE^{-1}\{T(\eta,\zeta)\} = f(\kappa,\rho),\ \ \kappa,\rho > 0$$

If ∃ a positive constant H such that $m > 0, n > 0$ in the region belong to the interval $0 \le \kappa < \infty, 0 \le \rho < \infty$ the function $f(\kappa,\rho)$ is said to have an exponential order:

$$|f(\kappa,\rho)| \le H\, e^{\left(\frac{\kappa}{m}+\frac{\rho}{n}\right)}$$

II.iii. The Double Elzaki Transform Standard Characteristics

This section will discuss a few features of the double Elzaki transform.:

LINEARITY PROPERTY: If $f(\kappa,\rho)$and $g(\kappa,\rho)$ be two function of $\kappa,\rho > 0$ such that $[f(\kappa,\rho)] = T_1(\eta,\zeta)$ and DE$[g(\kappa,\rho)] = T_2(\eta,\zeta)$, then

$$\text{DE}\{\, af(\kappa,\rho) + b\, g(\kappa,\rho)\} = a\, DE\{f(\kappa,\rho)\} + b\, DE\,\{g(\kappa,\rho)\} = a\, T_1(\eta,\zeta) + bT_2(\eta,\zeta)$$

CHANGE SHIFTING PROPERTY: If DE$\{f(\kappa,\rho)\} = T(\eta,\zeta)$, then

$$\text{DE}\{f(a\kappa, b\rho)\} = \frac{1}{ab}\, T(a\kappa, b\rho)$$

FIRST SHIFTING PROPERTY:

(a) If DE$\{f(\kappa,\rho)\} = T(\eta,\zeta)$, then

$$DE\{\, e^{a\kappa+b\rho}\, f(\kappa,\rho) = T\left[\frac{\eta}{1-a\eta}, \frac{\zeta}{1-b\zeta}\right]$$

(b) If DE$\{f(\kappa,\rho)\} = T(\eta,\zeta)$, then

$$DE\{\, e^{-a\kappa-b\rho}\, f(\kappa,\rho) = T\left[\frac{\eta}{1-a\eta}, \frac{\zeta}{1-b\zeta}\right]$$

II.iv. PARTIAL DERIVATIVES USING DOUBLE ELZAKI TRANSFORM

This section presents the double Elzaki transform of various partial derivatives:

a) $DE\,\{\frac{\partial}{\partial\kappa}\, f(\kappa,\rho)\} = \frac{1}{\eta}\, T(\eta,\zeta) - \eta\, T(0,\zeta)$
b) $\text{DE}\left\{\frac{\partial}{\partial\rho}\, f(\kappa,\rho)\right\} = \frac{1}{\zeta}\, T(\eta,\zeta) - \zeta\, T(\eta,0)$
c) $\text{DE}\{\frac{\partial^2}{\partial\kappa^2} f(\kappa,\rho)\,\} = \frac{1}{\eta^2} T(\eta,\zeta) - T(0,\zeta) - \eta\frac{\partial}{\partial\kappa}\, T(0,\zeta)$
d) $\text{DE}\,\{\frac{\partial^2}{\partial\rho^2} f(\kappa,\rho)\,\} = \frac{1}{\zeta^2} T(\eta,\zeta) - T(\eta,0) - \zeta\frac{\partial}{\partial\rho}\, T(\eta,0)$
e) $\text{DE}\,\{\frac{\partial^2}{\partial\kappa\partial\rho}\, f(\kappa,\rho)\,\} = \frac{1}{\eta\zeta}\, T(\eta,\zeta) - \frac{\zeta}{\eta}\, T(\eta,0) - \frac{\eta}{\zeta}\, T(0,0) + \eta\,\zeta\, T(0,0)$

III. Proposed Technique for Solving Models of PDEs

Examine the universal nonlinear partial differential equation's form.:

$$L\, u(\,\kappa,\rho) + N\, u(\,\kappa,\rho) = g(\kappa,\rho) \tag{2}$$

Under the initial condition

$$u(\kappa,0)=h(\kappa), \tag{3}$$

Here $g(\kappa,\rho)$ is the source term, and L stands for a linear differential operator $L=\frac{\partial}{\partial\kappa}$, and N for the nonlinear differential operator.

When using the single Elzaki transform on the initial condition, i.e., Equation (3), and the double Elzaki transform on Equation (2), we obtain

$$\mathrm{DE}\big(L\,u(\kappa,\rho)\big)+DE\,\big(N\,u(\kappa,\rho)\big)=DE\,(g(\kappa,\rho)) \tag{4}$$

and

$$E(u(\kappa,0))=\;E\big(h(\kappa)\big)=T(\eta,0) \tag{5}$$

From Equation (4), we obtain

$$\frac{1}{\zeta}T(\eta,\zeta)-\zeta\,T(\eta,0)=DE\,\big(g(\kappa,\rho)\big)-DE\,\big(N\,u(\kappa,\rho)\big)$$

This implies

$$T(\eta,\zeta)=\zeta^2T(\eta,0)+\zeta DE\,\big(g(\kappa,\rho)\big)-\{\zeta DE\,\big(N\,u(\kappa,\rho)\big)$$

Or

$$\mathrm{DE}\big(u(\kappa,\rho)\big)=\zeta^2\,E\big(h(\kappa)\big)+\zeta\,DE\,\big(g(\kappa,\rho)\big)-\big\{\zeta DE\big(N\,u(\kappa,\rho)\big)\big\} \tag{6}$$

The inverse double Elzaki transform applied to Equation (6), yields

$$u(\kappa,\rho)=G(\kappa,\rho)-DE^{-1}\big\{\zeta DE\big(N\,u(\kappa,\rho)\big)\big\} \tag{7}$$

where

$$G(\kappa,\rho)=DE^{-1}\big\{\zeta^2\,E\big(h(\kappa)\big)+\zeta DE\,\big(g(\kappa,\rho)\big)\big\}$$

Assume that the solution has the following form:

$$u(\kappa,\rho)=\sum_{n=0}^{\infty}u_n(\kappa,\rho) \tag{8}$$

The nonlinear term has the following form:

$$Nu(\kappa,\rho)=\sum_{n=0}^{\infty}A_n(u), \tag{9}$$

The Adomian polynomials $A_n(u)$ are represented here, and they may be computed as:

$$A_n=\frac{1}{n!}\frac{d^n}{d\epsilon^n}\big\{N\big(\textstyle\sum_{j=0}^{\infty}\epsilon^j u_j\big)\big\}_{\epsilon=0},n=0,1,2,3,\ldots$$

When the values from (8) and (9) are entered into (7), we get

$$\sum_{n=0}^{\infty}u_n(\kappa,\rho)=S(\kappa,\rho)-DE^{-1}\{\zeta DE(\textstyle\sum_{n=0}^{\infty}A_n(u))\} \tag{10}$$

From (10), we get

$$u_0(\kappa,\rho)=S(\kappa,\rho),$$

$$u_1(\kappa,\rho) = -DE^{-1}\{\zeta\, DE(A_0)\},$$

$$u_2(\kappa,\rho) = -DE^{-1}\{\zeta\, E_2(A_1)\},$$

$$\vdots$$

The problem's approximate solution is:

$$u(\kappa,\rho) = \lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(\kappa,\rho).$$

IV. Computational Work

In order to find the solutions of nonlinear PDEs that arise during the creation of liquid drops, we conduct a few test examples in this section.

Example 1: Examine the nonlinear Benjamin-Ono-like equation

$$u_{\rho\rho} + (u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa} = 2\rho^2 \tag{11}$$

In the initial condition $u(\kappa,0) = 0,\ u_\rho(\kappa,0) = \kappa$. The exact solution is:

$$u(\kappa,\rho) = \kappa\rho$$

Equation (11), when subjected to the double Elzaki transform, yields

$$\mathrm{DE}(u_{\rho\rho}) = DE(2\rho^2) - DE((u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})$$

This implies

$$\frac{1}{\zeta^2}T(\eta,\zeta) - T(\eta,0) - \zeta.\frac{\partial}{\partial\rho}T(\eta,0) = DE(2\rho^2) - DE((u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa}) \tag{12}$$

Using the initial conditions and a single Elzaki transform, we get

$$E(u(\kappa,0)) = T(\eta,0) = E(0) = 0$$

and

$$E\left(u_\rho(\kappa,0)\right) = \frac{\partial}{\partial\rho}T(\eta,0) = E(\kappa) = \eta^3$$

From (12), we obtain

$$\frac{1}{\zeta^2}T(\eta,\zeta) = \zeta.\eta^3 + DE(2\rho^2) - DE((u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})$$

This implies

$$T(\eta,\zeta) = \zeta^3.\eta^3 + \zeta^2.DE(2\rho^2) - \zeta^2.DE((u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})$$

When inverse double Elzaki transforms are implemented, we get

$$\mathrm{DE}^{-1}(T(\eta,\zeta)) = DE^{-1}\{\zeta^3.\eta^3 + \zeta^2.DE(2\rho^2) - \zeta^2.DE((u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})\}$$

This implies

$$u(\kappa,\rho) = \kappa\rho + \frac{\rho^4}{6} - DE^{-1}\{\zeta^2.DE((u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})\}$$

Applying the Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(\kappa,\rho) = \kappa\rho + \frac{\rho^4}{6} - DE^{-1}\left\{\zeta^2 . DE\left\{\sum_{n=0}^{\infty} A_n(u)\right\}\right\}$$

From the above Equation, we obtain

$$\begin{cases} u_0(\kappa,\rho) = \kappa\rho, \\ u_1(\kappa,\rho) = \frac{\rho^4}{6} - DE^{-1}(\zeta^2 . DE\{A_0\}), \\ u_2(\kappa,\rho) = -DE^{-1}(\zeta^2 . E_2\{A_1\}), \\ \vdots \end{cases}$$

Some of the Adomian polynomials are:

$$\begin{cases} A_0 = 2\rho^2, \\ A_1 = 0, \\ A_2 = 0, \\ \vdots \end{cases}$$

The values of $u_0, u_1, u_2, \ldots$ are given by

$$\begin{cases} u_0(\kappa,\rho) = \kappa\rho, \\ u_1(\kappa,\rho) = 0, \\ u_2(\kappa,\rho) = 0, \\ \vdots \end{cases}$$

The solution is:

$$u(\kappa,\rho) = u_0(\kappa,\rho) + u_1(\kappa,\rho) + u_2(\kappa,\rho) + \cdots$$

Or

$$u(\kappa,\rho) = \kappa\rho$$

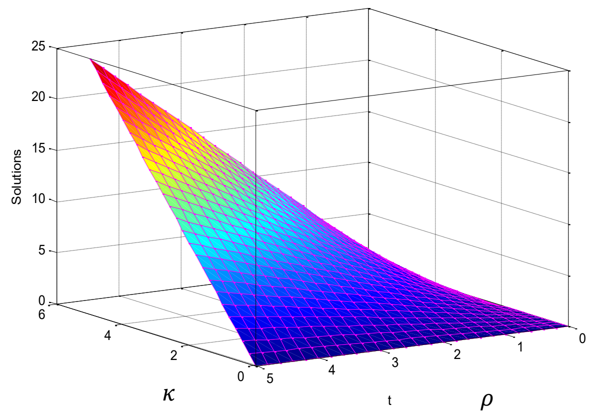

Fig. 1. Physical behavior of solutions of Example 1

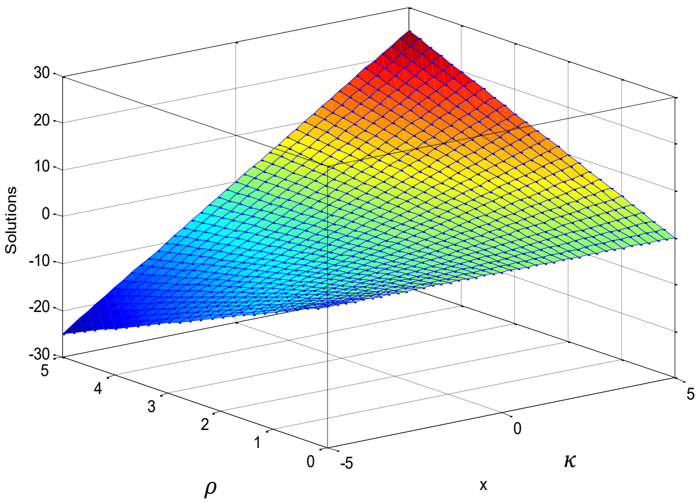

Fig. 2. Solutions of Example 1 for a different range of **κ**, ρ

The dynamical and physical characteristics of Example 1's analytical solutions generated by the Adomian decomposition method based on the double Elzaki transform are displayed in Figures 1 and 2 at various ranges of $\kappa$and ρ .

Variational Iteration Method (VIM) :

The formula for the variational iteration method is:

$$u_{n+1}(\kappa,\rho) = u_n(\kappa,\rho) + \lambda \int_0^\tau \left((u_n)_{\rho\rho} + (u_n^2)_{\kappa\kappa} + (u_n)_{\kappa\kappa\kappa\kappa} - 2\rho^2\right) d\tau \quad (13)$$

Here $\lambda = \tau - \rho$, and $u_0(\kappa,0) = 0,\ \ u_\rho(\kappa,0) = \kappa.$

Equation (13) can be written simply as:

$$u_{n+1}(\kappa,\rho) = u(\kappa,0) + \zeta . u_\rho(\kappa,0) + \int_0^\tau (\tau-\rho)((u_n^2)_{\kappa\kappa} + (u_n)_{\kappa\kappa\kappa\kappa} - 2\rho^2)d\tau$$

and the initial approximation is $u_0 = u\,(\kappa,0) + \rho . u_\rho(\kappa,0) = \kappa\rho.$

This can be written as:

$$u_{n+1}(\kappa,\rho) = \kappa\rho + \int_0^\tau (\tau-\rho)((u_n^2)_{\kappa\kappa} + (u_n)_{\kappa\kappa\kappa\kappa} - 2\rho^2)d\tau$$

For $n = 0$,

$$u_1 = \kappa\rho + \int_0^\tau (\tau-\rho)(2\rho^2 + 0 - 2\rho^2)d\rho = \kappa\rho$$

For $n = 1$,

$$u_2 = \kappa\rho + \int_0^\tau (\tau-\rho)((u_1^2)_{\kappa\kappa} + (u_1)_{\kappa\kappa\kappa\kappa} - 2\rho^2)d\rho = \kappa\rho$$

For $n = 2$,

$$u_3 = \kappa\rho + \int_0^{\tau} (\tau - \rho)((u_2^2)_{\kappa\kappa} + (u_2)_{\kappa\kappa\kappa\kappa} - 2\rho^2)d\rho = \kappa\rho$$

and so on. The way to solve this is:

$$u(\kappa, \rho) = \lim_{n\to\infty} u_n(\kappa, \rho) = \kappa\rho$$

Example 2: Consider the nonlinear Benjamin-Ono type equation

$$u_{\rho\rho} + 2(u^2)_{\rho\rho} + u_{\kappa\kappa\kappa\kappa} = 4 \tag{14}$$

with initial condition $u(\kappa, 0) = \kappa,\ u_\rho(\kappa, 0) = 1$. The precise solution is:

$$u(\kappa, \rho) = \kappa + \rho$$

When the double Elzaki transform is applied to Equation (14), we get

$$\mathrm{DE}(u_{\rho\rho}) = DE(4) - DE(2(u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})$$

This implies

$$\frac{1}{\zeta^2} T(\eta, \zeta) - T(\eta, 0) - \zeta . \frac{\partial}{\partial\rho} T(\eta, 0) = DE(4) - DE(2(u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa}) \tag{15}$$

Using the initial conditions and a single Elzaki transform, we get

$$E(u(\kappa, 0)) = T(\eta, 0) = E(\kappa) = \eta^3$$

and

$$E\left(u_\rho(\kappa, 0)\right) = \frac{\partial}{\partial\rho} T(\eta, 0) = E(1) = \eta^2$$

From (15), we obtain

$$\frac{1}{\zeta^2} T(\eta, \zeta) = \eta^3 + \zeta . \eta^2 + DE(4) - DE(2(u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})$$

This implies

$$T(\eta, \zeta) = \zeta^2 . \eta^3 + \zeta^3 . \eta^2 + \zeta^2 . DE(4) - \zeta^2 . DE(2(u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})$$

When inverse double Elzaki transforms are implemented, we get

$$\mathrm{DE}^{-1}(T(\eta, \zeta)) = DE^{-1}\{\zeta^2 . \eta^3 + \zeta^3 . \eta^2 + \zeta^2 . DE(4) - \zeta^2 . DE(2(u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})\}$$

This implies

$$u(\kappa, \rho) = \kappa + \rho + 2\rho^2 - DE^{-1}\{\zeta^2 . DE(2(u^2)_{\kappa\kappa} + u_{\kappa\kappa\kappa\kappa})\}$$

Applying the Adomian decomposition method, we get

$$\sum_{n=0}^{\infty} u_n(\kappa, \rho) = \kappa + \rho + 2\rho^2 - DE^{-1}\left\{\zeta^2 . DE\left\{\sum_{n=0}^{\infty} A_n(u)\right\}\right\}$$

From the above Equation, we get

$$\begin{cases} u_0(\kappa,\rho) = \kappa + \rho, \\ u_1(\kappa,\rho) = 2\rho^2 - DE^{-1}(\zeta^2 . DE\{A_0\}), \\ u_2(\kappa,\rho) = -DE^{-1}(\zeta^2 . DE\{A_1\}), \\ \vdots \end{cases}$$

Some of the Adomian polynomials are:

$$\begin{cases} A_0 = 4, \\ A_1 = 0, \\ A_2 = 0, \\ \vdots \end{cases}$$

The values of $u_0, u_1, u_2, \ldots$ are given by

$$\begin{cases} u_0(\kappa,\rho) = \kappa + \rho, \\ u_1(\kappa,\rho) = 0, \\ u_2(\kappa,\rho) = 0, \\ \vdots \end{cases}$$

The solution is:

$$u(\kappa,\rho) = u_0(\kappa,\rho) + u_1(\kappa,\rho) + u_2(\kappa,\rho) + \cdots$$

Or

$$u(\kappa,\rho) = \kappa + \rho$$

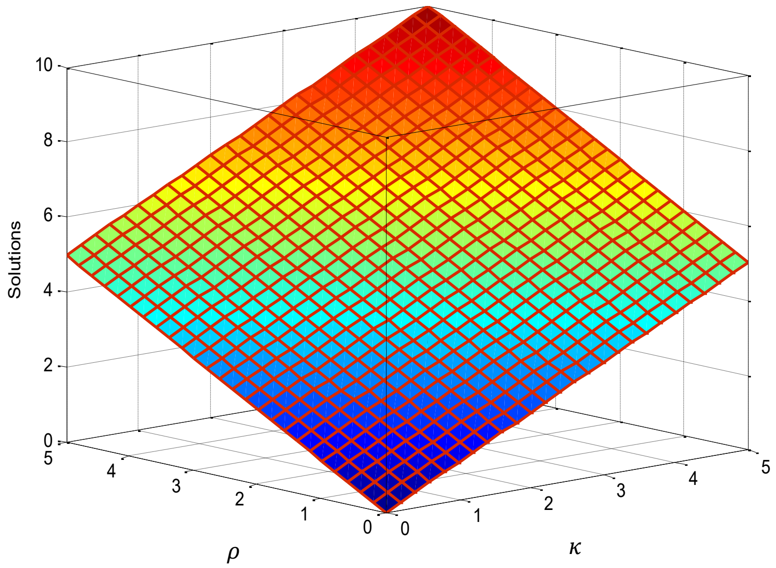

Fig. 3. Physical characteristics of Example 2's solutions

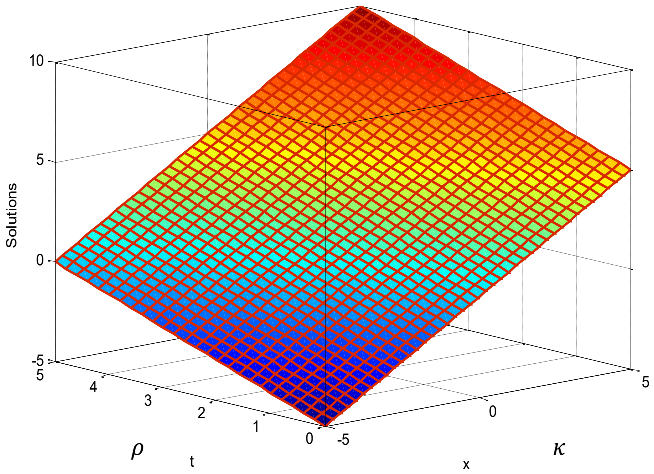

Fig. 4. Solutions of Example 2 for different ρ

The dynamical and physical behavior of analytical solutions derived from the Adomian decomposition approach based on the double Elzaki transform at various ranges of κ and ρ, and of Example 2, is depicted in Figures 3 and 4.

Variational Iteration Method (VIM) :

The formula for the variational iteration method is:

$$u_{n+1}(\kappa,\rho) = u_n(\kappa,\rho) + \lambda \int_0^{\tau} \left((u_n)_{\rho\rho} + 2(u_n^2)_{\kappa\kappa} + (u_n)_{\kappa\kappa\kappa\kappa} - 4\right) d\tau \qquad (16)$$

Here $\lambda = \tau - \rho$, and $u_0(\kappa, 0) = \kappa$.
Equation (16) can be written simply as:

$$u_{n+1}(\kappa,\rho) = u(\kappa,0) + \rho . u_\rho(\kappa,0) + \int_0^{\tau} (\tau-\rho)(2(u_n^2)_{\kappa\kappa} + (u_n)_{\kappa\kappa\kappa\kappa} - 4) d\tau$$

This can be written as:

$$u_{n+1}(\kappa,\rho) = (\kappa+\rho) + \int_0^{\tau} (\tau-\rho)(2(u_n^2)_{\kappa\kappa} + (u_n)_{\kappa\kappa\kappa\kappa} - 4) d\tau$$

For $n = 0$,

$$u_1 = (\kappa+\rho) + \int_0^{\rho} (\tau-\rho)(2(u_0^2)_{\kappa\kappa} + (u_0)_{\kappa\kappa\kappa\kappa} - 4) d\rho = \kappa + \rho$$

For $n = 1$,

$$u_2 = (\kappa+\rho) + \int_0^{\rho} (\tau-\rho)(2(u_1^2)_{\kappa\kappa} + (u_1)_{\kappa\kappa\kappa\kappa} - 4) d\rho = \kappa + \rho$$

For $n = 2$,

$$u_3 = (\kappa + \rho) + \int_0^{\rho} (\tau - \rho)(2(u_2^2)_{\kappa\kappa} + (u_2)_{\kappa\kappa\kappa\kappa} - 4)d\rho = \kappa + \rho$$

and so on. The way to solve this is:

$$u(\kappa, \rho) = \lim_{n\to\infty} u_n(\kappa, \rho) = \kappa + \rho$$

Example 3: Examine the nonlinear Buckmaster equation, which has the following form:

$$u_\rho = (u^4)_{\kappa\kappa} + (u^3)_\rho, \tag{17}$$

with initial condition

$$u(\kappa, 0) = \kappa$$

Rewrite the Equation (17),

$$u_\rho = \{(u^4)_{\kappa\kappa} + (u^3)_\kappa\}$$

Applying both sides of the double Elzaki transform, we get

$$\mathrm{DE}(u_\rho) = DE\{(u^4)_{\kappa\kappa} + (u^3)_\kappa\}$$

This implies

$$\frac{1}{\zeta}T(\eta, \zeta) - \zeta.T(\eta, 0) = DE\{(u^4)_{\kappa\kappa} + (u^3)_\kappa\}$$

After simplifications, we obtain

$$T(\eta, \zeta) = \zeta^2.T(\eta, 0) + \zeta.DE\{(u^4)_{\kappa\kappa} + (u^3)_\kappa\}$$

Applying initial conditions, we obtain

$$T(\eta, \zeta) = \zeta^2.\eta^3 + \zeta.DE\{(u^4)_{\kappa\kappa} + (u^3)_\kappa\}$$

Taking the inverse double Elzaki transform, we obtain

$$u(\kappa, \rho) = \kappa + DE^{-1}\{\zeta.DE\{(u^4)_{\kappa\kappa} + (u^3)_\kappa\}\}$$

Using the Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(\kappa, \rho) = \kappa + DE^{-1}\left(\zeta.DE\left\{\sum_{n=0}^{\infty} A_n(u)\right\}\right)$$

Comparing the different powers, we obtain

$$\begin{cases} u_0(\kappa, \rho) = \kappa, \\ u_1(\kappa, \rho) = DE^{-1}(\zeta.DE\{A_0\}), \\ u_2(\kappa, \rho) = DE^{-1}(\zeta.DE\{A_1\}), \\ \vdots \end{cases}$$

and so on. Some of the Adomian components are:

$$\begin{cases} A_0 = 15\kappa^2, \\ A_1 = -1380\kappa^3\rho, \\ A_2 = 45675\kappa^4\rho^2, \\ \vdots \end{cases}$$

and so on. Components of solutions are:

$$\begin{cases} u_0(\kappa,\rho) = \kappa, \\ u_1(\kappa,\rho) = -15\kappa^2\rho, \\ u_2(\kappa,\rho) = 690\kappa^3\rho^2, \\ \vdots \end{cases}$$

The way to solve this is:

$$u(\kappa,\rho) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots$$

Or

$$u(\kappa,\rho) = \kappa - 15\kappa^2\rho + 690\kappa^3\rho^2 + \cdots$$

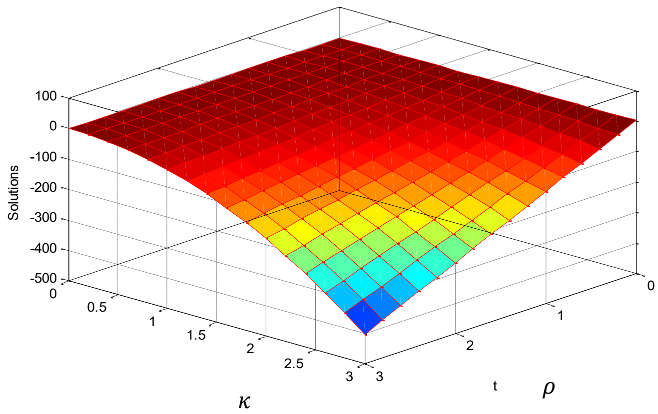

Fig. 5 demonstrates the dynamical and physical behavior of solutions at various κ and ρ ranges

Variational Iteration Method (VIM):

The formula for the variational iteration method is:

$$u_{n+1} = u_n + \lambda \int_0^\rho \left(\frac{\partial u_n}{\partial \rho} + (u_n^4)_{\kappa\kappa} + (u_n^3)_\kappa \right) d\rho$$

Here $\lambda = -1$, and $u_0(\kappa, 0) = \kappa$.

For $n = 0$,

$$u_1 = u_0 - \int_0^\rho \left(\frac{\partial u_0}{\partial \rho} + (u_0^4)_{\kappa\kappa} + (u_0^3)_\kappa \right) d\rho = \kappa - 15\kappa^2 \rho,$$

For $n = 1$,

$$u_2 = u_1 - \int_0^\rho \left(\frac{\partial u_1}{\partial \rho} + (u_1^4)_{\kappa\kappa} + (u_1^3)_\kappa \right) d\rho \qquad (18)$$

For simplifications, we use two terms after expanding $(u_1^4)_{\kappa\kappa}$ and $(u_1^3)_\kappa$ such as

$$u_1^4 \approx (\kappa - 15\kappa^2\rho)^4 \approx \kappa^4 + 4\kappa^3(-15\kappa^2\rho) \approx \kappa^4 - 60\kappa^5\rho,$$

and

$$u_1^3 \approx (\kappa - 15\kappa^2\rho)^3 \approx \kappa^3 + 3\kappa^2(-15\kappa^2\rho) \approx \kappa^3 - 45\kappa^4\rho$$

From (18), we obtain

$$u_2 = (\kappa - 15\kappa^2\rho) - \int_0^\rho (-15\kappa^2 + 12\kappa^2 - 1200\kappa^3\rho + 3\kappa^2 - 90\kappa^3\rho) d\rho$$

After simplifications, we obtain

$$u_2 = (\kappa - 15\kappa^2\rho) + \int_0^\rho (1290\kappa^3\rho) d\rho = \kappa - 15\kappa^2\rho + 645\kappa^3\rho^2,$$

and so on.

The physical and dynamical behavior of solutions of Example 3 at various κ and ρ ranges is depicted in Figure 5.

V. Convergence and Result Discussion

The Adomian Decomposition Method (ADM) expresses the solution of a nonlinear differential equation in the form of an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

The convergence of the above infinite series is not automatic, particularly for nonlinear differential equations. According to classical Adomian theory, convergence is guaranteed if the nonlinear operator $N(u)$ satisfies a Lipschitz condition in a suitable Banach space and the associated linear operator is contractive. Under these assumptions, an infinite series arising in ADM converges absolutely and uniformly to the exact solution. For a class of nonlinear differential equations as considered in this work, convergence depends strongly on the initial conditions, the nonlinearity strength, and parameter regimes. In particular, for sufficiently small initial data and bounded nonlinear terms, the ADM series converges rapidly.

Truncation Error and Residual Analysis: In various computations, an infinite series arising in ADM is truncated after N terms:

$$u^{(N)}(x,t) = \sum_{n=0}^{N} u_n(x,t)$$

and

$$E_N(x,t) = u(x,t) - u^{(N)}(x,t)$$

When the series is convergent, the truncation error satisfies

$$\|E_N\| \le \sum_{n=N+1}^{\infty} \|u_n\|,$$

and

$$R^{(N)}(x,t) = \mathcal{L}\left(u^{(N)}\right) + \mathbb{N}\left(u^{(N)}\right) - f(x,t)$$

Parameter Sensitivity and Convergence Behavior: The convergence behavior of the ADM solution is sensitive to physical and mathematical parameters appearing in the governing equation. Higher nonlinearity coefficients or dispersive parameters may slow convergence or restrict the interval of validity of the series solution. Conversely, moderate parameter values lead to fast convergence with only a few terms required for high accuracy. Numerical experiments demonstrate that increasing the number of ADM terms improves agreement with known exact or numerical solutions, confirming the stability and robustness of the method within the admissible parameter ranges. (see [I], [XXVI])

In particular, we assume that the solution $u(x,t)$ belongs to suitable Sobolev spaces $H^s(R)$, with $s > \frac{3}{2}$ and is exponentially bounded in time. These assumptions guarantee the existence of the Elzaki transform with respect to both spatial and temporal variables. For the Benjamin-Ono equation, the dispersive term involves the nonlocal Hilbert transform. It is well known that the Hilbert transform is a bounded linear operator on $L^2(R)$ and on Sobolev spaces $H^s(R)$. Therefore, the nonlocal dispersive operator is admissible under the double Elzaki transform. (see [XVIII, XIX, XXII])

The above analysis establishes that the ADM solutions obtained in this study are mathematically reliable within appropriate parameter regimes. The convergence discussion, residual analysis, and parameter sensitivity collectively enhance the mathematical completeness of the work and justify the effectiveness of the proposed semi-analytical solutions.

VI. Conclusion

According to the computational data above, the powerful mathematical method known as the double Elzaki transform can be applied in combination with the Adomian decomposition approach to solve the Benjamin-Ono and Buckmaster equations. When an infinite series has varying terms, the solutions are nearer the precise answer. This method will be applied in the future to determine the semi-analytical solutions of fractional nonlinear PDEs that come up in a range of technical and scientific applications.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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