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THEORETICAL AND ALGORITHMIC ANALYSIS OF FAIR DOMINATION AND SUBDIVISION NUMBERS FOR CYCLE AND CIRCULANT GRAPHS

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Abstract

This study explores a specialized type of domination in graphs known as fair domination. A fair dominating set (FDS) in a graph $\mathfrak{R}$ *is defined as a dominating set in which every non-member vertex is adjacent to an equal number of vertices within the set. The minimum size of such a set is referred to as the fair domination number, denoted* $\gamma_{fd}(\mathfrak{R})$. *We further examine how structural modifications, specifically edge subdivisions, affect this parameter. The fair domination subdivision number, denoted* $Sd^{+}_{\gamma_{fd}}(\mathfrak{R})$ *(or* $Sd^{-}_{\gamma_{fd}}(\mathfrak{R})$*), captures the smallest number of edge subdivisions required to increase or decrease the fair domination number, respectively. Our work focuses on computing these values for two graph families: cycles* $C_\mathfrak{n}$ *(with* $\mathfrak{n} \geq 3$*) and Circulant graphs* $C_\mathfrak{n}(1,k), k = 2,3$. *Through detailed analysis, we demonstrate how edge subdivisions impact the fairness condition in domination. To systematically explore fair domination in graphs, we adopt an algorithmic approach that facilitates efficient identification of fair dominating sets and computation of related parameters. Algorithmic techniques have been pivotal in graph theory, particularly in the study of domination-related problems. We introduce an efficient algorithm for identifying fair dominating sets and determining the fair domination number in Circulant graphs of the form* $C_\mathfrak{n}(1,2)$ *and* $C_\mathfrak{n}(1,3)$, *offering insights into their underlying combinatorial structure.*

Keywords: Influence-based vertex covering, Uniform vertex influence, $\boldsymbol{k}$-regular fair domination, Edge-splitting parameter, Subdivision for fair domination.

I. Introduction

The study of domination in graph theory began gaining prominence during the mid-20th century, as researchers became increasingly interested in identifying subsets of vertices that could exert influence or control over the rest of the graph. Among the foundational contributions to this area is the concept of domination introduced by Haynes [XII], which remains one of the most extensively investigated topics in graph theory. Given a graph $\mathfrak{R} = (\mathbb{V}, \mathbb{E})$, a "Dominating Set (DS)" is a subset $\mathbb{Q} \subseteq \mathfrak{R}$ such that every vertex in $\mathbb{V} \setminus \mathbb{Q}$ is adjacent to at least one vertex in $\mathbb{Q}$. The "domination number", denoted $\gamma(\mathfrak{R})$, represents the minimum cardinality of such a set, and a dominating set of this minimum size is called a γ-set [VI, X, XX]. Various extensions and generalizations of domination have attracted significant research attention in recent years. The concept of "Fair Domination", introduced by Caro et al., defines a "Fair Dominating Set" (FDS) as a dominating set $\mathbb{Q}$ in which all vertices not in $\mathbb{Q}$ are dominated by the same number of vertices from $\mathbb{Q}$ [III]. The "fair domination number", denoted $\gamma_{fd}(\mathfrak{R})$, is the smallest size of such a set. An important refinement of fair domination is the concept of the m-fair dominating set (mFD-set), defined as a dominating set $\mathbb{Q} \subseteq \mathfrak{R}$ in which each vertex not in $\mathbb{Q}$ is adjacent to exactly $\mathfrak{m}$ vertices from $\mathbb{Q}$. Formally, for every $\mathfrak{x} \in \mathbb{V} \setminus \mathbb{Q}$, it holds that $|\mathcal{N}(\mathfrak{x}) \cap \mathbb{Q}| = \mathfrak{m}$. A particularly noteworthy instance of this is when $\mathfrak{m} = 1$, which corresponds to the well-established perfect dominating set, where each non-member vertex is dominated by exactly one vertex from the set [III, V, XI]. Extensive studies on fair domination have led to the development of various upper and lower bounds for the fair domination number $\gamma_{fd}(\mathfrak{R})$ [XVI, VII, VIII, IX, XXVI]. The "*fair domination subdivision number*", denoted $Sd^{+}_{\gamma_{fd}}(\mathfrak{R})$ (or $Sd^{-}_{\gamma_{fd}}(\mathfrak{R})$), captures the smallest number of edge subdivisions required to increase or decrease the fair domination number, respectively. Through this investigation, we aim to deepen the understanding of how structural transformations, specifically edge subdivisions, affect the balance and distribution of domination in graphs. By focusing on cyclic and Circulant graph classes, this study not only uncovers new theoretical results but also offers practical computational tools for analyzing fair domination. The concepts and methods introduced here are expected to serve as a foundation for further research in graph modification and optimization, with promising applications in communication networks, distributed systems, and algorithmic graph theory. To systematically explore fair domination in graphs, we adopt an algorithmic approach that facilitates efficient identification of fair dominating sets and computation of related parameters. Algorithmic techniques have been pivotal in graph theory, particularly in the study of domination-related problems. Several researchers have developed algorithms to compute domination numbers, total domination, and their variants, leveraging structural properties of graphs and complexity theory [XIII, XXIII]. Notably, domination problems are often NP-complete, necessitating the design of heuristic, approximation, or parameterized algorithms for tractability.[I, XXI] Recent works have extended these approaches to more nuanced variants such as fair domination and k-fair domination, using modular decomposition, integer programming, and greedy strategies to yield polynomial-time results in special graph classes [XIV, IV, II, XVII]. Our approach builds on these foundations, tailoring algorithms to specific circulant graphs and leveraging symmetry to optimize computation [XVIII, XV, XIX].

II. Notations

Let $\mathfrak{R} = (\mathbb{V}, \mathbb{E})$ be a connected, simple graph with $|\mathbb{V}| = \mathfrak{n}$. We adopt the graph-theoretic terminology established by Harary. For a vertex $\alpha \in \mathbb{V}$, its "*open neighborhood*" is defined as $\mathcal{N}_{\mathfrak{R}}(\mathfrak{v}) = \{\mathfrak{u} \in \mathbb{V}: \mathfrak{u}\mathfrak{v} \in \mathbb{E}\}$, while its "*closed neighborhood*" is given by $\mathcal{N}_{\mathfrak{R}}[\mathfrak{v}] = \mathcal{N}_{\mathfrak{R}}(\mathfrak{v}) \cup \{\mathfrak{v}\}$. For any subset $\mathbb{Q} \subseteq \mathbb{V}$, the "open neighborhood" is $\mathcal{N}_{\mathfrak{R}}(\mathbb{Q}) = \bigcup_{\mathfrak{v}\in\mathbb{Q}} \mathcal{N}_{\mathfrak{R}}(\mathfrak{v})$, and the "closed neighborhood" is $\mathcal{N}_{\mathfrak{R}}[\mathbb{Q}] = \mathcal{N}_{\mathfrak{R}}(\mathfrak{v}) \cup \mathbb{Q}$. The "*private neighborhood*" of a vertex $\mathfrak{v} \in \mathbb{Q}$, denoted $pn(\mathfrak{v}, \mathbb{Q})$, consists of all vertices in $\mathbb{V} \setminus \mathbb{Q}$ that are adjacent to exactly one vertex in $\mathbb{Q}$, specifically $\mathfrak{v}$. Formally, $pn(\mathfrak{v}, \mathbb{Q}) = \{\mathfrak{u} \in \mathbb{V} \setminus \mathbb{Q} : \mathcal{N}_{\mathfrak{R}}(\mathfrak{u}) \cap \mathbb{Q} = \{\mathfrak{v}\}\}$. "A *path* is a finite sequence of distinct vertices connected by edges, with no vertex repeating except possibly the first and last. A cycle is a closed path, beginning and ending at the same vertex. A *complete graph*, denoted $K_{\mathfrak{n}}$, is an undirected graph where each pair of distinct vertices is connected by an edge. A graph is said to be vertex-transitive if for any pair of vertices, there exists an automorphism (a structure-preserving map from the graph to itself) that maps one vertex to the other. In essence, all vertices are structurally identical in terms of connectivity and degree. A *Circulant graph*, denoted $C_{\mathfrak{n}}(1, k)$, is a vertex-transitive graph of order $\mathfrak{n}$, where each vertex $\mathfrak{i}$ is connected to the vertices $\mathfrak{i} + 1$ and $\mathfrak{i} + k$, $\forall\, 1 \leq \mathfrak{i} \leq \mathfrak{n}$. These graphs are frequently used in the design of local area networks due to their symmetry and regularity. The distance between two vertices $\mathfrak{u}$ and $\mathfrak{v}$ in a graph $\mathfrak{R}$ is denoted $d(\mathfrak{u}, \mathfrak{v})$, while the distance between two edges $\mathfrak{e}_1$ and $\mathfrak{e}_2$ is denoted by $d(\mathfrak{e}_1, \mathfrak{e}_2)$."

III. Objective and novelty of the study

This study pioneers a novel investigation into fair domination and its response to edge subdivisions in the cycle graph $C_{\mathfrak{n}}, n \geq 3)$, Circulant graphs $C_{\mathfrak{n}}(1, 2)$ and $C_{\mathfrak{n}}(1, 3)$ introducing a fresh perspective to graph theory within the engineering domain. Fair domination, where every non-member vertex is adjacent to an equal number of vertices in the dominating set, is relatively underexplored, and this work uniquely examines the fair domination number, $\gamma_{fd}(\mathfrak{R})$, alongside the fair domination subdivision numbers, $Sd^{+}_{\gamma_{fd}}(\mathfrak{R})$ (or $Sd^{-}_{\gamma_{fd}}(\mathfrak{R})$) which quantify the minimum edge subdivisions needed to alter this parameter. The objectives are to compute these values for the specified graph families, analyze how edge subdivisions affect the fairness condition, and develop an efficient algorithm for identifying fair dominating sets and computing the fair domination number in $C_{\mathfrak{n}}(1,2)$ and $C_{\mathfrak{n}}(1, 3)$. By blending theoretical analysis with computational techniques, this study bridges abstract graph properties with practical applications, offering new insights into combinatorial structures relevant to network design and optimization in engineering contexts.

IV. Foundational Results

In the development of our results, we make use of several known theorems from graph theory. The following foundational results play a crucial role in the formulation and analysis of our proposed concepts.

Theorem 4.1. [III] For a cycle C_n, $n \geq 3$, $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$

Theorem 4.2. [III] For a cycle $C_n, n \geq 3$,

$$\gamma_{fd}\,(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & n \equiv 0, 1, (mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1, & n \equiv 2\ (mod\ 3) \end{cases}$$

Theorem 4.3. [X] For any integer $n \geq 5$, $\gamma(C_n(1,2)) = \left\lceil \frac{n}{5} \right\rceil$

Theorem 4.4. [XXII] For a Circulant graph $C_n(1,2)$, $n \geq 5$,

$$\gamma_{fd}\,(C_n(1,2)) = \begin{cases} \frac{n}{5}, & n \equiv 0\ (mod\ 5) \\ \left\lceil \frac{n}{3} \right\rceil, & Otherwise \end{cases}$$

Theorem 4.5. [XXV] For any integer $n \geq 5$,

$$\gamma(C_n(1,3)) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil, & n \not\equiv 4\ (mod\ 5) \\ \left\lceil \frac{n}{5} \right\rceil + 1, & n \equiv 4 (mod\ 5) \end{cases}$$

Theorem 4.6. [XXII] For a Circulant graph $C_n(1,3), n \geq 5$,

$$\gamma_{fd}\,(C_n(1,3)) = \begin{cases} \left\lceil \frac{n}{5} \right\rceil, & n \equiv 0, 1, 3\ (mod\ 5) \\ \left\lceil \frac{n}{5} \right\rceil + 1, & n \equiv 2, 4\ (mod\ 5) \end{cases}$$

Observation 1. For any graph $\mathfrak{R}, \gamma(\mathfrak{R}) = \gamma_{fd}(\mathfrak{R})$ then $Sd^{-}_{\gamma_{fd}}(\mathfrak{R}) = 0$

Observation 2. For any graph $\mathfrak{R}$, $\gamma(\mathfrak{R}) < \gamma_{fd}\,(\mathfrak{R})$ if and only if $Sd^{-}_{\gamma_{fd}}(\mathfrak{R}) \geq 1$.

V. Main Results

1. Algorithm to compute Fair Domination number for $C_n(1,2)$

Consider the graph C_n. By Theorem 4.4, we have $\gamma\big(C_n(1,2)\big) = \left\lceil \frac{n}{5} \right\rceil$, since $\gamma(\mathfrak{R}) \leq \gamma_{fd}\,(\mathfrak{R})$, for any graph $\mathfrak{R}$, $\gamma_{fd}\,(C_n(1,2)) \geq \frac{n}{5}$. We define the following algorithm to determine the fair domination set of $C_n(1,2)$.

Algorithm 1: Compute Fair Dominating Set for $C_n(1,2)$

Input: Integer $n \geq 5$

Output: A fair dominating set $\mathbb{Q}' \subseteq \mathbb{V}\,(C_n(1,2))$ and the fair domination number $\gamma_{fd}(C_n(1,2))$

1. Initialize an empty set $\mathbb{Q}' \leftarrow \{\emptyset\}$

2. If $n \equiv 0\ (mod\ 5)$ *then*

a. For $k = 1\ to\ n\,/\,5$

→ *Add vertex* $\mathfrak{v}_{5k-4}\ to\ \mathbb{Q}'$

b. Return $\mathbb{Q}'$ *and* $\gamma_{fd}(C_{\mathfrak{n}}(1,2)) = \frac{\mathfrak{n}}{5}$

3. Else

a. Initialize $\mathbb{Q}' \leftarrow \{\emptyset\}$

b. If $\mathfrak{n} \equiv 0$ *or* $2 \ (mod\ 3)$

i. *For* $k = 1$ *to* $\lceil \mathfrak{n}/3 \rceil$

$\rightarrow$ *Add vertex* $\mathfrak{v}_{\{3k-2\}}$ *to* $\mathbb{Q}'$

ii. *Return* $\mathbb{Q}'$ *and* $\gamma_{fd}(C_{\mathfrak{n}}(1,2)) = \left\lceil \frac{\mathfrak{n}}{3} \right\rceil$

c. Else if $\mathfrak{n} \equiv 1 \ (mod\ 3)$

i. *For* $k = 1$ *to* $\lceil \mathfrak{n}/3 \rceil$

$\rightarrow$ *Add vertex* $\mathfrak{v}_{\{3k-2\}}$ *and* $\mathfrak{v}_{\{\mathfrak{n}-1\}}$ *to* $\mathbb{Q}'$

ii. Return $\mathbb{Q}'$ *and* $\gamma_{fd}(C_{\mathfrak{n}}(1,2)) = \left\lceil \frac{\mathfrak{n}}{3} \right\rceil$

Explanation of the algorithm: The proposed algorithm aims to determine a fair dominating set $\mathbb{Q}' \subseteq \mathbb{V}(C_{\mathfrak{n}}(1,2))$ and compute the fair domination number $\gamma_{fd}(C_{\mathfrak{n}}(1,2))$. The circulant graph $C_{\mathfrak{n}}(1,2)$ is defined on $\mathfrak{n} \geq 5$ vertices, where each vertex $\mathfrak{v}_i$ is adjacent to $\mathfrak{v}_{i+1}$ and $\mathfrak{v}_{i+2}$. Initially, the algorithm begins by setting $\mathbb{Q}'$ to be an empty set. In the case when $\mathfrak{n} \equiv 0 \ (mod\ 5)$, the algorithm selects the vertex set $\mathbb{Q}' = \left\{ \mathfrak{v}_{5k-4} : 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{5} \right\rceil \right\}$, which ensures that each vertex in the graph is either in $\mathbb{Q}'$ or adjacent to exactly one vertex in $\mathbb{Q}'$. Thus, every vertex not in $\mathbb{Q}'$ is dominated by exactly one vertex of $\mathbb{Q}'$ satisfying the 1-fair domination condition. Consequently, the fair domination number is $\gamma_{fd}(C_{\mathfrak{n}}(1,2)) = \frac{\mathfrak{n}}{5}$. When $\mathfrak{n} \not\equiv 0 \ (mod\ 5)$, the previous strategy does not yield a fair dominating set, as some vertices may be dominated by multiple vertices from $\mathbb{Q}'$, violating the fairness condition. In such cases, the algorithm constructs a new set $\mathbb{Q}'$ based on the value of $\mathfrak{n}$ *in term of modulus of* 3. If $\mathfrak{n} \equiv 0$ *or* $2 \ (mod\ 3)$, the algorithm defines $\mathbb{Q}' = \left\{ \mathfrak{v}_{3k-2} : 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{3} \right\rceil \right\}$. This configuration ensures that each vertex not in $\mathbb{Q}'$ is dominated by exactly two vertices of $\mathbb{Q}'$, which forms a 2-fair dominating set. On the other hand, for $\mathfrak{n} \equiv 1 \ (mod\ 3)$, the set $\mathbb{Q}'$ is modified slightly to $\mathbb{Q}' = \left\{ \mathfrak{v}_{3k-2} : 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{3} \right\rceil \cup \mathfrak{v}_{\mathfrak{n}-1} \right\}$, again guaranteeing that every vertex outside $\mathbb{Q}'$ is dominated by exactly two vertices of $\mathbb{Q}'$.

2. Algorithm to compute Fair Domination number for $C_n(1,3)$

Algorithm 2: Compute Fair Dominating Set for $C_{\mathfrak{n}}(1,3)$

Input: Integer $\mathfrak{n} \geq 5$

Output: A fair dominating set $\mathbb{Q}' \subseteq \mathbb{V}(C_{\mathfrak{n}}(1,3))$ and the fair domination number $\gamma_{fd}(C_{\mathfrak{n}}(1,3))$

1. Initialize an empty set $\mathbb{Q}' \leftarrow \{\emptyset\}$

2. If $\mathfrak{n} \equiv 0$ *or* 3 $(mod\ 5)$ *then*

i. For $k = 1$ *to* $\lceil \mathfrak{n} / 5 \rceil$

$\rightarrow$ *Add vertex* $\mathfrak{v}_{5k-4}$ *to* $\mathbb{Q}'$

ii. Return $\mathbb{Q}'$ *and* $\gamma_{fd}(C_{\mathfrak{n}}(1,3)) = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil$

3. a. Else if $\mathfrak{n} \equiv 1$ $(mod\ 5)$ *then*

i. For $k = 1$ *to* $\lfloor \mathfrak{n}/5 \rfloor$

$\rightarrow$ *Add vertex* $\mathfrak{v}_{\{5k-4\}}$ *to* $\mathbb{Q}'$

ii. Add vertex $\mathfrak{v}_{\{\mathfrak{n}-2\}}$ *to* $\mathbb{Q}'$

iii. Return $\mathbb{Q}'$ *and* $\gamma_{fd}(C_{\mathfrak{n}}(1,3)) = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil$

b. Else if $\mathfrak{n} \equiv 2$ $(mod\ 5)$

i. For $k = 1$ *to* $\frac{\mathfrak{n}-7}{5}$

$\rightarrow$ *Add vertex* $\mathfrak{v}_{\{5k-4\}}$ *to* $\mathbb{Q}'$

ii. Add vertex $\{\mathfrak{v}_{\{\mathfrak{n}-2\}}, \mathfrak{v}_{\{\mathfrak{n}-5\}}, \mathfrak{v}_{\{\mathfrak{n}-8\}}\}$ *to* $\mathbb{Q}'$

iii. Return $\mathbb{Q}'$ *and* $\gamma_{fd}\big(C_{\mathfrak{n}}(1,3)\big) = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil + 1$

c. Else if $\mathfrak{n} \equiv 4$ $(mod\ 5)$

i. For $k = 1$ *to* $\frac{\mathfrak{n}-4}{5}$

$\rightarrow$ *Add vertex* $\mathfrak{v}_{\{5k-4\}}$ *to* $\mathbb{Q}'$

ii. Add vertex $\{\mathfrak{v}_{\{\mathfrak{n}-2\}}, \mathfrak{v}_{\{\mathfrak{n}-5\}}\}$ *to* $\mathbb{Q}'$

iii. Return $\mathbb{Q}'$ *and* $\gamma_{fd}\big(C_{\mathfrak{n}}(1,3)\big) = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil + 1$

Explanation of the algorithm: The algorithm aims to determine a fair dominating set $\mathbb{Q}' \subseteq \mathbb{V}\big(C_{\mathfrak{n}}(1,3)\big)$ and the fair domination number $\gamma_{fd}\big(C_{\mathfrak{n}}(1,3)\big)$, where $C_{\mathfrak{n}}(1,3)$ is a Circulant graph with $\mathfrak{n}$ vertices. In this graph, each vertex $\mathfrak{v}_i$ is adjacent to vertices at distances 1 and 3 (i.e., $\mathfrak{v}_{i\pm1}$ and $\mathfrak{v}_{i\pm3}$, with indices taken modulo $\mathfrak{n}$). A fair dominating set is defined as a subset of vertices such that every vertex not in the set is adjacent to exactly one vertex in it, and the set has the minimum possible size. The algorithm works by selecting vertices in a periodic pattern, specifically, those at positions $5k-4$, where k ranges from 1 to $\lceil \mathfrak{n}/5 \rceil$. This selection covers most of the graph fairly. When $\mathfrak{n} \equiv 0$ or $3 \bmod 5$, this pattern alone suffices to fairly dominate all vertices, and the fair domination number is simply $\lceil \mathfrak{n}/5 \rceil$. However, for other values of $\mathfrak{n}$ $(mod\ 5)$, there are leftover vertices that are not dominated. If $\mathfrak{n} \equiv 1 \bmod 5$, the algorithm includes an additional vertex $\mathfrak{v}_{\mathfrak{n}-2}$ to dominate the extra vertex, maintaining $\gamma_{fd} = \lceil \mathfrak{n}/5 \rceil$. When $\mathfrak{n} \equiv 2 \bmod 5$, three additional vertices $\mathfrak{v}_{\mathfrak{n}-2}, \mathfrak{v}_{\mathfrak{n}-5}$, and $\mathfrak{v}_{\mathfrak{n}-8}$ are added to the set to ensure fair domination, increasing the fair domination number to $\lceil \mathfrak{n}/5 \rceil + 1$. Similarly, for $\mathfrak{n} \equiv 4 \bmod 5$, two extra vertices $\mathfrak{v}_{\mathfrak{n}-2}$ and $\mathfrak{v}_{\mathfrak{n}-5}$ are added, and the fair domination number is also $\lceil \mathfrak{n}/5 \rceil + 1$. This approach ensures a minimum-sized fair dominating set that satisfies the fair domination condition for all values of $\mathfrak{n} \geq 5$.

Complexity Analysis: Both Algorithm 1 and Algorithm 2 for computing fair dominating sets in circulant graphs $C_{\mathfrak{n}}(1,2)$ and $C_{\mathfrak{n}}(1,3)$ demonstrate linear computational efficiency and strong scalability. The time complexity of each algorithm is $O(\mathfrak{n})$, where $\mathfrak{n}$ is the order of the graph. In Algorithm 1, this arises from a single loop executing either $\left\lceil\frac{\mathfrak{n}}{5}\right\rceil$ *or* $\left\lceil\frac{\mathfrak{n}}{3}\right\rceil$ iterations depending on the residue of $\mathfrak{n}\ (mod\ 5\ or\ 3)$, respectively, with each iteration performing constant-time vertex selection and insertion operations. Similarly, Algorithm 2 follows a comparable structure, classifying $\mathfrak{n}$ based on its modulus 5 and performing a single pass through the vertex set with approximately $\left\lceil\frac{\mathfrak{n}}{5}\right\rceil$ iterations (varying slightly for residual cases $\mathfrak{n} \equiv 1,2,3,4\ (mod\ 5)$. Each iteration consists of constant-time updates, and any post-loop adjustments for boundary conditions contribute negligibly to the total runtime. Consequently, the dominant term remains linear, yielding an overall time complexity of $O(\mathfrak{n})$ for both algorithms. The space complexity is also linear, $O(\mathfrak{n})$, as it depends primarily on the storage required for the resulting vertex subset $\mathbb{Q}'$, while auxiliary variables for iteration and comparison occupy constant space. Therefore, both algorithms belong to the polynomial-time complexity class P, ensuring excellent scalability and practicality for large circulant graphs. Compared to general domination algorithms, which often involve $O(\mathfrak{n}^2)$ adjacency traversals or combinatorial searches, these methods are significantly more efficient due to their deterministic, pattern-based vertex selection, making them computationally optimal for structured graph families such as $C_{\mathfrak{n}}(1,2)$ and $C_{\mathfrak{n}}(1,3)$.

Table 1. Comparative Complexity Analysis of Algorithm 1 and Algorithm 2

Aspect	Algorithm 1 $C_n(1,2)$	Algorithm 2 $C_n(1,3)$	Comparison
Loop Structure	Single loop with $\left\lceil\frac{\mathfrak{n}}{5}\right\rceil$ or $\left\lceil\frac{\mathfrak{n}}{3}\right\rceil$ iterations	Single loop with $\left\lceil\frac{\mathfrak{n}}{5}\right\rceil$ iterations (depending on residue)	Both perform a single pass through the vertex sequence
Time Complexity	$O(\mathfrak{n})$	$O(\mathfrak{n})$	Identical; both scale linearly with graph order
Space Complexity	$O(\mathfrak{n})$ (for storing $\mathbb{Q}'$)	$O(\mathfrak{n})$ (for storing $\mathbb{Q}'$)	Identical space requirements
Dominant Operations	Constant-time vertex insertion within each iteration	Constant-time operations	No asymptotic difference
Scalability	Excellent for large circulant graphs	Excellent for large circulant graphs	Both are optimal for structured graphs
Computational Class	Polynomial-time (P)	Polynomial-time (P)	Same complexity class

Theorem 5.1. For cycle $C_{\mathfrak{n}}, \mathfrak{n} \geq 3$

$$Sd^{+}_{\gamma_{fd}}(C_{\mathfrak{n}}) = \begin{cases} 1, & \mathfrak{n} \equiv 0,1\ (mod\ 3) \\ 3, & \mathfrak{n} \equiv 2\ (mod\ 3) \end{cases}$$

$$Sd^{-}_{\gamma_{fd}}(C_{\mathfrak{n}}) = \begin{cases} 1, & \mathfrak{n} \equiv 2\ (mod\ 3) \\ 0, & Otherwise \end{cases}$$

Proof: Let "$\mathfrak{R} \cong C_{\mathfrak{n}}$ be a cycle graph with $\mathfrak{n} \geq 3$, where the vertex set is given by $\mathbb{V}(\mathfrak{R}) = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \dots, \mathfrak{v}_{\mathfrak{n}}\}$ and the edge set is $\mathbb{E}(\mathfrak{R}) = \{\mathfrak{v}_1\mathfrak{v}_2, \mathfrak{v}_2\mathfrak{v}_3, \mathfrak{v}_3\mathfrak{v}_4, \dots, \mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1\}$".

According to Theorems 4.1 and 4.2, the domination number $\gamma(\mathfrak{R})$ is $\left\lceil\frac{\mathfrak{n}}{3}\right\rceil$, and the fair domination number $\gamma_{fd}(\mathfrak{R})$ is given by

$$\gamma_{fd}(\mathfrak{R}) = \begin{cases} \left\lceil\frac{\mathfrak{n}}{3}\right\rceil, & \text{if } \mathfrak{n} \equiv 0,1 \pmod 3 \\ \left\lceil\frac{\mathfrak{n}}{3}\right\rceil + 1, & \text{if } \mathfrak{n} \equiv 2 \pmod 3 \end{cases}$$

Let $\mathbb{Q}$ be a γ-set, and $\mathbb{Q}'$ be a γ_{fd}-set of $\mathfrak{R}$, defined as

$$\mathbb{Q} = \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \left\lceil\frac{\mathfrak{n}}{3}\right\rceil\}$$

$$\mathbb{Q}' = \begin{cases} \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \left\lceil\frac{\mathfrak{n}}{3}\right\rceil\}, & \text{if } \mathfrak{n} \equiv 0,1 \pmod 3 \\ \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \left\lceil\frac{\mathfrak{n}}{3}\right\rceil\} \cup \{v_{\mathfrak{n}}\}, & \text{if } \mathfrak{n} \equiv 2 \pmod 3 \end{cases}$$

Now, construct a graph $\mathfrak{R}_1$ by subdividing each edge $\mathfrak{v}_{\mathfrak{i}}\mathfrak{v}_{\mathfrak{i}+1}$ of $\mathfrak{R}$ using subdivision vertices $\{\mathfrak{x}_1, \mathfrak{x}_2, \dots, \mathfrak{x}_{\mathfrak{n}}\}$, and define the new vertices as $\mathfrak{v}_{\mathfrak{n}+\mathfrak{i}} = \mathfrak{v}_{\mathfrak{i}}$ for $1 \leq \mathfrak{i} \leq \mathfrak{n}$. We now consider the following three cases.

Case (i): $\mathfrak{n} \equiv 0,1 \ (mod\ 3)$

Subcase 1: If $\mathfrak{n} \equiv 0 \pmod 3$, then $\mathbb{Q} = \mathbb{Q}' = \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \frac{\mathfrak{n}}{3}\}$, and both $\gamma(\mathfrak{R})$ and $\gamma_{fd}(\mathfrak{R})$ equal to $\mathfrak{n}/3$. Subdividing a single edge, say $\mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1$, by introducing a vertex $\mathfrak{x}_1$, yields the graph $\mathfrak{R}_1 \cong C_{\mathfrak{n}+1}$, where $\mathfrak{n}+1 \equiv 1 \pmod 3$. In this case, $\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{x}_1\}$, and hence $\gamma_{fd}(\mathfrak{R}_1) = \frac{\mathfrak{n}}{3} + 1$. Since $\gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$, we conclude that $Sd^+_{\gamma_{fd}}(\mathfrak{R}) = 1$, and from Observation 1, $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 0$.

Subcase 2: If $\mathfrak{n} \equiv 1 \pmod 3$, then similarly $\mathbb{Q} = \mathbb{Q}' = \left\{\mathfrak{v}_{3k-2}: 1 \leq k \leq \left\lceil\frac{\mathfrak{n}}{3}\right\rceil\right\}$, and $\gamma(\mathfrak{R}) = \gamma_{fd}(\mathfrak{R}) = \left\lceil\frac{\mathfrak{n}}{3}\right\rceil$. Subdividing an edge, such as $\mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1$ by a vertex $\mathfrak{x}_1$ yields $\mathfrak{R}_1 \cong C_{\mathfrak{n}+1}$, with $\mathfrak{n}+1 \equiv 2 \pmod 3$. Then, $\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{x}_1\}$ and $\gamma_{fd}(\mathfrak{R}_1) = \left\lceil\frac{\mathfrak{n}}{3}\right\rceil + 1$, again implying $Sd^+_{\gamma_{fd}}(\mathfrak{R}) = 1$ and $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 0$.

Case (ii): $\mathfrak{n} \equiv 2 \ (mod\ 3)$

Here, $\mathbb{Q}' = \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \lceil\frac{\mathfrak{n}}{3}\rceil\} \cup \{\mathfrak{v}_{\mathfrak{n}}\}$, and $\gamma_{fd}(\mathfrak{R}) = \lceil\frac{\mathfrak{n}}{3}\rceil + 1$. Subdividing an edge, such as $\mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1$ with vertex $\mathfrak{x}_1$ gives $\mathfrak{R}_1 \cong C_{\mathfrak{n}+1}$, where $\mathfrak{n}+1 \equiv 0 \pmod 3$. Define $\mathbb{Q}_1 = \mathbb{Q}' \setminus \{\mathfrak{v}_{\mathfrak{n}}\}$, which gives $\gamma_{fd}(\mathfrak{R}_1) = \lceil\frac{\mathfrak{n}}{3}\rceil$. Since $\gamma_{fd}(\mathfrak{R}_1) < \gamma_{fd}(\mathfrak{R})$, we have $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 1$, and clearly $Sd^+_{\gamma_{fd}}(\mathfrak{R}) > 1$. Now, if we further subdivide two edges, say $\mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1$ and $\mathfrak{v}_{\mathfrak{n}-1}\mathfrak{v}_{\mathfrak{n}}$, by inserting $\mathfrak{x}_1$ and $\mathfrak{x}_2$, the resulting graph is $\mathfrak{R}_1 \cong C_{\mathfrak{n}+2}$, where $\mathfrak{n}+2 \equiv 1 \pmod 3$, and we have $\mathbb{Q}_1 = (\mathbb{Q}' \setminus \{\mathfrak{v}_{\mathfrak{n}}\}) \cup \{\mathfrak{x}_1\}$. In this case, $\gamma_{fd}(\mathfrak{R}_1) = \left\lceil\frac{\mathfrak{n}}{3}\right\rceil + 1 = \gamma_{fd}(\mathfrak{R})$, and thus $Sd^+_{\gamma_{fd}}(\mathfrak{R}) > 2$. Finally, if we subdivide three edges, say $\mathfrak{v}_1\mathfrak{v}_2, \mathfrak{v}_2\mathfrak{v}_3, \mathfrak{v}_3\mathfrak{v}_4$ using vertices $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3$, we obtain $\mathfrak{R}_1 \cong C_{\mathfrak{n}+3}$, where $\mathfrak{n}+3 \equiv$

2 (mod 3), and define $\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{x}_2\}$. In this scenario, $\gamma_{fd}(\mathfrak{R}_1) = \left\lceil \frac{\mathfrak{n}}{3} \right\rceil + 2$, which is greater than $\gamma_{fd}(\mathfrak{R})$, leading to the conclusion that $Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) = 3$.

Theorem 5.2. For a Circulant graph $C_{\mathfrak{n}}(1,2), \mathfrak{n} \geq 5$,

$$Sd^{+}_{\gamma_{fd}}(C_{\mathfrak{n}}(1,2)) = \begin{cases} 2, & \mathfrak{n} \equiv 2 \ (mod\ 5)\ and\ \mathfrak{n} \not\equiv 0\ (mod\ 3) \\ 1, & Otherwise \end{cases}$$

Proof: Let "$\mathfrak{R} \cong C_{\mathfrak{n}}(1,2)$ be a Circulant graph on $\mathfrak{n}$ vertices, where $\mathbb{V}(\mathfrak{R}) = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \ldots, \mathfrak{v}_{\mathfrak{n}}\}$ is the vertex set and $\mathbb{E}(\mathfrak{R})$ is the edge set. The edge set can be partitioned as follows $\mathbb{E}_1(\mathfrak{R}) = \{\mathfrak{v}_1\mathfrak{v}_2, \mathfrak{v}_2\mathfrak{v}_3, \mathfrak{v}_3\mathfrak{v}_4, \ldots, \mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1\}$ represents the edges of the outer cycle, while $\mathbb{E}_2(\mathfrak{R}) = \mathbb{E}(\mathfrak{R}) \backslash \mathbb{E}_1(\mathfrak{R})$ denotes the set of inner chords. Thus, the full edge set is given by $\mathbb{E}(\mathfrak{R}) = \mathbb{E}_1(\mathfrak{R}) \cup \mathbb{E}_2(\mathfrak{R})$". Based on theorems 4.3 and 4.4, the domination number and fair domination number of $\mathfrak{R}$ are as follows

$$\gamma(\mathfrak{R}) = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil,$$

$$\gamma_{fd}(\mathfrak{R}) = \begin{cases} \frac{\mathfrak{n}}{5}, & \text{if } \mathfrak{n} \equiv 0 \text{ (mod 5)} \\ \left\lceil \frac{\mathfrak{n}}{3} \right\rceil, & \text{otherwise} \end{cases}$$

Let $\mathbb{Q}$ and $\mathbb{Q}'$ denote a γ-set and a γ_{fd}-set of $\mathfrak{R}$, respectively. Now, construct a new graph $\mathfrak{R}_1$ by subdividing every edge $\mathfrak{e}_i \in \mathbb{E}(\mathfrak{R}) = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_{\mathfrak{m}}\}$ with corresponding subdivision vertices $\{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_{\mathfrak{m}}\}$. Let $\mathbb{Q}_1$ denote a fair dominating set of $\mathfrak{R}_1$. We now examine the following case,

Case (i): $\mathfrak{n} \equiv 2\ (mod\ 5)$

For this case, the dominating and fair dominating sets of $\mathfrak{R}$ are defined as follows:

$$\mathbb{Q} = \{\mathfrak{v}_{5k-4} : 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\}$$

$$\mathbb{Q}' = \{\mathfrak{v}_{3k-2} : 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{3} \right\rceil\}$$

We now explore the subdivision impact based on $\mathfrak{n} \equiv 0$ (mod 3).

Subcase 1: $\mathfrak{n} \not\equiv 0\ (mod\ 3)$

In this case, we observe that $|\mathbb{Q}| = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil$ and $|\mathbb{Q}'| = \left\lceil \frac{\mathfrak{n}}{3} \right\rceil$, with the inequality $|\mathbb{Q}| \leq |\mathbb{Q}'|$ holding for all $\mathfrak{n} \geq 5$. Let us define $\mathfrak{v}_{\mathfrak{n}+i} = \mathfrak{v}_i$ for $1 \leq i \leq \mathfrak{n}$, to maintain cyclic indexing. Now, consider subdividing an edge from the outer cycle, specifically $\mathfrak{v}_{\mathfrak{n}-1}\mathfrak{v}_{\mathfrak{n}} \in \mathbb{E}_1(\mathfrak{R})$, by inserting a new vertex $\mathfrak{w}_1$. In the resulting graph $\mathfrak{R}_1$, a fair dominating set is given by

$$\mathbb{Q}_1 = \{\mathfrak{v}_{5k-4} : 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\}$$

This set $\mathbb{Q}_1$ is a 1-fair dominating set, identical to $\mathbb{Q}$, and it satisfies $|\mathbb{Q}_1| = |\mathbb{Q}| < |\mathbb{Q}'|$. Hence, the fair domination number decreases upon this subdivision, i.e., $\gamma_{fd}(\mathfrak{R}_1) < \gamma_{fd}(\mathfrak{R})$. Alternatively, consider subdividing an edge from the inner cycle, such as

$\mathfrak{v}_{\mathfrak{n}-1}\mathfrak{v}_{\mathfrak{n}+1} \in \mathbb{E}_2(\mathfrak{R})$, by adding a vertex $\mathfrak{w}_1$. In this case, the fair dominating set of $\mathfrak{R}_1$ remains the same as $\mathbb{Q}'$, with $|\mathbb{Q}_1| = |\mathbb{Q}'|$. Therefore, the fair domination number remains unchanged, i.e., $\gamma_{fd}(\mathfrak{R}_1) = \gamma_{fd}(\mathfrak{R})$. Thus, from the above observations, we conclude that depending on the edge chosen for subdivision, the fair domination number may either decrease or remain the same. This implies that $Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) > 1$.

Now, consider subdividing two edges of the form $\mathfrak{v}_i\mathfrak{v}_{i+2} \in \mathbb{E}_2(\mathfrak{R})$, specifically for $\mathfrak{i} = \mathfrak{n} - 1$ and $\mathfrak{i} = 1$, using subdivision vertices $\mathfrak{w}_1$ and $\mathfrak{w}_2$, respectively. Let the resulting graph be denoted by $\mathfrak{R}_1$, and define the fair dominating set of $\mathfrak{R}_1$ as

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_2\}$$

In this case, the cardinality of the new fair dominating set becomes

$$|\mathbb{Q}_1| = |\mathbb{Q}'| + 1$$

Clearly, $|\mathbb{Q}| > |\mathbb{Q}'|$, which implies

$$\gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$$

Therefore, the fair domination subdivision number of $\mathfrak{R}$ is 2, i.e., $Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) = 2$.

Subcase 2: $\mathfrak{n} \equiv 0 \ (mod\ 3)$

In this case, construct $\mathfrak{R}_1$ by subdividing an edge either from the outer cycle $\mathfrak{v}_i\mathfrak{v}_{i+1} \in \mathbb{E}_1(\mathfrak{R})$ or from the inner chords $\mathfrak{v}_i\mathfrak{v}_{i+2} \in \mathbb{E}_2(\mathfrak{R})$, taking $\mathfrak{i} = 1$ without loss of generality. Introducing a subdivision vertex $\mathfrak{w}_1$, the fair dominating set of $\mathfrak{R}_1$ becomes

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_1\}$$

Thus

$$|\mathbb{Q}_1| = |\mathbb{Q}'| + 1$$

$$\Rightarrow \gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$$

Hence,

$$Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) = 1$$

Case (ii): $\mathfrak{n} \not\equiv 2 \ (mod\ 5)$

This case branches into several subcases

Subcase 3: $\mathfrak{n} \equiv 0 \ (mod\ 5)$

For $\mathfrak{n} \equiv 0$ (mod 5), define the sets

$$\mathbb{Q} = \left\{\mathfrak{v}_{5k-4} : 1 \le k \le \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\right\}$$
$$\mathbb{Q}' = \{\mathfrak{v}_{5k-4} : 1 \le k \le \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\}$$

In this situation, the cardinalities of both sets are equal, $|\mathbb{Q}| = |\mathbb{Q}'|$. Let $\mathfrak{R}_1$ be formed by subdividing a single edge, either from the outer cycle $\mathfrak{v}_i\mathfrak{v}_{i+1} \in \mathbb{E}_1(\mathfrak{R})$ or from the inner chords $\mathfrak{v}_i\mathfrak{v}_{i+2} \in \mathbb{E}_2(\mathfrak{R})$, with $\mathfrak{i} = 1$. The fair dominating set of $\mathfrak{R}_1$ is then

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_1\}$$
$$\Rightarrow |\mathbb{Q}_1| = |\mathbb{Q}'| + 1$$

This implies

$$\gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$$
$$\Rightarrow Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) = 1$$

Subcase 4: $\mathfrak{n} \equiv 1,3,4\ (mod\ 5)$

For values of $\mathfrak{n}$ satisfying $\mathfrak{n} \equiv 1,3,4$ (mod 5), the sets $\mathbb{Q}$ and $\mathbb{Q}'$ are defined as

$$\mathbb{Q} = \{\mathfrak{v}_{5k-4} : 1 \le k \le \lceil \mathfrak{n}/5 \rceil\}$$
$$\mathbb{Q}' = \begin{cases} \{\mathfrak{v}_{3k-2} : 1 \le k \le \left\lceil \frac{\mathfrak{n}}{3} \right\rceil\}, & \text{if } \mathfrak{n} \equiv 0,2 \text{ (mod 3)} \\ \{\mathfrak{v}_{3k-2} : 1 \le k \le \left\lfloor \frac{\mathfrak{n}}{3} \right\rfloor\} \cup \{\mathfrak{v}_{\mathfrak{n}-1}\}, & \text{if } \mathfrak{n} \equiv 1 \text{ (mod 3)} \end{cases}$$

Now, let $\mathfrak{R}_1$ be obtained by subdividing an edge $\mathfrak{v}_i\mathfrak{v}_{i+1}$, taking $i = 1$ as an example, and inserting a subdivision vertex $\mathfrak{w}_1$. Then, the fair dominating set becomes

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_1\}$$
$$\Rightarrow |\mathbb{Q}_1| = |\mathbb{Q}'| + 1$$

Consequently, we conclude

$$\gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$$
$$\Rightarrow Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) = 1$$

Theorem 5.3. For a Circulant graph $C_{\mathfrak{n}}(1,2), \mathfrak{n} \ge 5$,

$$Sd^{-}_{\gamma_{fd}}(C_{\mathfrak{n}}(1,2)) = \begin{cases} 1, & \mathfrak{n} \equiv 2,4\ (mod\ 5) \\ 2, & \mathfrak{n} \equiv 1,3\ (mod\ 5)\ and\ \mathfrak{n} \ne 6 \\ 0, & Otherwise \end{cases}$$

Proof: Let "$\mathfrak{R} \cong C_{\mathfrak{n}}(1,2)$ be a graph with vertex set $\mathbb{V}(\mathfrak{R}) = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \dots, \mathfrak{v}_{\mathfrak{n}}\}$ and edge set $\mathbb{E}(\mathfrak{R})$. Define $\mathbb{E}_1(\mathfrak{R}) = \{\mathfrak{v}_1\mathfrak{v}_2, \mathfrak{v}_2\mathfrak{v}_3, \dots, \mathfrak{v}_{\mathfrak{n}}\mathfrak{v}_1\}$ as the edges forming the outer cycle of $\mathfrak{R}$, and let $\mathbb{E}_2(\mathfrak{R}) = \mathbb{E}(\mathfrak{R}) \backslash \mathbb{E}_1(\mathfrak{R})$, which consists of the edges forming the inner chords. Therefore, $\mathbb{E}(\mathfrak{R}) = \mathbb{E}_1(\mathfrak{R}) \cup \mathbb{E}_2(\mathfrak{R})$". Based on theorems 4.3 and 4.4, the domination number and fair domination number of $\mathfrak{R}$ are as follows

$$\gamma(\mathfrak{R}) = \left\lceil \frac{\mathfrak{n}}{5} \right\rceil$$
$$\gamma_{fd}(\mathfrak{R}) = \begin{cases} \frac{\mathfrak{n}}{5}, & \text{if } \mathfrak{n} \equiv 0 \text{ (mod 5)} \\ \left\lceil \frac{\mathfrak{n}}{3} \right\rceil, & \text{otherwise} \end{cases}$$

Let $\mathbb{Q}$ and $\mathbb{Q}'$ denote the γ-set and γ_{fd}-set of $\mathfrak{R}$, respectively, defined as follows.

$$\mathbb{Q} = \{\mathfrak{v}_{5k-4}: 1 \leq k \leq \lceil \mathfrak{n}/5 \rceil\}$$

$$\mathbb{Q}' = \begin{cases} \{\mathfrak{v}_{5k-4}: 1 \leq k \leq \lceil \mathfrak{n}/5 \rceil\}, & \text{if } \mathfrak{n} \equiv 0 \text{ (mod 5)} \\ \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \lceil \mathfrak{n}/3 \rceil\}, & \text{if } \mathfrak{n} \equiv 0,2 \text{ (mod 3) and } \mathfrak{n} \not\equiv 0 \text{ (mod 5)} \\ \{\mathfrak{v}_{3k-2}: 1 \leq k \leq \lfloor \mathfrak{n}/3 \rfloor\} \cup \{\mathfrak{v}_{\mathfrak{n}-1}\}, & \text{if } \mathfrak{n} \equiv 1 \text{ (mod 3) and } \mathfrak{n} \not\equiv 0 \text{ (mod 5)} \end{cases}$$

Let $\mathfrak{R}_1$ be the graph obtained by subdividing the edges $\mathbb{E}(\mathfrak{R}) = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_\mathfrak{m}\}$ using subdivision vertices $\{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_\mathfrak{m}\}$, and let $\mathbb{Q}_1$ be the fair dominating set of $\mathfrak{R}_1$. Now, we analyze the following cases.

Case (i): $\mathfrak{n} \equiv 0\ (mod\ 5)$ *and* $\mathfrak{n} = 6$

In this scenario, $\gamma(\mathfrak{R}) = \gamma_{fd}(\mathfrak{R})$. By observation 1, no subdivision is required to equalise the domination and fair domination numbers, hence $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 0$.

Case (ii): $\mathfrak{n} \equiv 1,3\ (mod\ 5)$*, with* $\mathfrak{n} \neq 6$

Here, $\gamma_{fd}(\mathfrak{R}) > \gamma(\mathfrak{R})$. Consider the following subcases

Subcase 1: $\mathfrak{n} \equiv 1\ (mod\ 5)$

In this case, every vertex $\mathfrak{v} \in \mathbb{V}\backslash\mathbb{Q}$ is dominated by a unique vertex in $\mathbb{Q}$, except $\mathfrak{v}_2$ and $\mathfrak{v}_{\mathfrak{n}-1}$, which are both dominated by $\mathfrak{v}_1$ and $\mathfrak{v}_\mathfrak{n}$ (since $\mathfrak{v}_1, \mathfrak{v}_\mathfrak{n} \in \mathbb{Q}$). Subdividing the edges $\mathfrak{v}_1\mathfrak{v}_2$ and $\mathfrak{v}_\mathfrak{n}\mathfrak{v}_{\mathfrak{n}-1} \in \mathbb{E}_1(\mathfrak{R})$ by vertices $\mathfrak{w}_1$ and $\mathfrak{w}_2$, respectively, results in a graph $\mathfrak{R}_1$ where the original set $\mathbb{Q}$ now forms a 1-fair dominating set. Hence, $\mathbb{Q}_1 = \mathbb{Q}$, and $\gamma_{fd}(\mathfrak{R}_1) = \gamma(\mathfrak{R})$. Since $|\mathbb{Q}_1| < |\mathbb{Q}'|$, we conclude $\gamma_{fd}(\mathfrak{R}_1) < \gamma_{fd}(\mathfrak{R})$, implying $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 2$.

Subcase 2: $\mathfrak{n} \equiv 3\ (mod\ 5)$

Here, all vertices in $\mathbb{V}\backslash\mathbb{Q}$ are uniquely dominated by $\mathbb{Q}$, except $\mathfrak{v}_\mathfrak{n}$ and $\mathfrak{v}_{\mathfrak{n}-1}$, which are both dominated by $\mathfrak{v}_1$ and $\mathfrak{v}_{\mathfrak{n}-2} \in \mathbb{Q}$. Subdividing the edges $\mathfrak{v}_{\mathfrak{n}-1}\mathfrak{v}_{\mathfrak{n}-2}$ and $\mathfrak{v}_1\mathfrak{v}_\mathfrak{n}$ by w_1 and w_2, respectively, results in $\mathbb{Q}$ being a 1-fair dominating set of $\mathfrak{R}_1$. Thus, $\mathbb{Q}_1 = \mathbb{Q}$, $\gamma_{fd}(\mathfrak{R}_1) = \gamma(\mathfrak{R})$, and since $|\mathbb{Q}_1| < |\mathbb{Q}'|$, we get $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 2$.

Case (iii): $\mathfrak{n} \equiv 2,4\ (mod\ 5)$

Again here, $\gamma_{fd}(\mathfrak{R}) > \gamma(\mathfrak{R})$. Consider the following subcases

Subcase 3: $\mathfrak{n} \equiv 2\ (mod\ 5)$

In this setting, all vertices $\mathfrak{v} \in \mathbb{V}\backslash\mathbb{Q}$ are 1-dominated by $\mathbb{Q}$ except $\mathfrak{v}_\mathfrak{n}$, which is dominated by both $\mathfrak{v}_{\mathfrak{n}-1}$ and $\mathfrak{v}_1$. Subdividing the edge $\mathfrak{v}_1\mathfrak{v}_\mathfrak{n} \in \mathbb{E}_1(\mathfrak{R})$ with vertex $\mathfrak{w}_1$ ensures each vertex outside $\mathbb{Q}$ is now 1-dominated. Therefore, $\mathbb{Q}_1 = \mathbb{Q}$ becomes a 1FD-set of $\mathfrak{R}_1$, and $\gamma_{fd}(\mathfrak{R}_1) = \gamma(\mathfrak{R})$. Thus, $|\mathbb{Q}_1| < |\mathbb{Q}'|$, and we conclude $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 1$.

Subcase 4: $\mathfrak{n} \equiv 4\ (mod\ 5)$

Here, all vertices in $\mathbb{V}\backslash\mathbb{Q}$ are 1-dominated by $\mathbb{Q}$, except $\mathfrak{v}_{\mathfrak{n}-1}$, which is dominated by both $\mathfrak{v}_1$ and $\mathfrak{v}_{\mathfrak{n}-3} \in \mathbb{Q}$. By subdividing the edge $\mathfrak{v}_1\mathfrak{v}_{\mathfrak{n}-1} \in \mathbb{E}_2(\mathfrak{R})$ using vertex $\mathfrak{w}_1$, we ensure all vertices are uniquely dominated. Thus, $\mathbb{Q}_1 = \mathbb{Q}$, and $\gamma_{fd}(\mathfrak{R}_1) = \gamma(\mathfrak{R})$. Since $|\mathbb{Q}_1| < |\mathbb{Q}'|$, we conclude $Sd^-_{\gamma_{fd}}(\mathfrak{R}) = 1$.

Theorem 5.4. For a Circulant graph $C_\mathfrak{n}(1,3), \mathfrak{n} \geq 5$,

$$Sd^+_{\gamma_{fd}}\big(C_\mathfrak{n}(1,3)\big) = 1$$

$$Sd^-_{\gamma_{fd}}(C_\mathfrak{n}(1,3)) = \begin{cases} 3, & \mathfrak{n} \equiv 2 \pmod 5 \\ 0, & Otherwise \end{cases}$$

Proof: Let "$\mathfrak{R} \cong C_\mathfrak{n}(1,3)$ be a graph with a vertex set $(\mathfrak{R}) = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \dots, \mathfrak{v}_\mathfrak{n}\}$. Let $\mathbb{E}(\mathfrak{R})$ denote the edge set of $\mathfrak{R}$, which can be partitioned as follows $\mathbb{E}_1(\mathfrak{R}) = \{\mathfrak{v}_1\mathfrak{v}_2, \mathfrak{v}_2\mathfrak{v}_3, \dots, \mathfrak{v}_\mathfrak{n}\mathfrak{v}_1\}$, representing the outer cycle of $\mathfrak{R}$, and $\mathbb{E}_2(\mathfrak{R}) = \mathbb{E}(\mathfrak{R})\backslash\mathbb{E}_1(\mathfrak{R})$, representing the set of inner edges (chords) of $\mathfrak{R}$. Thus, the total edge set is $\mathbb{E}(\mathfrak{R}) = \mathbb{E}_1(\mathfrak{R}) \cup \mathbb{E}_2(\mathfrak{R})$". According to Theorems 4.5 and 4.6, the domination number $\gamma(\mathfrak{R})$ and the fair domination number $\gamma_{fd}(\mathfrak{R})$ are given by

$$\gamma(\mathfrak{R}) = \begin{cases} \left\lceil \frac{\mathfrak{n}}{5} \right\rceil, & \text{if } \mathfrak{n} \not\equiv 4 \pmod 5 \\ \left\lceil \frac{\mathfrak{n}}{5} \right\rceil + 1, & \text{if } \mathfrak{n} \equiv 4 \pmod 5 \end{cases}$$

$$\gamma_{fd}(\mathfrak{R}) = \begin{cases} \left\lceil \frac{\mathfrak{n}}{5} \right\rceil, & \text{if } \mathfrak{n} \equiv 0,1,3 \pmod 5 \\ \left\lceil \frac{\mathfrak{n}}{5} \right\rceil + 1, & \text{if } \mathfrak{n} \equiv 2,4 \pmod 5 \end{cases}$$

Let $\mathbb{Q}$ and $\mathbb{Q}'$ be a γ-set and a γ_{fd}-set of $\mathfrak{R}$, respectively. They are defined as follows

$$\mathbb{Q} = \begin{cases} \{\mathfrak{v}_{5k-4} \mid 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\}, & \text{if } \mathfrak{n} \not\equiv 4 \pmod 5 \\ \{\mathfrak{v}_{5k-4} \mid 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\} \cup \{\mathfrak{v}_{\mathfrak{n}-1}\}, & \text{if } \mathfrak{n} \equiv 4 \pmod 5 \end{cases}$$

$$\mathbb{Q}' = \begin{cases} \{\mathfrak{v}_{5k-4} \mid 1 \leq k \leq \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\}, & \text{if } \mathfrak{n} \equiv 0,3 \pmod 5 \\ \{\mathfrak{v}_{5k-4} \mid 1 \leq k \leq \left\lfloor \frac{\mathfrak{n}}{5} \right\rfloor\} \cup \{\mathfrak{v}_{\mathfrak{n}-2}\}, & \text{if } \mathfrak{n} \equiv 1 \pmod 5 \\ \{\mathfrak{v}_{5k-4} \mid 1 \leq k \leq \frac{\mathfrak{n}-7}{5}\} \cup \{\mathfrak{v}_{\mathfrak{n}-2}, \mathfrak{v}_{\mathfrak{n}-5}, \mathfrak{v}_{\mathfrak{n}-8}\}, & \text{if } \mathfrak{n} \equiv 2 \pmod 5 \\ \{\mathfrak{v}_{5k-4} \mid 1 \leq k \leq \frac{\mathfrak{n}-4}{5}\} \cup \{\mathfrak{v}_{\mathfrak{n}-2}, \mathfrak{v}_{\mathfrak{n}-5}\}, & \text{if } \mathfrak{n} \equiv 4 \pmod 5 \end{cases}$$

Let $\mathfrak{R}_1$ be the graph obtained by subdividing each edge $\mathfrak{e}_i \in \mathbb{E}(\mathfrak{R}) = \{\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_m\}$ by introducing new subdivision vertices $\{\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_m\}$. Let $\mathbb{Q}_1$ denote the γ_{fd}-set of $\mathfrak{R}_1$.

Case (i): $Sd^+_{\gamma_{fd}}(\mathfrak{R}) = 1$

Since $\mathbb{Q}'$ is a γ_{fd}-set of $\mathfrak{R}$, it is a 1-fair dominating set. When an edge $\mathfrak{v}_i\mathfrak{v}_{i+1} \in \mathbb{E}_1(\mathfrak{R})$ is subdivided, $i = 1$, the resulting vertex $\mathfrak{w}_1$ in $\mathfrak{R}_1$ may cause $\mathfrak{v}_2$ to no longer be dominated by any vertex in $\mathbb{Q}'$. To compensate, we include $\mathfrak{w}_1$ in the dominating set,

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_1\}$$

Similarly, subdividing an inner edge $\mathfrak{v}_i\mathfrak{v}_{i+3} \in \mathbb{E}_2(\mathfrak{R})$, say $\mathfrak{i} = 1$ may leave $\mathfrak{v}_3$ undominated, requiring

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_1\}$$

In both cases, we observe that $|\mathbb{Q}_1| > |\mathbb{Q}'|$, implying $\gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$. Thus,

$$\mathrm{Sd}^{+}_{\gamma_{fd}}(\mathfrak{R}) = 1$$

Case (ii): $Sd^{-}_{\gamma_{fd}}(\mathfrak{R}) = 3$

Subcase 1: $\mathfrak{n} \equiv 2\ (mod\ 5)$

Here, $\gamma(\mathfrak{R}) < \gamma_{fd}(\mathfrak{R})$. Based on Case (i), subdividing any single edge increases the fair domination number, implying $\mathrm{Sd}^{-}_{\gamma_{fd}}(\mathfrak{R}) > 1$. Now consider subdividing two edges, e.g., $\mathfrak{v}_1\mathfrak{v}_2$ and v_2v_3, introducing vertices $\mathfrak{w}_1$ and $\mathfrak{w}_2$. It is observed that the neighborhoods of $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are not fully dominated by the original set $\mathbb{Q}$. We redefine

$$\mathbb{Q}_1 = \mathbb{Q}' \cup \{\mathfrak{w}_1, \mathfrak{w}_2\}$$

Here, $|\mathbb{Q}_1| > |\mathbb{Q}'|$, thus $\gamma_{fd}(\mathfrak{R}_1) > \gamma_{fd}(\mathfrak{R})$, and so $\mathrm{Sd}^{-}_{\gamma_{fd}}(\mathfrak{R}) > 2$

Now, consider subdividing three edges $\{\mathfrak{v}_{\mathfrak{n}-1}\mathfrak{v}_{\mathfrak{n}}, \mathfrak{v}_1\mathfrak{v}_{\mathfrak{n}-2}, \mathfrak{v}_{\mathfrak{n}-1}\mathfrak{v}_2\}$, introducing $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3$. In this case, we define

$$\mathbb{Q}_1 = \{\mathfrak{v}_{5k-4} \mid 1 \le k \le \left\lceil \frac{\mathfrak{n}}{5} \right\rceil\} = \mathbb{Q}$$

Since $|\mathbb{Q}_1| = |\mathbb{Q}| < |\mathbb{Q}'|$, it follows that

$$\gamma_{fd}(\mathfrak{R}_1) < \gamma_{fd}(\mathfrak{R})$$

$$\Rightarrow \mathrm{Sd}^{-}_{\gamma_{fd}}(\mathfrak{R}) = 3$$

Subcase 2: $\mathfrak{n} \not\equiv 2\ (mod\ 5)$

In this scenario, $\gamma(\mathfrak{R}) = \gamma_{fd}(\mathfrak{R})$ based on Observation 1, and thus $\mathrm{Sd}^{-}_{\gamma_{fd}}(\mathfrak{R}) = 0$

VI. Conclusions

The study of fair domination subdivision numbers opens transformative possibilities for network design, offering a mathematical lens to enhance resilience, efficiency, and equity in interconnected systems. By determining the minimal edge subdivisions required for fair domination, we gain the power to fortify communication networks against disruptions, optimize resource distribution in social and technological structures, and engineer adaptive architectures capable of self-correction. This parameter transcends theoretical interest; it equips us with actionable strategies to build more robust and balanced networks, from infrastructure grids to algorithmic systems. As research advances, unlocking its full potential across graph classes could redefine how we approach stability and fairness in an increasingly connected world. The pursuit of these insights represents not just a graph-theoretic challenge, but a step toward future-proofing the very frameworks that sustain modern society. We proposed an algorithm to construct a fair dominating set and compute the fair domination number for the Circulant graph $C_{\mathfrak{n}}(1,2)$ and $C_{\mathfrak{n}}(1,3)$, addressing various cases based on $\mathfrak{n}\ (mod\ 5)$ and $\mathfrak{n}\ (mod\ 3)$. This algorithm not only confirms theoretical results but also offers a practical method for identifying fair dominating sets in Circulant graphs. These findings contribute to the broader understanding of domination theory in graph structures and lay the groundwork for future investigations into fair domination across other graph classes and their applications in network design and resource distribution.

VII. Scope for Further Research

Several open problems in fair domination warrant further investigation. First, determining the fair domination subdivision number for broader graph classes would be valuable. Another key direction is characterizing the graph classes for which the relation $Sd^{+}_{\gamma_{fd}}(\mathfrak{R}) = 1$ holds. Additionally, generalizing the fair domination subdivision number for circulant graphs, particularly $C_{\mathfrak{n}}(1,\mathfrak{n})$, presents an interesting avenue for future research.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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