



GENERATING FUNCTION FOR BERNOULLI NUMBERS AND ITS GENERALIZATIONS

Gasanov Magomedysuf

¹Department of Higher Mathematics, Moscow State University of Civil
Engineering, Moscow, 129337 Russia.

Email: vonasag6991@mail.ru

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Abstract

This paper explores certain generalizations of the generating function of Bernoulli numbers, the computation of integrals, and the investigation of the convergence of integrals from these functions. The primary tools employed in the research include the use of Taylor series, theorems on uniform continuity (such as Weierstrass's and Dini's theorems), as well as special functions such as the gamma function, incomplete gamma function, Riemann zeta function, and Lambert function. Various examples for specific parameter values are considered in the article. The obtained results can be strengthened in subsequent works and generalized to a broader class of functions. The derived estimates can be applied in various tasks related to the assessment of similar integrals.

Keywords: Bernoulli numbers, generating function, Taylor series, uniform convergence, special functions, incomplete gamma function, Riemann zeta function, Lambert function.

I. Introduction

Swiss mathematician Jacob Bernoulli and Japanese mathematician Seki Takakazu independently discovered Bernoulli numbers at approximately the same time while solving the same problem. Takakazu's discovery [III] was published in 1712, while Bernoulli's discovery was published in 1713.

The Bernoulli numbers refer to a sequence of rational numbers discovered when calculating the sum of identical powers of consecutive natural numbers:

$$\sum_{n=1}^{N-1} n^k = \frac{1}{k+1} \sum_{s=0}^k C_{k+1}^s B_s N^{k+1-s}$$

Another definition exists. Bernoulli numbers are formally defined as the coefficients in the expansion of the exponential generating function:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}$$

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Additionally, there exists a recurrent formula for Bernoulli numbers, expressed as:

$$B_n = -\frac{1}{n+1} \sum_{k=1}^n C_{n+1}^{k+1} B_{n-k}$$

The Bernoulli numbers find applications in various areas of mathematics, namely: in the expansion of different trigonometric and hyperbolic functions into Taylor series; in a certain representation of the digamma function; in topology, specifically in determining the order of the cyclic group of diffeomorphism classes; and in combinatorics. The Bernoulli numbers can be approached in different ways, as they have various definitions: through a recurrent relation, as an explicit formula, or using a generating function. This study investigates the characteristics of improper integrals arising from various modifications of the generating function. The main tasks involve finding the integrals directly or investigating their convergence. These problems will be addressed by introducing a series. Such an approach is quite commonly employed in the analysis and exploration of various problems, such as problems in complex analysis [V], approximation of functions to polynomials for simplified modeling in software complexes [II], [I], and the estimation of computation errors [IV].

This paper describes the solution of the integral of the generating function for Bernoulli numbers, as well as its various generalizations, for example, in the form $f(x, a, s) = \frac{x^s}{e^{ax}-1}$, where $s \geq 1, a \geq 1$. Various estimates are proven for improper integrals of functions of the form $\frac{f(x)}{e^{bx}-1}$, by expanding this function into a Taylor series and demonstrating its uniform convergence to the sum using the Weierstrass and Dini theorems. Geometric problems related to finding the volume of a solid generated by rotating a curve are also addressed. The results obtained may later be extended to a wider class of functions.

II. Main results

The generalized generating function of Bernoulli numbers is considered:

$$f(x, a, s) = \frac{x^s}{e^{ax}-1}$$

Let's start by considering the special case when $s = 1$ and $a = 1$. In this case, let's evaluate the integral:

$$S(x, 1, 1) = \int_0^\infty f(x, 1, 1) dx = \int_0^\infty \frac{x}{e^x-1} dx \quad (1)$$

Let's transform $f(x, 1, 1)$:

$$f(x, 1, 1) = \frac{x}{e^x-1} = \sum_{n \geq 0} x e^{-(n+1)x} = D(x, 1, 1) \quad (2)$$

Substituting (2) into (1), we obtain:

$$S(x, 1, 1) = \int_0^\infty \sum_{n \geq 0} x e^{-(n+1)x} dx \quad (3)$$

Let's prove that in the equality (3), the change of the integral and the sum sign is correct.

Theorem 1. The series $\sum_{n \geq 0} x e^{-(n+1)x}$ does not uniformly converge to its sum $D(x, 1, 1)$ on any interval $[0, a]$ for $a > 0$.

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Proof. Indeed, for $x = 0$, the sum $D(x, 1, 1) = 0$. However,

$$\lim_{x \rightarrow 0^+} \frac{x}{e^x - 1} = \lim_{x \rightarrow 0^+} \frac{x}{x + o(x)} = 1$$

Thus, the function is not continuous at $x = 0$, and consequently, the series $\sum_{n \geq 0} x e^{-(n+1)x}$ does not converge uniformly to $D(x, 1, 1)$ on any interval $[0, a]$.

Theorem 2. The series $\sum_{n \geq 0} x e^{-(n+1)x}$ uniformly converges to its sum $D(x, 1, 1)$ on any interval $[\varepsilon, a]$ for $0 < \varepsilon < a$.

Proof. Let's use the following theorem:

Dini's Theorem. Let the terms of the series $\sum_{n \geq 0} f_n(x)$ be continuous and nonnegative on the interval $[a, b]$, and suppose that the series converges to a function $F(x) \in C[a, b]$ on this interval. Then the series $\sum_{n \geq 0} f_n(x)$ converges uniformly to its sum on $[a, b]$.

Since the series and the sum from the statement of the theorem satisfy Dini's Theorem on the specified interval, the series $\sum_{n \geq 0} x e^{-(n+1)x}$ converges to the sum $D(x, 1, 1)$ on the interval $[\varepsilon, a]$.

As is known, under uniform convergence, interchanging the integral and the sum sign is valid. Therefore, applying **Theorem 2**, it follows that:

$$\sum_{n=0}^{\infty} \int_{\varepsilon}^a x e^{-(n+1)x} dx = \sum_{n=0}^{\infty} \left(\frac{\varepsilon e^{-\varepsilon n}}{n} + \frac{e^{-\varepsilon n}}{n^2} - \frac{a e^{-an}}{n} - \frac{e^{-an}}{n^2} \right) \quad (4)$$

Theorem 3. The series $T_n(x) = \sum_{n=0}^{\infty} \left(\frac{x e^{-xn}}{n} + \frac{e^{-xn}}{n^2} \right)$ uniformly converges on the ray $[0, +\infty)$.

Proof. Let's use the Weierstrass Uniform Convergence Theorem and the inequality $e^x > x \forall x \geq 0$:

$$\sum_{n=0}^{\infty} \left(\frac{x e^{-xn}}{n} + \frac{e^{-xn}}{n^2} \right) = \sum_{n=0}^{\infty} \frac{e^{-xn}(nx+1)}{n^2} < \sum_{n=0}^{\infty} \frac{e^{-xn} e^{xn+1}}{n^2} = \sum_{n=0}^{\infty} \frac{e}{n^2} = \frac{e\pi^2}{6}$$

Thus, the series uniformly converges on the ray.

Considering $\varepsilon \rightarrow 0^+$, $a \rightarrow +\infty$ we get:

$$\begin{aligned} S(x, 1, 1) &= \lim_{\varepsilon \rightarrow 0^+} \lim_{a \rightarrow +\infty} \sum_{n=0}^{\infty} \int_{\varepsilon}^a x e^{-(n+1)x} dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \sum_{n=0}^{\infty} \left(\frac{\varepsilon e^{-\varepsilon n}}{n} + \frac{e^{-\varepsilon n}}{n^2} \right) - \lim_{a \rightarrow +\infty} \sum_{n=0}^{\infty} \left(\frac{a e^{-an}}{n} - \frac{e^{-an}}{n^2} \right) \\ &= \sum_{n=0}^{\infty} \left(\lim_{\varepsilon \rightarrow 0^+} \left(\frac{\varepsilon e^{-\varepsilon n}}{n} + \frac{e^{-\varepsilon n}}{n^2} \right) \right) - \sum_{n=0}^{\infty} \left(\lim_{a \rightarrow +\infty} \left(\frac{a e^{-an}}{n} - \frac{e^{-an}}{n^2} \right) \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n^2} - \sum_{n=0}^{\infty} 0 = \frac{\pi^2}{6} \end{aligned}$$

Now, let's switch to the general form, namely, we will consider the integral:

$$S(x, a, s) = \int_0^\infty f(x, a, s) dx = \int_0^\infty \frac{x^s}{e^{ax} - 1} dx$$

Using the theorems mentioned above, let's calculate this integral:

$$f(x, a, s) = \frac{x^s}{e^{ax} - 1} = \sum_{n \geq 0} x^s e^{-(n+1)ax} = D(x, a, s).$$

$$\lim_{x \rightarrow 0^+} \frac{x^s}{e^{ax} - 1} = \lim_{x \rightarrow 0^+} \frac{x^s}{ax + o(x)} = \lim_{x \rightarrow 0^+} \frac{x^{s-1}}{a} = \begin{cases} 0, & \text{if } s > 1 \\ \frac{1}{a}, & \text{if } s = 1 \end{cases}.$$

In the case of $s > 1$, the series $\sum_{n \geq 0} x^s e^{-(n+1)ax}$ uniformly converges to the sum $D(x, a, s)$ on the set $[0, t]$ for $t > 0$, in the case of $s = 1$ continuity is not maintained at $x = 0$, so we need to use **Theorem 2**.

Let's consider the case $s > 1$:

$$S_t(x, a, s) = \int_0^t \sum_{n \geq 0} x^s e^{-(n+1)ax} dx = \sum_{n \geq 0} \int_0^t x^s e^{-(n+1)ax} dx = |x = z(n+1)a|$$

$$= \frac{1}{a^{s+1}} \sum_{n \geq 0} \frac{\gamma\left(s+1, \frac{t}{n+1}\right)}{(n+1)^{s+1}} \leq \frac{1}{a^{s+1}} \sum_{n \geq 0} \frac{\Gamma(s+1)}{(n+1)^{s+1}} = \frac{\Gamma(s+1)}{a^{s+1}} \sum_{n \geq 0} \frac{1}{(n+1)^{s+1}}$$

The last series uniformly converges, so it is permissible to perform a limit transition $\rightarrow +\infty$:

$$S(x, a, s) = \lim_{t \rightarrow +\infty} S_t(x, a, s) = \lim_{t \rightarrow +\infty} \frac{1}{a^{s+1}} \sum_{n \geq 0} \frac{\gamma\left(s+1, \frac{t}{n+1}\right)}{(n+1)^{s+1}}$$

$$= \frac{1}{a^{s+1}} \lim_{t \rightarrow +\infty} \sum_{n \geq 0} \frac{\gamma\left(s+1, \frac{t}{n+1}\right)}{(n+1)^{s+1}} = \frac{1}{a^{s+1}} \sum_{n \geq 0} \frac{\Gamma(s+1)}{(n+1)^{s+1}}$$

$$= \frac{\Gamma(s+1)}{a^{s+1}} \sum_{n \geq 0} \frac{1}{(n+1)^{s+1}} = \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}}$$

where

$\zeta(s+1)$ is the Riemann zeta function.

For numerical implementation, consider the estimate of the remainder term

$$R_N = \sum_{n=N+1}^\infty \frac{\Gamma(s+1)}{(an)^{s+1}}$$

We use an analogue of Taylor's theorem for a monotonically decreasing series to estimate the remainder:

$$R_N \leq \int_N^\infty \frac{\Gamma(s+1)}{(ax)^{s+1}} dx = \frac{\Gamma(s+1)}{a^{s+1} s N^s}$$

It follows that the error decreases like $1/N^s$. The formula for the convergence rate is

$$|S(x, a, s) - S_N(x, a, s)| \lesssim \frac{\Gamma(s+1)}{a^{s+1} s N^s}.$$

- The larger s , the faster the error decreases.
- The larger a , the smaller the absolute value of the integral and the error.

The table and plots of the numerical calculation are presented below

Table 1: Numerical calculation of S_N

N	$S_N(a=2, s=3)$	Error Estimate	Relative Error
1	0.375	0.030871	7.6%
2	0.398438	0.007433	1.8%
5	0.404011	0.001860	0.46%
10	0.405263	0.000608	0.15%
20	0.405701	0.000170	0.042%
50	0.405857	0.000014	0.0034%

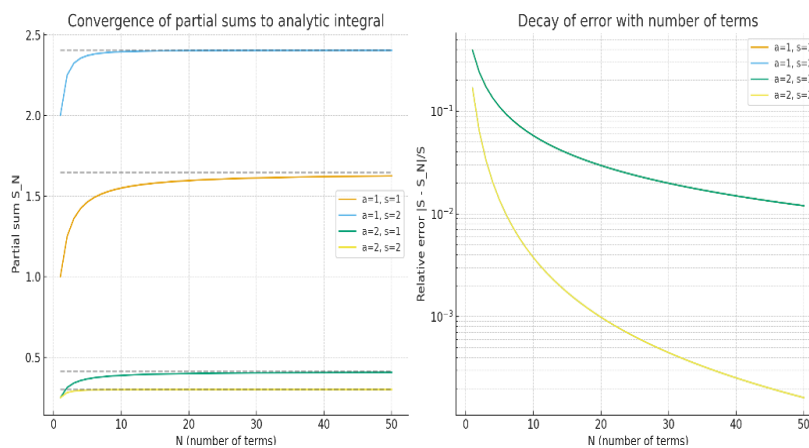


Fig. 1. Numerical analysis convergence of $S_N(x, a, s)$

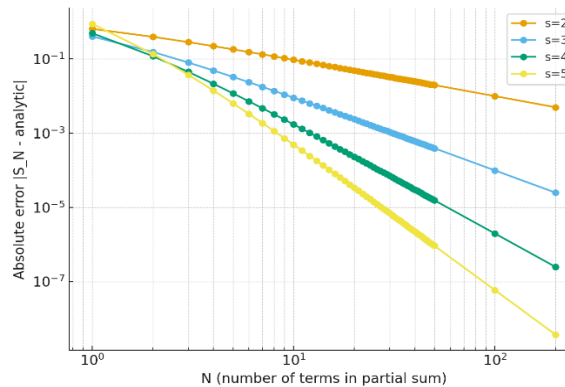


Fig. 2. Absolute error $S(x, a, s)$

Let's consider another generalization of the generating function of Bernoulli numbers.

Theorem 4. Suppose the following conditions are satisfied:

1. $f(x) \geq 0 \forall x \in [0, +\infty)$,
2. $f(0) = 0$,
3. $f(x)$ is a regular function on $\mathbb{R}$,
4. $f^{(n)}(0) < \frac{1}{\zeta(n+1)} \forall n \in \mathbb{N}_0$,
5. $\lim_{x \rightarrow 0^+} \frac{f(x)}{e^{bx}-1} = 0 \forall b > 1$,

Then the following estimate holds:

$$\left| \int_0^\infty \frac{f(x)}{e^{bx}-1} dx \right| < \frac{1}{b-1}$$

Proof. Let's use the classical triangle inequality from mathematical analysis, which also works well for integrals:

$$\begin{aligned} \left| \int_0^\infty \frac{f(x)}{e^{bx}-1} dx \right| &< \int_0^\infty \frac{|f(x)|}{e^{bx}-1} dx = \int_0^\infty |f(x)| e^{-b(n+1)x} dx \\ &= \int_0^\infty \sum_{n \geq 0} \left| \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} x^k \right| e^{-b(n+1)x} dx < \sum_{n \geq 0} \sum_{k \geq 0} \frac{1}{\zeta(k+1)k!} \int_0^\infty x^k e^{-b(n+1)x} dx \\ &= \sum_{k \geq 0} \sum_{n \geq 0} \frac{1}{\zeta(k+1)k!} \int_0^\infty x^k e^{-b(n+1)x} dx = \sum_{k \geq 0} \left(\frac{\Gamma(k+1)}{b^{k+1}\zeta(k+1)k!} \sum_{n \geq 0} \frac{1}{(n+1)^{k+1}} \right) \\ &= \sum_{k \geq 0} \left(\frac{\Gamma(k+1)}{b^{k+1}\zeta(k+1)k!} \zeta(k+1) \right) = \sum_{k \geq 0} \left(\frac{\Gamma(k+1)}{b^{k+1}k!} \right) = \sum_{k \geq 0} \frac{1}{b^{k+1}} = \frac{1}{b-1} \end{aligned}$$

Theorem 5 (strengthening of Theorem 4). Suppose conditions 1-4 of **Theorem 4** are satisfied, and $\lim_{x \rightarrow 0^+} \frac{f(x)}{e^{bx}-1} = \lambda < \infty \forall b > 1$, then the estimate from **Theorem 4** holds.

This fact is proven using **Theorem 2** and the auxiliary **Dini's Theorem**.
Theorem 6. Suppose the following conditions are satisfied:

1. $f^{(k)}(x) \geq 0 \forall x \in [0, +\infty), \forall k \in \mathbb{N}_0$,
2. $f(0) = 0$,
3. $f(x)$ is a regular function on $\mathbb{R}$,

$$4. \lim_{n \rightarrow \infty} \frac{f^{(n+1)}(0)\zeta(n+1)}{f^{(n)}(0)\zeta(n)} < b,$$

$$5. \lim_{x \rightarrow 0^+} \frac{f(x)}{e^{bx}-1} = 0 \forall b > 1,$$

Then the integral $\int_0^\infty \frac{f(x)}{e^{bx}-1} dx$ converges.

Proof. Based on the methods described earlier and the presented theorems, we obtain:

$$\begin{aligned} \int_0^\infty \frac{f(x)}{e^{bx}-1} dx &= \int_0^\infty f(x) e^{-b(n+1)x} dx = \int_0^\infty \sum_{n \geq 0} \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} x^k e^{-b(n+1)x} dx \\ &= \sum_{n \geq 0} \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} \int_0^\infty x^k e^{-b(n+1)x} dx = \sum_{k \geq 0} \sum_{n \geq 0} \frac{f^{(k)}(0)}{k!} \int_0^\infty x^k e^{-b(n+1)x} dx \\ &= \sum_{k \geq 0} \left(\frac{f^{(k)}(0) \Gamma(k+1)}{b^{k+1} k!} \sum_{n \geq 0} \frac{1}{(n+1)^{k+1}} \right) = \sum_{k \geq 0} \left(\frac{f^{(k)}(0) \Gamma(k+1)}{b^{k+1} k!} \zeta(k+1) \right) \\ &= \sum_{k \geq 0} \frac{f^{(k)}(0) \zeta(k+1)}{b^{k+1}} \end{aligned}$$

Let's use the classical criteria for the convergence of constant sign series, from which we obtain:

$$\lim_{k \rightarrow \infty} \frac{\frac{f^{(k+1)}(0)\zeta(k+2)}{b^{k+1}}}{\frac{f^{(k)}(0)\zeta(k+1)}{b^k}} = \frac{1}{b} \lim_{k \rightarrow \infty} \frac{f^{(k+1)}(0)\zeta(k+2)}{f^{(k)}(0)\zeta(k+1)} < 1,$$

from which we derive the required by the theorem.

Theorem 7. For the generating function of Bernoulli numbers $f(x, a, s)$, the following inequalities hold:

1. $\int_0^\infty f^n(x, a, s) dx < \chi \left(\frac{\chi^s}{e^{a\chi}-1} \right)^n + \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}}$, if $s > 1, \chi \leq 1$,
2. $\int_0^\infty f^n(x, a, s) dx < 2\chi \left(\frac{\chi^s}{e^{a\chi}-1} \right)^n + \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}}$, if $s > 1, \chi > 1$,

$$3. \int_0^\infty f^n(x, a, s) dx < \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}}, \quad \text{if} \quad s = 1, \\ \text{where } \chi: f(\chi, a, s) = f_{\max}.$$

Proof. Let's prove the first statement of the theorem by examining the generating function of Bernoulli numbers for monotonicity:

$$\int_0^\infty f^n(x, a, s) dx = \int_0^\infty \left(\frac{x^s}{e^{ax} - 1} \right)^n dx = \int_0^\infty \frac{x^{sn}}{(e^{ax} - 1)^n} dx$$

Note that $f(x, 1, 1)$ monotonically decreases over the entire interval. Indeed,

$$f'(x, a, s) = \frac{sx^{s-1}(e^{ax} - 1) - ax^s e^{ax}}{(e^{ax} - 1)^2} = 0, \\ sx^{s-1}(e^{ax} - 1) - ax^s e^{ax} = 0, \\ e^{ax-s}(ax - s) = -se^{-s}, \\ W(e^{ax-s}(ax - s)) = W(-se^{-s}), \\ x = \frac{W_n(-se^{-s}) + s}{a} = \chi,$$

Where $W(x)$ is the Lambert function.

The first solution is obvious, $x = 0$. The sign of the second root depends on the parameter s as follows:

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } s > 1 \\ 0, & \text{if } s = 1 \\ -1, & \text{if } s \in (0, 1) \end{cases}$$

From this, two cases arise: $s > 1$ and $s = 1$. In the first case, we obtain that on the interval $(\chi, +\infty)$ the function $f(x, a, s)$ monotonically decreases. Next, we consider two subcases:

1. $\chi \leq 1$. Under this condition, the following estimate holds:

$$\begin{aligned} \int_0^\infty f^n(x, a, s) dx &= \int_0^\infty \left(\frac{x^s}{e^{ax} - 1} \right)^n dx \\ &= \int_0^\chi \left(\frac{x^s}{e^{ax} - 1} \right)^n dx + \int_\chi^{+\infty} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx \\ &< \int_0^\chi \left(\frac{x^s}{e^{ax} - 1} \right)^n dx + \int_\chi^{+\infty} \frac{x^s}{e^{ax} - 1} dx \\ &\leq \chi \left(\frac{\chi^s}{e^{a\chi} - 1} \right)^n + \int_0^{+\infty} \frac{x^s}{e^{ax} - 1} dx \\ &= \chi \left(\frac{\chi^s}{e^{a\chi} - 1} \right)^n + \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}} \end{aligned}$$

2. $\chi > 1$. Due to the fact that the function $f(x, a, s)$ monotonically decreases on the interval $(\chi, +\infty)$ and $\lim_{x \rightarrow +\infty} f(x, a, s) = 0$, then $\exists \tilde{\chi} (\tilde{\chi} > \chi): f(x, a, s) \leq 1 \forall x \in [\tilde{\chi}, +\infty)$. Taking this into account, the integral estimate has the following form:

$$\begin{aligned} \int_0^{\infty} f^n(x, a, s) dx &= \int_0^{\infty} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx \\ &= \int_0^{\chi} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx + \int_{\chi}^{\tilde{\chi}} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx + \int_{\tilde{\chi}}^{+\infty} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx \\ &< 2 \int_0^{\chi} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx + \int_{\tilde{\chi}}^{+\infty} \frac{x^s}{e^{ax} - 1} dx \leq 2\chi \left(\frac{\chi^s}{e^{a\chi} - 1} \right)^n + \int_0^{+\infty} \frac{x^s}{e^{ax} - 1} dx \\ &= 2\chi \left(\frac{\chi^s}{e^{a\chi} - 1} \right)^n + \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}} \end{aligned}$$

In the second case $s = 1$. Under this condition, the function $f(x, a, s)$ monotonically decreases on the interval $(0, +\infty)$, and $\lim_{x \rightarrow 0^+} f(x, a, s) = \frac{1}{a} \leq 1$, therefore, the integral estimate will be as follows:

$$\int_0^{\infty} f^n(x, a, s) dx = \int_0^{\infty} \left(\frac{x^s}{e^{ax} - 1} \right)^n dx < \int_0^{+\infty} \frac{x^s}{e^{ax} - 1} dx = \frac{\Gamma(s+1)\zeta(s+1)}{a^{s+1}}$$

These integrals play an important role in practical applications, including the computation of surface areas generated by rotating a curve, the determination of volumes of solids of revolution, and the calculation of the length of a curve segment.

Theorem 8. Let the function be given by $f(x, 1, 1) = \frac{x}{e^x - 1}$, which is the generating function of Bernoulli numbers. Then, the volume of the solid formed by rotating this curve around the x -axis, for $x \in (0, +\infty)$, is equal to:

$$V_x = \frac{(\pi^3 - 6\pi\zeta(3))}{3}$$

Proof. Let's use the classic method of integration by parts:

$$\begin{aligned} V_x &= \pi \int_0^{\infty} \frac{x^2}{(e^x - 1)^2} dx = \pi \int_0^{\infty} \frac{x^2 e^{-2x}}{(1 - e^{-x})^2} dx = \pi \int_0^{\infty} x^2 e^{-x} d\left(\frac{1}{e^x - 1}\right) \\ &= \pi \left(\left[\frac{x^2 e^{-x}}{e^x - 1} \right] \Big|_0^{\infty} + \int_0^{\infty} \frac{(2x - x^2)e^{-x}}{e^x - 1} dx \right) \end{aligned}$$

When substituting the limits of integration and finding the limit, the expression $\left[\frac{x^2 e^{-x}}{e^x - 1} \right] \Big|_0^{\infty}$ becomes zero.

Indeed, $\lim_{x \rightarrow +\infty} \frac{x^2 e^{-x}}{e^x - 1} = \lim_{x \rightarrow +\infty} \frac{x^2}{e^{2x} - e^x} = \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow +\infty} \frac{2x}{2e^{2x} - e^x} = \lim_{x \rightarrow +\infty} \frac{2}{4e^{2x} - e^x} = 0$

and $\lim_{x \rightarrow 0} \frac{x^2 e^{-x}}{e^x - 1} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} e^{-x} \cdot \frac{x^2}{e^x - 1}$.

Since $e^{-x} \rightarrow 1$ as $x \rightarrow 0$, it remains to analyze $\lim_{x \rightarrow 0} \frac{x^2}{e^x - 1}$ we can write

$$e^x - 1 = x + o(x) \quad (x \rightarrow 0)$$

hence $\lim_{x \rightarrow 0} \frac{x^2}{e^x - 1} = \lim_{x \rightarrow 0} \frac{x^2}{x + o(x)} = \lim_{x \rightarrow 0} \frac{x}{1 + o(1)} = 0$, thus, $\lim_{x \rightarrow 0} \frac{x^2 e^{-x}}{e^x - 1} = 0$.

With this in mind, we get

$$\begin{aligned} V_x &= \pi \int_0^\infty \frac{(2x - x^2)e^{-x}}{e^x - 1} dx = 2\pi \int_0^\infty \frac{xe^{-x}}{e^x - 1} dx - \pi \int_0^\infty \frac{x^2 e^{-x}}{e^x - 1} dx \\ &= 2\pi \int_0^\infty \frac{xe^{-2x}}{1 - e^{-x}} dx - \pi \int_0^\infty \frac{x^2 e^{-2x}}{1 - e^{-x}} dx = 2\pi \left(\frac{\pi^2}{6} - 1 \right) - 2\pi(\zeta(3) - 1) \\ &= \frac{(\pi^3 - 6\pi\zeta(3))}{3}. \end{aligned}$$

III. Conclusion

In this work, an investigation of generalized generating functions of Bernoulli numbers and some of their modifications was conducted. Integrals were calculated in specific cases, and more general cases were examined for convergence. The main method of investigation involved the use of the Taylor series. In addition to the Taylor series, special functions were utilized, such as the gamma function, the incomplete gamma function, the Riemann zeta function, and the Lambert function. The proofs of estimates were reduced to examining the uniform convergence of subintegral series to their sums, which was demonstrated using Weierstrass's theorem on the estimation of a functional series and Dini's theorem. The volumes of solids formed by rotating curves, represented as generating functions of Bernoulli numbers, were determined. In addition to the analytic derivations, we provided numerical evaluations confirming the correctness of the formulas. The tables demonstrate the agreement between truncated series and exact integrals, and error estimates show that the convergence is sufficiently fast for practical computations, even for moderate indices n .

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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