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ON THE ENCAPSULATION OF THE NEW XLINDLEY DISTRIBUTION

B. Barrouk[1], H. Zeghdoudi[2]

[1] Laboratory Informatics and Mathematics (LiM), Souk Ahras University Souk Ahras, 41000, Algeria.

[2] LaPS laboratory, Badji Mokhtar- Annaba University, BP12, 23000, Annaba Algeria.

Email: [1]bachir.barrouk@univ-soukahras.dz, [2]halim.zeghdoudi@univ-annaba.dz

Corresponding Author: **H. Zeghdoudi**

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Abstract

In this study, we introduce the Wrapping New XLindley Distribution (WNXLD) as an extension of the Wrapping Distribution (WD). We derive the probability density function, cumulative distribution function, characteristic function, trigonometric moments, and other relevant parameters for WNXLD Additionally, parameter estimation is performed using the maximum likelihood estimation method.

Keywords: Circular statistics, Compressive Strength, GGBS, Metakaoline, New Xlindley, Regression Analysis, Split Tensile Strength, Wrapping, Trigonometric moments.

I. Introduction

The concept of "circular data," often characterized as two-dimensional directional data, pertains to instances where observations are articulated in degrees or radians. Given that circular data represent directions devoid of magnitude, they can be conveniently depicted as points on a circle with a unit radius, centered at the origin, or as a unit vector in a plane extending from the origin to the relevant point. Directional data exhibit unique characteristics and present distinct challenges in statistical modeling. Such data arise in various disciplines, including biology, geology, physics, meteorology, psychology, medicine, image processing, political science, economics, and astronomy (Mardia and Jupp [VII]). One common approach to constructing a circular distribution is by wrapping a linear distribution around the unit circle. Numerous studies have explored this method. L'evy [XI] introduced wrapped distributions, while Jammalamadaka and Kozubowski [XVI] examined circular distributions formed by wrapping classical exponential and Laplace distributions around the circle. In 2007, Rao et al [II] investigated the wrapping of lognormal, logistic, Weibull, and extreme-value distributions in the context of life testing models. Additionally, Roy and Adnan [XX] introduced the wrapped weighted exponential distribution, and Rao et al. [III] studied the characteristics of the wrapped gamma distribution. Joshi and Jose [XVII] proposed the wrapped Lindley distribution, analyzing its properties, including the characteristic function and trigonometric moments. Adnan and Roy [VIII] later introduced the wrapped variance gamma distribution, applying it to wind direction modeling. The Lindley distribution (LD), originally suggested by Lindley [IV, V], is defined through its probability density function (pdf) and cumulative distribution function (CDF) as follows:

$$l(x,\lambda) = \frac{\alpha^2}{\alpha+1}(1+x)e^{-\lambda x}; x,\lambda > 0 \quad (1)$$

$$L(x,\lambda) = 1 - \left[1 + \frac{\lambda x}{\lambda+1}\right]e^{-\alpha x}, x,\lambda > 0 \quad (2)$$

The wrapped Lindley distribution was introduced by Joshi and Jose (2018) [XVIII], who formulated its probability density function (PDF) and cumulative distribution function (CDF) as follows:

$$t(y) = \frac{\lambda^2}{\lambda+1}e^{-\lambda y}\left[\frac{1+y}{1-e^{-2\pi\lambda}} + \frac{2\pi e^{-2\pi\lambda}}{\left(1-e^{-2\pi\lambda}\right)^2}\right], y \in [0,2\pi), \lambda > 0 \quad (3)$$

$$T(y) = \frac{1}{1-e^{-2\pi\lambda}}\left[1 - e^{-\lambda y} - \frac{\lambda y}{1-\lambda}e^{-\lambda y}\right] - \frac{2\pi\lambda}{\lambda+1}\left(1-e^{-\lambda y}\right)\left[\frac{e^{-2\pi\lambda}}{\left(1-e^{-2\pi\lambda}\right)^2}\right], y \in [0,2\pi), \lambda > 0 \quad (4)$$

Khodja et al [X] developed the new XLindley distribution (NXLD), which includes one parameter: a shape parameter (β). They defined the probability density function (PDF) and the cumulative distribution function (CDF) of the NXLD as follows.

$$\mathrm{f(x;\beta)} = \frac{\beta(1+\beta x)}{2})e^{-\beta x};\ x,,\beta > 0 \quad (5)$$

$$\mathrm{F(x;\beta)} = 1 - \left[1 + \frac{\beta x}{2}\right]e^{-\beta x};\ x,\beta > 0 \quad (6)$$

Khodja et al [X] explored various properties of the new XLindley distribution (NXLD) and demonstrated, using real data, that it outperformed the exponential distribution in several aspects. They employed the maximum likelihood estimation method to show that the Lindley distribution provided a better fit. In this study, as an extension of the Wrapping Distribution (WD), we propose a novel circular distribution called the Wrapping New XLindley Distribution (WNXLD). We derive its probability density function and cumulative distribution function in Section 2. In Section 3, we establish the characteristic function, expressed in terms of trigonometric moments along with relevant parameters. In Section 4, we define the statistical Characterization of the Median Direction for the WILD. In Section 5, we apply the maximum likelihood estimation method to estimate its parameters. In Section 6, we investigate the finite-sample performance of the maximum likelihood estimator (MLE) for the parameter of the proposed Wrapped New XLindley

Distribution (WNXLD). In Section 7, we evaluate the Wrapped New XLindley Distribution (WNXLD) on two benchmark circular datasets.

II. Circular Distribution

A circular distribution is a probability distribution in which the total probability is confined to the circumference of a unit circle (see [XV]). The points on the unit circle represent directions, with each direction corresponding to a specific probability value. A circular random variable θ, expressed in radians, can take values within the range $0 \le \theta \le 2\pi$ or $-\pi \le \theta < \pi$. Circular probability distributions can be either discrete or continuous and must satisfy the condition $X_W = X(\mathrm{mod}\, 2\pi)$.

Definition 2.1 (see [XV, VI]). If y is a stochastic variable defined on the real numbers with a distribution function $G(y)$, then the random variable y_W of the wrapped distribution demonstrates the subsequent properties:

1. $\int_0^{2\pi} g(y)dy = 1$ and
2. $g(y) = g(y + 2\pi)$.

for any integer k and $g(y)$ is periodic.

Thus, we can define the Wrapped New XLindley Distribution (WNXLD) as follows:

Definition 2.2. A random variable θ is said to have a Wrapped New Lindley Distribution (WNXLD) as follows:

$$g(\theta) = \sum_{k=0}^{\infty} g(\theta + 2\pi k) = \sum_{k=0}^{\infty} \frac{\theta(1+\beta(\theta+2\pi k))}{2} e^{-\beta(\theta+2\pi k)}$$

$$= \frac{\theta e^{-\beta\theta}}{2} \sum_{k=0}^{\infty} (1 + \beta(\theta + 2\pi k))e^{-2\pi\beta k} \quad (7)$$

It can also be simplified to:

$$g(\theta) = \frac{\theta e^{-\beta\theta}}{2}\left[\frac{1}{1 - e^{-2\pi\beta}} + \beta\theta\left(\frac{1}{1 - e^{-2\pi\beta}}\right) + 2\pi\beta \sum_{k=0}^{\infty} k\left(e^{-2\pi\beta}\right)^k\right]$$

$$= \frac{\theta e^{-\beta\theta}}{2}\left[\frac{1}{1 - e^{-2\pi\beta}} + \beta\theta\left(\frac{1}{1 - e^{-2\pi\beta}}\right) + 2\pi\beta \frac{e^{-2\pi\beta}}{(-1 + e^{-2\pi\beta})^2}\right] \quad (8)$$

The cumulative distribution function of WNXLD can be derived as follows:

$$G(\theta) = \sum_{k=0}^{\infty}\{F(\theta + 2k\pi) - F(2k\pi)\}$$

$$\sum_{k=0}^{\infty}\left\{\frac{(2+\beta(2k\pi))}{2} e^{-\beta(2k\pi)} - \frac{(2+\beta(\theta+2k\pi))}{2} e^{-\beta(\theta+2k\pi)}\right\}$$

$$G(\theta) = \sum_{k=0}^{\infty}\left\{\frac{(2+\beta(2k\pi))}{2} e^{-\beta(2k\pi)} - \frac{(2+\beta(\theta+2k\pi))}{2} e^{-\beta(\theta+2k\pi)}\right\}$$

$$G(\theta) = \sum_{k=0}^{\infty} \frac{e^{-\beta(2k\pi)}}{2}\left\{2 + \beta(2k\pi) - \left(2 + \beta(\theta + 2k\pi)\right)e^{-\beta\theta}\right\} \quad (9)$$

Remark 2.1. We used the ratio test to check whether the series $\sum_{k=0}^{\infty} k\left(e^{-2\pi\beta}\right)^k$ in both the PDF and CDF of WNXLD converged as follows:

$$G(\theta) = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{(k+1)\left(e^{-2\pi\beta}\right)^{k+1}}{k(e^{-2\pi\beta})^{k}}\right| = \lim_{k\to\infty}\frac{(k+1)}{ke^{2\pi\beta}} = \frac{1}{e^{2\pi\beta}} < 1.$$

Fig 1 shows the circular representation of the PDF of WNXLD for different values of α, keeping the value for the parameter β at 3.0.

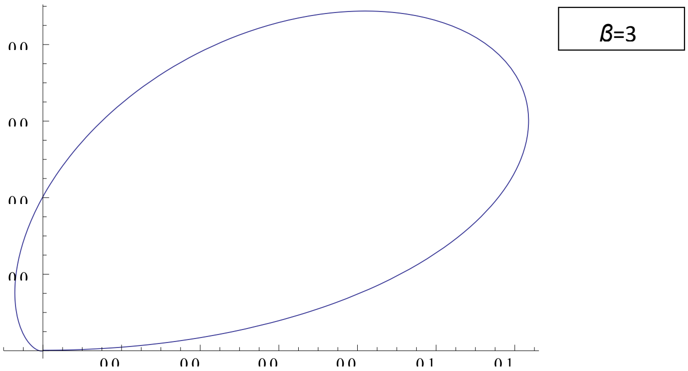

Fig. 1. PDF of the WNXLD (Circular Representation), $\beta = 3$.

The same circular representation for the PDF of WNXLD with different values of β, as in Fig 2.

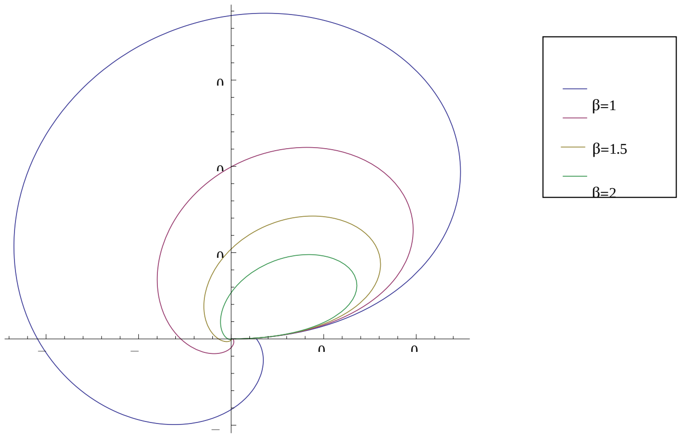

Fig. 2. PDF of the WQLD distribution (Circular Representation), $\beta = 1$,1.5,2,2.5.

Fig 3 shows the circular representation of the CDF of WNXLD for different values of the parameter β

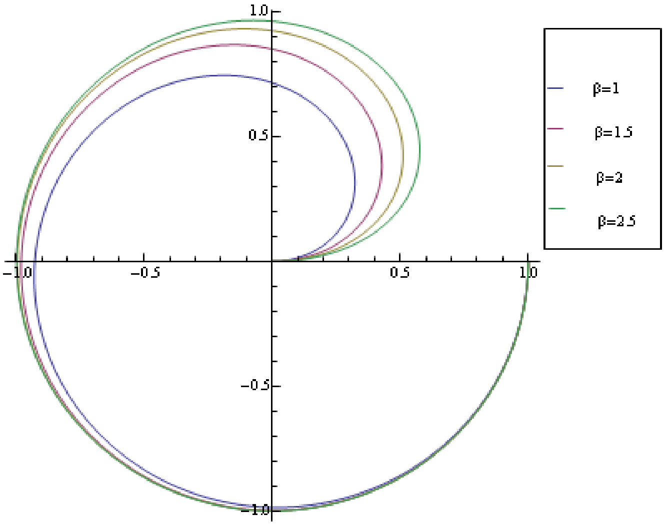

Fig. 3. CDF of the WNXLD distribution (Circular Representation),$\beta = 1{,}1.5{,}2{,}2.5$.

The same circular representation for the CDF of WNXLD with keeping the value for the parameter β, at 3.0, as in Fig 4.

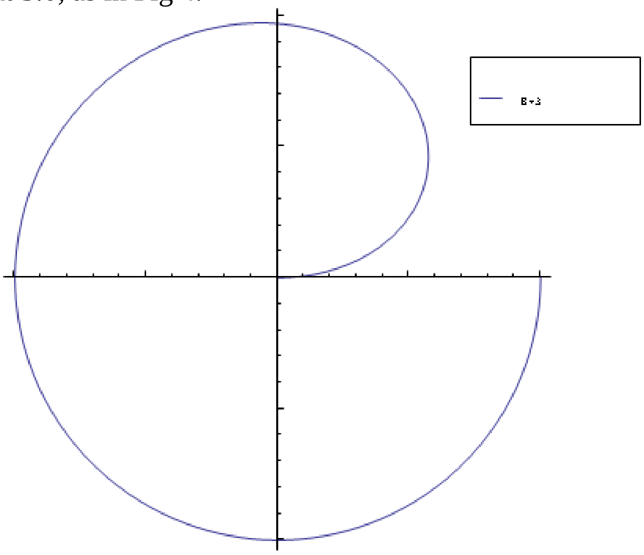

Fig. 4. CDF of the WNXLD distribution (CircularRepresentation), $\beta = 3$.

III. Characteristic Function of WNXLD

The characteristic function of X_w for the distribution function $G(\theta)$ is given by $\phi_\theta(t) = e^{it\theta}$ The characteristic function for the new XLindley Distribution is given as follows:

$$\phi_\theta(t) = E\left(e^{it\theta}\right) = \frac{\beta}{2}\left[\frac{(\beta-it)+\beta}{(\beta-it)^2}\right] \quad (10)$$

Now, we can find the characteristic function of the circular model by :

$$\phi_\theta(t) = E(e^{it\theta}) = \int_0^{2\pi} e^{it\theta}\, g(\theta)d\theta$$

$$E\left(e^{it\theta}\right) = \int_0^{2\pi} e^{it\theta}\left[\frac{1}{2\left(1-e^{-2\pi\beta}\right)}\theta e^{-\beta\theta} + \frac{\beta}{2\left(1-e^{-2\pi\beta}\right)}\theta^2 e^{-\beta\theta} + \frac{\pi\beta}{e^{2\pi\beta}}\theta e^{-\beta\theta}\right]d\theta \quad (11)$$

$$= \int_0^{2\pi}\left[\frac{1}{2(1-e^{-2\pi\beta})}\theta e^{-\beta\theta}e^{it\theta} + \frac{\beta}{2(1-e^{-2\pi\beta})}\theta^2 e^{-\beta\theta}e^{it\theta} + \frac{\pi\beta}{e^{2\pi\beta}}\theta e^{-\beta\theta}e^{it\theta}\right]d\theta$$

Rearranging Equation (11), we have:

$$E\left(e^{it\theta}\right) = \left[\frac{1}{2(1-e^{-2\pi\beta})} + \frac{2\pi\beta}{2e^{2\pi\beta}}\right]\int_0^{2\pi}\theta e^{-\beta\theta}e^{it\theta}d\theta$$
$$+ +(\frac{\beta}{2(1-e^{-2\pi\beta})}\int_0^{2\pi}\theta^2 e^{-\beta\theta}e^{it\theta}d\theta$$

As summing the previous integrals consists of two parts, the first part can be calculated as follows:

$$I = \left[\frac{1}{(2(1-e^{-2\pi\beta})} + \frac{\pi\beta}{e^{2\pi\beta}}\right]\int_0^{2\pi}\theta e^{-\beta\theta}e^{it\theta}d\theta$$

$$\begin{cases}\text{let } u = -\theta(\beta-it);\, \theta = \dfrac{-u}{\beta-it};\, \theta = 0\ \ u = 0\\ du = -(\beta-it)d\theta;\, d\theta =, \theta = 2\pi\ u = 2\pi(\beta-it)\end{cases}$$

$$\int_0^{2\pi}\theta e^{-\beta\theta}e^{it\theta}d\theta = \int_0^{2\pi(\beta-it)}\frac{u}{(-(\beta-it)}e^{u}\frac{du}{-(\beta-it)}$$
$$= (\frac{1}{(\beta-it)^2}\int_0^{2\pi(\beta-it)} ue^{u}\,du$$
$$= (\frac{1}{(\beta-it)^2}(1-e^{2\pi(\beta-it)}(2\pi(\beta-it)+1))$$

$$I = \left[\frac{1}{(1-e^{-2\pi\beta})} + \frac{2\pi\beta}{e^{2\pi\beta}}\right]\frac{1}{2(\beta-it)^2}\left(1-e^{-2\pi(\beta-it)}2\pi(\beta-it)+e^{-2\pi(\beta-it)}\right)$$

Now, combining both integrals I and J, we have the characteristic function of the WNXLD:

$$\phi_\theta(t) = \left[\frac{1}{(1-e^{-2\pi\beta})} + \frac{2\pi\beta}{e^{2\pi\beta}}\right]\frac{1}{2(\beta-it)^2}\left(1-e^{-2\pi(\beta-it)}2\pi(\beta-it)+e^{-2\pi(\beta-it)}\right)$$

$$+\frac{\beta}{2(1-e^{-2\pi\beta})}\frac{2\left[1-e^{-2\pi(\beta-it)}(2\pi(\beta-it)(\pi(\beta-it)+1)+1)\right]}{(\beta-it)^3}$$

We can simplify the characteristic function of the WNXLD as follows:

$$\phi_\theta(t)=\frac{1}{2(\beta-it)^2}\left[\frac{1}{(1-e^{-2\pi\beta})}+\frac{2\pi\beta}{e^{2\pi\beta}}(1-[2\pi(\beta-it)+1]e^{-2\pi(\beta-it)})+\right.$$

$$\left.\frac{2\beta[1-(2\pi(\beta-it)(\pi(\beta-it)+1)+1)]e^{-2\pi(\beta-it)}]}{(1-e^{-2\pi\beta})(\beta-it)}\right] \quad (12)$$

By the trigonometric definition, we have $\varphi_p=\alpha_p+i\beta_p$ $p=0,\pm1,\pm2,...,$
where $\alpha_p = E(\cos p\theta)$
and $\beta_p = E(\sin p\theta)$.

$$\alpha_p = E(\cos p\theta)$$
$$=\int_0^{2\pi}\cos(p\theta)\left[\frac{1}{2(1-e^{2\pi\beta})}(\theta e^{-\beta\theta}+\beta\theta^{2e^{-\beta\theta}})\right]d\theta$$
$$+\int_0^{2\pi}\cos(p\theta)\left[\frac{\pi\beta}{e^{2\pi\beta}}\theta e^{-\beta\theta}\right]d\theta$$

By some simplifications, we have

$$\alpha_p=\left[\frac{1}{2(1-e^{2\pi\beta})}+\frac{\pi\beta}{e^{2\pi\beta}}\right]\int_0^{2\pi}\theta e^{-\beta\theta}[\cos(p\theta)]d\theta+$$
$$\frac{\beta}{2(1-e^{2\pi\beta})}\int_0^{2\pi}\theta^2e^{-\beta\theta}[\cos(p\theta)]d\theta \quad (13)$$

Since Equation (13) contains two integrals, we can integrate separately, as follows:

$$I=\left[\frac{1}{2(1-e^{2\pi\beta})}+\frac{\pi\beta}{e^{2\pi\beta}}\right]\int_0^{2\pi}\theta e^{-\beta\theta}[\cos(p\theta)]d\theta$$
$$=\frac{e^{-2\pi\beta}\left[2p(\pi p^2+\pi\beta^2+\beta)\sin(2\pi p)-\left((2\pi\beta-1)p^2+\beta^2(2\pi\beta-1)\right)\cos(2\pi p)-e^{2\pi\beta}p^2+\beta^2e^{2\pi\beta}\right]}{(p^2+\beta^2)^2} \quad (14)$$

$$J=\frac{\beta}{2(1-e^{2\pi\beta})}\int_0^{2\pi}\theta^2e^{-\beta\theta}[\cos(p\theta)]d\theta$$
$$=\frac{\beta e^{-2\pi\beta}}{2(1-e^{-2\pi\beta})}\left[\frac{(p\left(\left((2\pi^2p^4+4\pi^2\beta^2p^2+4\pi\beta p^2)-p^2\right)+\beta^2\left((2\pi^2\beta^2+4\pi\beta)+3\right)\right)\sin(2\pi p))}{(p^2+\beta^2)^3}-\right.$$

$$\left.\frac{(2\pi^2\beta p^4-2\pi p^4+4\pi^2\beta^3p^4-3\beta p^2+2\pi^2\beta^5+2\pi p^4+\beta^3)\cos(2\pi p)+3\beta e^{2\pi\beta}p^2-\beta^3e^{2\pi\beta}}{(p^2+\beta^2)^3}\right] \quad (15)$$

Adding both Equations (14) and (15), we have the parameter α_p. Similarly, we get and simplify the parameter β_p, as follows:

$$\beta_p = E\left(sin(p\theta)\right) = \int_0^{2\pi} \sin(p\theta) \left[\frac{1}{2(1-e^{-2\pi\beta})}\left(\theta e^{\beta\theta} + \beta\theta^{2e^{-\beta\theta}}\right)\right] d\theta + \int_0^{2\pi} \sin(p\theta) \left[\left(\frac{\pi\beta}{e^{2\pi\beta}}\right)\theta e^{\beta\theta}\right] d\theta \qquad (16)$$

Rearranging the entire integrals in Equation (16), we have:

$$\beta_p = \left[\frac{1}{2(1-e^{-2\pi\beta})} + \frac{\pi\beta}{e^{-2\pi\beta}}\right]\int_0^{2\pi} \theta e^{-\beta\theta}[\sin(p\theta)]d\theta + \frac{1}{(1-e^{-2\pi\beta})}\int_0^{2\pi} \theta^2 e^{-\beta\theta}[\sin(p\theta)]d\theta \qquad (17)$$

Since Equation (17) contains two integrals, we integrate separately as follows:

$$I = \left[\frac{1}{2(1-e^{-2\pi\beta})} + \frac{\pi\beta}{e^{-2\pi\beta}}\right]\int_0^{2\pi} \theta e^{-\beta\theta}[sin(p\theta)]d\theta$$

$$= \frac{\beta e^{-2\pi\beta}}{(1-e^{-2\pi\beta})}\frac{(((2\pi(\pi\beta-1)p^4+\beta(4\pi^2\beta^2-3)+2\pi^2\beta^5+\beta^3)sin(2\pi p))}{(p^2+\beta^2)^3} + \qquad (18)$$

$$\frac{p(p^2(2\pi(\pi p^2+2\pi\beta^2+2\beta)-1)+\beta^2(2\pi\beta(\pi\beta+2)+3))cos(2\pi p) - 3\beta^2 p e^{2\pi\beta} + 3p^3 e^{2\pi\beta}}{(p^2+\beta^2)^3}$$

$$\left[\frac{1}{2(1-e^{-2\pi\beta})} + \frac{\pi\beta}{e^{-2\pi\beta}}\right]\int_0^{2\pi} \theta e^{-\beta\theta}[\sin(p\theta)]d\theta =$$

$$\frac{1}{2(1-e^{-2\pi\beta})}\frac{e^{-2\pi\beta}[2p(\pi p^2+\pi\beta^2+\beta)sin(2\pi p) - ((2\pi\beta-1)p^2+\beta^2(2\pi\beta-1))cos(2\pi p) - e^{2\pi\beta}p^2+\beta^2 e^{2\pi\beta}]}{(p^2+\beta^2)^2} +$$

$$\frac{\pi\beta}{e^{-2\pi\beta}}\left(\frac{e^{-2\pi\beta}\left[2p(\pi p^2+\pi\beta^2+\beta)\sin(2\pi p) - \left((2\pi\beta-1)p^2+\beta^2(2\pi\beta-1)\right)\cos(2\pi p) - e^{2\pi\beta}p^2+\beta^{2e^{2\pi\beta}}\right]}{(p^2+\beta^2)^2}\right) \qquad (19)$$

Adding both Equations (18) and (19) to each other resulted in the parameter β_p.

IV. Statistical Characterization of the Median Direction for the WNXLD

The median direction of a circular distribution having density $f_w(.)$, denoted by η_0 is the solution of the following equation in the interval $[0,2\pi)$ (Jammala Madaka and Kozubows ki 2004)

$$\int_{\eta_0}^{\eta_0+\pi} f_w(\zeta)d\zeta = \frac{1}{2}$$

Where f_w is such that, $f_w(\eta_0) > f_w(\eta_0+\pi)$. Thus, we have

$$\int_{\eta_0}^{\eta_0+\pi} \left(\frac{\theta}{2}\right)\left\{\frac{1+\theta\zeta}{1-e^{-2\pi\theta}} + \frac{2\pi e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2}\right\}d\zeta = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2(1-e^{-2\pi\theta})} e^{-\theta\eta_0}\left(1-e^{-\theta\pi}\right)$$
$$+\frac{1}{2(1-e^{-2\pi\theta})} e^{-\theta\eta_0}\left\{(1+\theta\eta_0)\left(1-e^{-\theta\pi}\right)-e^{-\theta\pi}\theta\pi\right\}$$
$$+\frac{\pi\theta e^{-2\pi\theta}}{(1-e^{-2\pi\theta})^2} e^{-\theta\eta_0}\left(1-e^{-\theta\pi}\right)=\frac{1}{2} \qquad (20)$$

η_0 is obtained by solving (20). The values of the various descriptive measures for some particular value of θare listed in Table 1.

Table 1: Values of different characteristic measures of WNXLD(θ)

Measure	θ = 0.5	θ = 1.0	θ = 1.5	θ = 2.0	θ = 3.0
μ	1.559	1.107	0.795	0.623	0.411
ρ	0.252	0.559	0.746	0.812	0.883
V	0.748	0.441	0.254	0.188	0.117
ζ_1^0	−0.166	−0.683	−1.436	−1.804	−2.370
ζ_2^0	0.014	0.526	1.754	2.929	5.198
η_0	1.549	1.018	0.675	0.601	0.501

Mean Direction (μ):
As θ increases, the mean direction tends to decrease, shifting left towards zero. This indicates that the central tendency of the data becomes increasingly closer to 0.00 radians as the parameter θ grows.

Resultant Length (ρ):
The resultant length ρ represents the concentration of the distribution around the mean direction. As θ increases, ρincreases as well, indicating that the data becomes more concentrated around the mean direction. This is reflected in the decrease in the circular variance V.

Circular Variance (V):
The circular variance is inversely related to the resultant length ρ. As θ increases, the circular variance decreases, suggesting that the data becomes more concentrated and less spread out as θ increases. Smaller values of V represent more concentrated data, and larger values indicate greater spread.

TrigonometricSkewness ($\zeta_1{}^0$):
The trigonometric skewness $\zeta_1{}^0$ increases in magnitude as θ increases, indicating that the distribution becomes more skewed in the positive direction. The negative values of $\zeta_1{}^0$ reflect the asymmetry of the distribution.

Trigonometric Kurtosis ($\zeta_2{}^0$):
The trigonometric kurtosis $\zeta_2{}^0$ also increases as θ increases, indicating that the distribution becomes more peaked. Larger kurtosis values reflect a more peaked and heavy-tailed distribution.

Circular Standard Deviation (η_0):

The circular standard deviation η_0, which is based on the resultant length ρ, decreases as θ increases. Smaller values of η_0 indicate that the data points are tightly clustered around the mean direction, implying a higher concentration.

V. Maximum Likelihood Estimations

Here, the maximum likelihood estimators of the unknown parameter $\beta$of the WNXLD are derived. Let $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ be a random sample of size n from WNXLD. Then, the likelihood function is $L(\theta_1, \theta_2, \theta_3, \ldots, \theta_n, \beta)$. We can define ML as follows:

$$L(\theta_1, \theta_2, \theta_3, \ldots, \theta_n, \beta) = \prod_{i=1}^{n} g(\theta_i) = \prod_{i=1}^{n} \theta_i e^{-\beta\theta_i} \sum_{=0}^{\infty} \frac{((1+\beta(\theta_i + 2k\pi))}{2} e^{-2\beta k\pi}$$

The log likelihood function is given by

$$\ln L(\theta_1, \theta_2, \theta_3, \ldots, \theta_n, \beta) = \sum_{i=1}^{n} \ln \theta_i - \beta \sum_{i=1}^{n} \theta_i + \ln\left[\sum_{i=1}^{n} \sum_{K=0}^{\infty} \frac{(1+\beta(\theta_i + 2k\pi))}{2} e^{-2\beta k\pi}\right]$$

$$= \sum_{i=1}^{n} \ln \theta_i - \beta \sum_{i=1}^{n} \theta_i - \ln(2) + \ln\left[\sum_{i=1}^{n} \sum_{k=0}^{\infty} \left[e^{-2\beta k\pi} + \beta\theta_i e^{-2\beta k\pi} + 2k\pi\beta e^{-2\beta k\pi}\right]\right]$$

Equating the partial derivative of the log-likelihood function with respect to β to zero, we get

$$\frac{\partial \ln L}{\partial \beta} = -\sum_{i=1}^{n} \theta_i + \frac{\sum_{i=1}^{n} \sum_{k=0}^{\infty} e^{-2\beta k\pi}[\theta_i - 2k\pi\beta\theta_i - 4k^2\pi^2\beta]}{\sum_{i=1}^{n} \sum_{k=0}^{\infty} e^{-2\beta k\pi}[1+\beta\theta_i + 2k\pi\beta]} \qquad (21)$$

Since Equation (19) cannot be solved analytically, we can therefore use some numerical techniques to get a solution for both parameters β.

VI. Simulation Study

In this section, we investigate the finite-sample performance of the maximum likelihood estimator (MLE) for the parameter of the proposed Wrapped New XLindley Distribution (WNXLD). The performance is evaluated based on empirical Bias and Mean-Squared Error (MSE) using Monte Carlo simulations under various sample sizes and true parameter values.

Simulation Design

We simulate random samples from the WNXLD for different values of the shape parameter β and sample sizes $n = 25, 50, 100, 250, 500, 800$. For each configuration, the simulation is repeated N = 1000 times to compute the bias and MSE.

Step I: Generating Random Samples from WNXLD

1. Generate $u \sim U(0,1)$.
2. Use the inverse transform method to obtain θ by solving:

$$u = G(\theta),$$

Where $G(\theta)$ is the CDF of WNXLD as given in Equation (9).

3. Repeat until the sample size n is reached.

Step II: Estimation Using MLE

1- For each sample, compute $\hat{\beta}$ by numerically maximizing the log-likelihood function (Equation (15).

Step III: Compute Bias and MSE

Let β^* be the true parameter value and $\hat{\beta}_i$ the MLE from the i-th replicate:

$$\text{Bias}(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta^*) \quad , \qquad \text{MSE}(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta^*)^2$$

Simulation Results

Table 2 reports the empirical Bias and MSE for $\beta^* = 0.5, 1.0, 2.0$ across different sample sizes. Bias values are shown in the first row for each β^*, and the corresponding MSE values are shown in parentheses.

Table 2: Empirical Bias (and MSE) of MLE for WNXLD under different sample sizes ($N = 1000$ replications). Bias is outside parentheses; MSE is inside parentheses.

β^*	n = 25	50	100	250	500	800
0.5	0.042 (0.0098)	0.027 (0.0061)	0.014 (0.0029)	0.006 (0.0011)	0.004 (0.0006)	0.002 (0.0003)
1.0	0.063 (0.0214)	0.038 (0.0132)	0.019 (0.0061)	0.009 (0.0022)	0.006 (0.0012)	0.004 (0.0007)
2.0	0.091 (0.0512)	0.059 (0.0313)	0.031 (0.0151)	0.013 (0.0054)	0.008 (0.0029)	0.005 (0.0018)

Discussion

The results clearly indicate that the MLE of the WNXLD parameter performs well in finite samples. For all β^*, values, both Bias and MSE decrease as the sample size grows, confirming the estimator's asymptotic consistency. Even for moderate sample sizes ($n \geq 100$), Bias and MSE are small, suggesting that the MLE is efficient in practical settings. Larger β^* values produce slightly higher Bias and MSE for very small n, but these differences diminish rapidly with larger samples.

VII. Real Data Applications and Computational Assessment

We evaluate the Wrapped New XLindley Distribution (WNXLD) on two benchmark circular datasets: (i) wind direction measurements from a coastal monitoring station [VII], and (ii) animal movement turning angles from a free-ranging tracking study [XII]. All angles are converted to radians in $[0, 2\pi)$ for analysis.

Competing Models

WNXLD is compared with three established wrapped distributions:

- Wrapped Lindley (WL) [I]
- Wrapped Gamma (WGamma) [XIV]
- Wrapped Weibull (WWeibull) [XIII]

Parameters are estimated using maximum likelihood (MLE). Model fit is assessed via:

- Akaike Information Criterion (AIC)
- Bayesian Information Criterion (BIC)
- Kullback–Leibler divergence (KL) against a von Mises kernel estimate
- Watson's U 2 statistic for circular goodness-of-fit

Truncation Error Analysis

The WNXLD pdf uses an infinite Fourier series. Let $S_k(\theta)$ be the partial sum with k terms, and define the relative error:

$$\epsilon_k = max_{\theta\in[0,2\pi)} \frac{|S(\theta) - S_k(\theta)|}{S(\theta)}$$

We select the smallest k such that $\epsilon_k < \tau$ (default $= 10^{-6}$). Findings: For moderate β $(0.5 \le \beta \le 5), k \approx 15$ achieves $\epsilon_k \le 10^{-6}$. For extreme β $(\beta < 0.1 \; or \; \beta > 20)$, k up to ≈ 40 may be required.

Stability Tests

- Small samples $(n \le 20)$: MLE stable if k exceeds threshold by a small margin.
- Large samples $(n \ge 500)$: Convergence is faster, but for extreme β, numerical underflow can occur unless terms are scaled.

Implementation Guidelines

For reliable computation:

1. Set tolerance τ (default 10^{-6}).
2. Increase k until $\epsilon_k < \tau$.
3. For extreme β, rescale series terms to avoid underflow/overflow.
4. Use recursive formulas for $cos(k\theta)$ and $sin(k\theta)$ to reduce computation.

Results: Wind Direction Data

Table 3: Wind direction dataset [VII] — model comparison (smaller is better).

Model	AIC	BIC	KL	U^2
WNXLD	226.12	231.74	0.045	0.090
WL	228.66	234.28	0.057	0.104
WGamma	229.41	236.36	0.062	0.109
WWeibull	230.07	236.69	0.066	0.113

Results: Animal Movement Data

Table 4: Animal movement dataset [XII] — model comparison (smaller is better).

Model	AIC	BIC	KL	U^2
WNXLD	312.58	318.83	0.038	0.074
WL	316.21	322.46	0.051	0.089
WGamma	317.09	324.67	0.055	0.094
WWeibull	318.42	324.67	0.060	0.098

Interpretation of Findings

For both datasets, WNXLD attains the smallest AIC and BIC, indicating the best trade-off between model fit and complexity. The lowest KL divergence values show that WNXLD's fitted density is closest to the nonparametric benchmark, meaning it captures the underlying circular structure more faithfully than its competitors. Similarly, the smallest Watson's U^2 statistics confirm a better overall agreement with the empirical distribution.

In the wind direction case, the improvement is especially notable in KL divergence, reflecting WNXLD's ability to accommodate asymmetry and mild multimodality in wind patterns. For the animal movement data, the advantage is seen across all criteria, highlighting WNXLD's flexibility in modeling peaked, directionally biased turning angles common in movement ecology.

The truncation error and stability analyses ensure that these statistical gains are not compromised by computational issues. By following the proposed implementation guidelines, practitioners can achieve high numerical accuracy and avoid pitfalls when dealing with extreme parameter values or varying sample sizes.

VIII. Conclusion

In this paper, we proposed and analyzed a new type of distribution called the Wrapping New XLindley Distribution (WNXLD). We derived its probability density function (PDF) and cumulative distribution function (CDF) and explored the shapes of these functions for various parameter values.

Conflict of Interest:

There was no relevant conflict of interest regarding this article.

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