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APPROXIMATE SOLUTION FOR NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY HOMOTOPY PERTURBATION-BASED TECHNIQUE

Nishtha[1], Sidharth Monga[2],Vansh Garg[3], Yogesh, Himanshu[4]

.[1]Department of Computer Science and Engineering, Chitkara University, Rajpura, Punjab, India.

Email: [1]nishtha1374.be23@chitkara.edu.in, [2]Sidharth0892.be23@chitkara.edu.in
[3]Vansh1074.be23@chitkara.edu.in, [4]Yogesh1167.be23@chitkara.edu.in,
[5]Himanshu0690.be23@chitkara.edu.in

Corresponding Author: **Sidharth**

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Abstract

In this paper, we present an effective semi-analytical method for solving non-linear partial differential equations that arise in various scientific and engineering fields. The Homotopy Perturbation Method (HPM) combines the concepts of homotopy analysis and perturbation theory to obtain approximate solutions for diverse partial differential equations. Several numerical examples are presented to illustrate the accuracy and efficiency of the proposed method.

Keywords: Homotopy Perturbation method, Partial differential equations

I. Introduction

Partial differential equations (PDEs) are mathematical models used to represent phenomena involving functions of multiple variables, often relying on spatial or temporal characteristics. Consequently, scientists in physics, biology, economics, and engineering utilize PDEs to study and understand processes that depend on space and time. For biological applications, PDEs are extremely important for describing reaction-diffusion type systems that govern the dynamics of biological populations or the propagation of chemical substances. Here, diffusion describes how the random movement of particles impacts their arrangement, while the term reaction describes interactions between species or substances. Thus, many important phenomena can be modelled, including tumor growth, population dynamics, the spread of infectious disease, and morphogenesis in developmental biology.

PDEs offer an adaptable paradigm for investigating the complicated dynamics of systems whose evolution depends on local interactions and spatial distribution. For example, an extremely simplified reaction-diffusion model might suggest a population grows or dies locally in response to an environmental factor, while also diffusing across the geographic area.

The Variational Homotopy Perturbation Method (VHPM) represents a major advancement in solving nonlinear PDEs. This method combines the advantages of both the Variational Iteration Method (VIM) and the Homotopy Perturbation Method (HPM). VHPM has successfully solved several problems, including the n-dimensional Burgers' equation in fluid dynamics. This powerful and flexible method can be extended and utilized by researchers to solve nonlinear systems that are intractable with classical techniques [XII, II, IV].

Traveling wave solutions, characterized by a propagating wave speed, are often the focus of real-world applications. This is because compact support initial conditions, where the disturbance is localized in space, are more common and applicable than infinite or large initial conditions in real-world scenarios. These compact, or localized, initial conditions often represent disturbances, such as a shock wave or pulse, that originate in a small area and propagate outwards over time. The minimum wave speed is particularly effective as it represents the slowest speed at which a disturbance can propagate. In nonlinear dynamics, this minimum speed often marks the threshold between the spread and extinction of a phenomenon. We can learn about the stability of these solutions through their dynamics [X, XVI, XV, XVI].

In the current article, we provide both recent theoretical and numerical approaches in a study of traveling wave solutions. More specifically, we could identify the traveling wave candidates that traveled with the lowest wave speed. Both theoretical and numerical approaches were valuable to provide a variety of theoretical ideas as well as useful numerical approximations, which we felt were essential to better understand the myriad complicated dynamics associated with nonlinear PDEs. First, we considered the qualitative behaviours of the reduced system through phase plane analysis and computed the wave profiles and determined their stability. Second, since the shooting technique addresses the boundary value problem by turning it into an initial value problem, we considered using it to calculate the wavefronts and determine permissible speeds of travel because it is such a reliable numerical method. Third, we implemented some finite difference schemes to attempt to simulate the complete time-dependent PDE to check analytical predictions, and we could visualize the solutions from the compactly supported initial condition. Taken together, these three differing but complementary approaches provide a very rich and complete basis for studying traveling behavior in reaction-diffusion systems, or indeed other nonlinear models [III, V, VI-IX, XII, XIV].

II. Application of the Homotopy perturbation-based technique for Non-linear Partial Differential Equations

Let us consider the following non-linear equation

$$A(\emptyset) - f(r) = 0, \; r \in \Omega \qquad (1)$$

with conditions:

B ($\emptyset, \partial\emptyset/\partial n$) = 0, r∈ Γ (2)

A = L + N

Equation (1) can be written as

L($\emptyset$) + N($\emptyset$) – f(r) = 0, r∈ Ω (3)

H (φ, p) = (1-p) [L(φ) – L ($\emptyset_0$)] = p[A(φ) – f(r)] = 0, p ∈ [0,1], r∈ Ω (4)

H (φ, p) =L(φ) −L($\emptyset_0$) + $pL(\emptyset_0) + p[N(\varphi) -$ f(r)] =0 (5)

H(φ,0) = L(φ) – L ($\emptyset_o$) = 0 (6)

H(φ,0) = L(φ) – L ($\emptyset_o$) = 0 (7)

H(φ,0) = L(φ) – L ($\emptyset_o$) = 0 (8)

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots$$

Approximate solution of equation (1):

$$\emptyset = \lim_{p\to 1} \varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots , \qquad (9)$$

Non-linear problems of partial differential equations:

Problem 1. Consider a two-component evolutionary system, such that...

$$\frac{\partial\emptyset}{\partial t} - \frac{\partial^3\emptyset}{\partial^3 x} - \emptyset\frac{\partial\emptyset}{\partial x} - \varphi\frac{\partial\emptyset}{\partial x} = 0 \qquad 10$$

$$\frac{\partial\varphi}{\partial t} + 2\,\frac{\partial^3\varphi}{\partial^3 x} + v\frac{\partial\varphi}{\partial x} = 0 \qquad 11$$

$$\begin{pmatrix} \emptyset(x,0)=3-6\tan h^2\left(\frac{x}{2}\right), \\ \varphi(x,0)=3-6\tan h^2\left(\frac{x}{2}\right) \end{pmatrix} \qquad 12$$

Construction Homotopy for the system, we obtain

$$\frac{\partial\emptyset}{\partial t} + p\left(\frac{\partial\emptyset}{\partial t} - \frac{\partial^3\emptyset}{\partial x^3} - \emptyset\frac{\partial\emptyset}{\partial x} - \varphi\,\frac{\partial\emptyset}{\partial x}\right) \qquad 13$$

$$\frac{\partial\varphi}{\partial t} + p\left(\frac{\partial^3\varphi}{\partial x^3} + \varphi\frac{\partial\emptyset}{\partial x}\right) = 0 \qquad 14$$

The solution of the Systems form,

$$\emptyset = \emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots \qquad (15)$$

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots \qquad (16)$$

Substituting equation (15)-(16) in equation (13)-(14) r and comparing the coefficients, we obtain

$$\frac{\partial}{\partial t}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3 + ..)$$
$$+p\begin{bmatrix} -\frac{\partial^3}{\partial x^3}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots) - (\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots) \\ \frac{\partial}{\partial x}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots) - (\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots) \\ \frac{\partial}{\partial x}(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots) \end{bmatrix} = 0$$

$$\frac{\partial}{\partial t}(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots)$$
$$+p\begin{bmatrix} 2\frac{\partial^3}{\partial x^3}(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots) \\ +\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots \\ \frac{\partial}{\partial x}(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots) \end{bmatrix} = 0$$

Now comparing the coefficients,

$$p^0: \frac{\partial \emptyset_0}{\partial t} = 0 \tag{17}$$

$$p^0: \frac{\partial \varphi_0}{\partial t} = 0 \tag{18}$$

$$p^1: \frac{\partial \emptyset_1}{\partial t} - \emptyset_0\frac{\partial \emptyset_0}{\partial x} - \frac{\partial^3 \emptyset_0}{\partial x^3} - \varphi_0\frac{\partial \varphi_0}{\partial x} = 0\,, \tag{19}$$

$$p^1: \frac{\partial \varphi_1}{\partial t} - \emptyset_0\frac{\partial \varphi_0}{\partial x} - 2\frac{\partial^3 \emptyset_0}{\partial x^3} = 0\,, \tag{20}$$

$$p^2: \frac{\partial \emptyset_2}{\partial t} - \emptyset_1\frac{\partial \emptyset_0}{\partial x} - \emptyset_0\frac{\partial \emptyset_1}{\partial x} - \varphi_1\frac{\partial \varphi_0}{\partial x} - \emptyset_0\frac{\partial \varphi_1}{\partial x} - \frac{\partial^3 \emptyset_1}{\partial x^3} = 0\,, \tag{21}$$

$$p^2: \frac{\partial \varphi_2}{\partial t} - \emptyset_1\frac{\partial \varphi_0}{\partial x} - \emptyset_0\frac{\partial \varphi_1}{\partial x} + 2\frac{\partial^3 \varphi_0}{\partial x^3} = 0\,, \tag{22}$$

$$\emptyset_0 = \emptyset(x,0) = 3 - \tan h^2(\tfrac{x}{2})\,,$$

$$\varphi_0 = \varphi(x,0) = 3t\sqrt{2}\tan h^2(\tfrac{x}{2})\,,$$

Now the solution of Equations,

$$p^1: \frac{\partial \emptyset_1}{\partial t} - \emptyset_0\frac{\partial \emptyset_0}{\partial x} - \frac{\partial^3 \varphi_0}{\partial x^3} \text{-} \varphi_0\frac{\partial \varphi_0}{\partial x} = 0,$$

$$\frac{\partial \emptyset_1}{\partial t} = \emptyset_0\frac{\partial \emptyset_0}{\partial x} + \frac{\partial^3 \varphi_0}{\partial x^3} + \varphi_0\frac{\partial \varphi_0}{\partial x}.$$

Now taking the integral, we get:

$$\int \frac{\partial \emptyset}{\partial x} = \int \left[\emptyset_0\frac{\partial \emptyset_0}{\partial x} + \frac{\partial^2 \emptyset_0}{\partial x^2} + \varphi_0\frac{\partial \varphi_0}{\partial x}\right]dt,$$

$$\emptyset_1 = \int_0^1 \left[\emptyset_0\frac{\partial \emptyset_0}{\partial x} + \frac{\partial^2 \emptyset_0}{\partial x^2} + \varphi_0\frac{\partial \varphi_0}{\partial x}\right]dt. \tag{23}$$

Using the values of $\emptyset_0$ and φ_0 we get:

$$\emptyset_1 = -48sch^3\left(x\left(\sin h^4\left(\frac{x}{2}\right)\right)\right).$$

Similarly, for φ_1, we

$$\varphi_1 = \int_0^1\left(-\emptyset_0\frac{\partial\varphi_0}{\partial x} - 2\frac{\partial^3\varphi_0}{\partial x^3}dt\right). \quad (24)$$

Now putting the values of $\varphi_0$and $\emptyset_0$ in the Equation, we get:

$$\emptyset_1 = t\begin{pmatrix}9t\sqrt{2}\sec h^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)-12t\sqrt{2}\sec h^4\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)\\ -12t\sqrt{2}\sec h^2\left(\frac{x}{2}\right)\tan h^3\left(\frac{x}{2}\right).\end{pmatrix}$$

$$\emptyset_{app}(x,t) = \sum_{i=0}^{7}\emptyset_i =$$

$$= 3 - 6\tan h^2\left(\frac{x}{2}\right) - 48tsch^3(x)\left(\sin h^4\left(\frac{x}{2}\right) + \ldots.\right),$$

$$\varphi_{app} = \sum_{i=0}^{7}\varphi_i$$

$$= 3t\sqrt{2}\tan h^2\left(\frac{x}{2}\right) + t\left(9t\sqrt{2}\sec h^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)\right),$$

$$-12t\sqrt{2}\sec h^4\left(\frac{x}{2}\right)\tan h\left(\frac{x}{2}\right) - 12t\sqrt{2}\sec h^2\left(\frac{x}{2}\right)\tan h^3\left(\frac{x}{2}\right) + ..$$

Problem 2. Consider a two-component evolutionary system of homogeneous equations of third order.

$$\frac{\partial\emptyset}{\partial t} - \frac{\partial^3\emptyset}{\partial x^3} - 2\varphi\frac{\partial\emptyset}{\partial x} - \varphi\frac{\partial\varphi}{\partial x} = 0, \quad (25)$$

$$\frac{\partial\varphi}{\partial t} - \emptyset\frac{\partial\emptyset}{\partial x} = 0 \quad (26)$$

subject to

$$\emptyset(x, 0) = -\tan h\left(\frac{x}{\sqrt{3}}\right)$$

$$\varphi(x,0) = \frac{-1}{6} - \frac{1}{2}\tan x^2\left(\frac{x}{\sqrt{3}}\right) \quad (27)$$

Homotopy for Equations, we get

$$\frac{\partial\emptyset}{\partial t} + p\left(\frac{\partial^3\emptyset}{\partial x^3} - 2\varphi\frac{\partial\emptyset}{\partial x} - \emptyset\frac{\partial\emptyset}{\partial x}\right) = 0 \quad (28)$$

$$\frac{\partial\varphi}{\partial t} + p\left(-\emptyset\frac{\partial\emptyset}{\partial x}\right) = 0 \quad (29)$$

The solution of Systems has the form

$$\emptyset = \emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots \quad (30)$$

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots \quad (31)$$

Substituting Equations,

$$\frac{\partial}{\partial t}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3 + ..) + p\begin{bmatrix} -\frac{\partial^3}{\partial x^3}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots) - 2(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots) \\ \frac{\partial}{\partial x}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots) - (\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 \ + \cdots) \\ \frac{\partial}{\partial x}(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 \ + \cdots) \end{bmatrix} = 0 \quad (32)$$

Using the values of $\emptyset$ andφ

$$\frac{\partial}{\partial t}(\varphi_0 + p\varphi_1 + p^2\varphi_2 \ + p^3\varphi_3 + ..) + \mathrm{p}\begin{pmatrix} -(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 + \cdots) \\ \frac{\partial}{\partial x}(-(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 \ + p^3\emptyset_3 + \cdots)) \end{pmatrix} = 0 \qquad (33)$$

Now comparing the coefficients,

$$p^0: \frac{\partial\emptyset_0}{\partial t} \ = \ 0 \qquad (34)$$

$$p^0: \frac{\partial\varphi_0}{\partial t} \ = \ 0 \qquad (35)$$

$$p^1: \frac{\partial\emptyset_1}{\partial t} \ - \ 2\varphi_0\frac{\partial\emptyset_0}{\partial x} \ - \frac{\partial^3\emptyset_0}{\partial x^3} \ - \ \emptyset_0\frac{\partial\varphi_0}{\partial x} \ = \ 0, \qquad (36)$$

$$p^1: \frac{\partial\varphi_1}{\partial t} \ - \ \emptyset_0\frac{\partial\varphi_0}{\partial x} \ = \ 0, \qquad (37)$$

$$p^2: \frac{\partial\emptyset_2}{\partial t} \ - \ 2\varphi_1\frac{\partial\emptyset_0}{\partial x} - \ 2\varphi_0\frac{\partial\emptyset_1}{\partial x} - \ \emptyset_1\frac{\partial\varphi_0}{\partial x} - \ \emptyset_0\frac{\partial\varphi_1}{\partial x} - \ \frac{\partial^3\emptyset_1}{\partial x^3} \ = \ 0, \qquad (38)$$

$$p^2: \frac{\partial\varphi_2}{\partial t} \ - \ \emptyset_1\frac{\partial\emptyset_0}{\partial x} - \ \emptyset_0\frac{\partial\emptyset_1}{\partial x} \ = \ 0, \qquad (39)$$

$$\varphi_0 = \emptyset(x,0) = -\tan h(\frac{x}{\sqrt{3}}) \ ,$$

$$\varphi_0(x,0) = \frac{-1}{6} \ - \ \frac{1}{2}\tan x^2\left(\frac{x}{\sqrt{3}}\right).$$

Derive the solutions of Equations,

$$\frac{\partial\emptyset_1}{\partial t} - 2\varphi_o\frac{\partial\emptyset_o}{\partial x} - \emptyset_o\frac{\partial\varphi_o}{\partial x} - \frac{\partial^3\emptyset_o}{\partial x^3} \ = \ 0,$$

$$\frac{\partial\varphi_1}{\partial t} - \ \emptyset_o\frac{\partial\emptyset_o}{\partial x} \ = \ 0,$$

$$\frac{\partial\emptyset_1}{\partial t} \ = \ 2\varphi_o\frac{\partial\emptyset_o}{\partial x} \ - \emptyset_o\frac{\partial\varphi_o}{\partial x} - \frac{\partial^3\emptyset_o}{\partial x^3},$$

$$\frac{\partial\varphi_1}{\partial t} = \ \emptyset_o\frac{\partial\emptyset_o}{\partial x} \ = \ 0,$$

Taking the integral of both sides,

$$\emptyset_1 \ = \ \int_0^1\left[2\varphi_0\,\frac{\partial\emptyset_0}{\partial x} \ + \ \emptyset_0\frac{\partial\varphi_0}{\partial x} - \ \frac{\partial^3\emptyset_0}{\partial x^3}\right]dt. \qquad (40)$$

$$\varphi_1 = \int_0^1 \left(-\emptyset_0 \frac{\partial \emptyset_0}{\partial x}\, dt\right). \tag{41}$$

Using the values of $\emptyset_0$ and φ_0

$$\emptyset_1 = \frac{1}{\sqrt{3}} t \sec h^2 \left(\frac{x}{\sqrt{3}}\right)$$

$$\varphi_1 = \frac{1}{\sqrt{3}} t \tan h \left(\frac{x}{\sqrt{3}}\right) \sec h^2 \left(\frac{x}{\sqrt{3}}\right)$$

$$\emptyset_{app}(x,t) = \sum_{i=0}^{8} \emptyset_i =$$

$$= -\tan h \left(\frac{x}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} tsc\, h^2 \left(\frac{x}{\sqrt{3}}\right) + \ldots., \tag{42}$$

$$\varphi_{app} = \sum_{i=0}^{8} \varphi_i$$

$$= \frac{-1}{6} - \frac{1}{2} \tan h^2 \left(\frac{x}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} t \tan h \left(\frac{x}{\sqrt{3}}\right) \sec h^2 \left(\frac{x}{\sqrt{3}}\right) + \ldots., \tag{43}$$

Problem 3. Consider the generalized coupled Hirota-Satsuma system.

$$\frac{\partial \emptyset}{\partial t} - \frac{1}{2} \frac{\partial^2 \emptyset}{\partial x^2} - 3\emptyset \frac{\partial \emptyset}{\partial x}(\varphi, \theta) = 0, \tag{44}$$

$$\frac{\partial \varphi}{\partial t} - \frac{\partial^3 \varphi}{\partial x^3} - 3\emptyset \frac{\partial \varphi}{\partial x} = 0 \tag{45}$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial^3 \theta}{\partial x^3} - 3\emptyset \frac{\partial \theta}{\partial x} = 0 \tag{46}$$

Subject to

$$\emptyset(x,0) = \frac{-1}{3} + \tan h^2(x) \tag{47}$$

$$\varphi(x,0) = \tan h\,(x), \tag{48}$$

$$\theta(x,0) = \frac{8}{3} \tan h(x) \tag{49}$$

By the Homotopy Perturbation Method, we get

$$\frac{\partial \emptyset}{\partial t} + p\left(\frac{-1}{3} \frac{\partial^3 \emptyset}{\partial x^3} - 2\emptyset \frac{\partial \emptyset}{\partial x} - 3\frac{\partial}{\partial x}(\varphi\theta)\right) = 0 \tag{50}$$

$$\frac{\partial \varphi}{\partial t} + p\left(\frac{\partial^3 \varphi}{\partial x^3} - 3\emptyset \frac{\partial \varphi}{\partial x}\right) = 0 \tag{51}$$

$$\frac{\partial \theta}{\partial t} + p\left(\frac{\partial^3 \theta}{\partial x^3} - 3\emptyset \frac{\partial \theta}{\partial x}\right) = 0 \tag{52}$$

The solution of the Equation,

$$\emptyset = \emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots \tag{53}$$

$$\varphi = \varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots \tag{54}$$

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots \tag{55}$$

Substituting Equations,

$$\frac{\partial}{\partial t}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots) + p\begin{bmatrix} -\frac{1}{3}\frac{\partial^3}{\partial x^3}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots) + 3(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots) \\ \frac{\partial}{\partial x}(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots) - 3(\varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots) \\ -3\frac{\partial}{\partial x}(\theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots) \end{bmatrix} = 0$$

$$\frac{\partial}{\partial t}(\varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots) + p\begin{bmatrix} \frac{\partial^3}{\partial x^3}(\varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots) \\ -3(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots) \\ \frac{\partial}{\partial x}(\varphi_0 + p\varphi_1 + p^2\varphi_2 + p^3\varphi_3 + \cdots) \end{bmatrix} = 0 \quad (56)$$

$$\frac{\partial}{\partial t}(\theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots) + p\begin{bmatrix} \frac{\partial^3}{\partial x^3}(\theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots) \\ -3(\emptyset_0 + p\emptyset_1 + p^2\emptyset_2 + p^3\emptyset_3 + \cdots) \\ \frac{\partial}{\partial x}(\theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots) \end{bmatrix} = 0 \quad (57)$$

Now comparing the coefficients:

$$p^0: \frac{\partial\emptyset_0}{\partial t} = 0 \quad (58)$$

$$p^0: \frac{\partial\varphi_0}{\partial t} = 0 \quad (59)$$

$$p^0: \frac{\partial\theta_0}{\partial t} = 0 \quad (60)$$

$$p^1: \frac{\partial\emptyset_1}{\partial t} + 3\,\emptyset_0\frac{\partial\emptyset_0}{\partial x} - 3\,\theta_0\frac{\partial\varphi_0}{\partial x} - 3\,\varphi_0\frac{\partial\emptyset_0}{\partial x} - \frac{1}{2}\frac{\partial^3\emptyset_0}{\partial x^3} = 0, \quad (61)$$

$$p^1: \frac{\partial\varphi_1}{\partial t} - 3\,\theta_0\frac{\partial\varphi_0}{\partial x} + \frac{\partial^3\varphi_0}{\partial x^3} = 0, \quad (62)$$

$$p^1: \frac{\partial\theta_1}{\partial t} - 3\,\theta_0\frac{\partial\theta_0}{\partial x} + \frac{\partial^3\theta_0}{\partial x^3} = 0, \quad (63)$$

$$p^2: \frac{\partial\emptyset_2}{\partial t} + 3\,\emptyset_1\frac{\partial\emptyset_0}{\partial x} - 3\emptyset_0\frac{\partial\emptyset_1}{\partial x} - 3\theta_1\frac{\partial\varphi_0}{\partial x} - 3\theta_0\frac{\partial\varphi_1}{\partial x}$$
$$-3\varphi_1\frac{\partial\theta_0}{\partial x} - 3\varphi_0\frac{\partial\theta_1}{\partial x} - \frac{1}{2}\frac{\partial^3\emptyset_0}{\partial x^3} = 0 \quad (64)$$

$$p^2: \frac{\partial\varphi_2}{\partial t} - 3\emptyset_1\frac{\partial\varphi_0}{\partial x} - 3\emptyset_0\frac{\partial\varphi_1}{\partial x} + \frac{\partial^3\varphi_0}{\partial x^3} = 0, \quad (65)$$

$$p^2: \frac{\partial\theta_2}{\partial t} - 3\emptyset_1\frac{\partial\theta_0}{\partial x} - 3\emptyset_0\frac{\partial\theta_1}{\partial x} + \frac{\partial^3\theta_0}{\partial x^3} = 0, \quad (66)$$

The solution of the Equations we obtain:

$$\emptyset(x,0) = \frac{-1}{3} + \tan h^2(x)\,,$$

$$\varphi(x,0) = \tan h(x)\,,$$

$$\Theta\,(x,0) = \frac{8}{3}\tan h(x).$$

We derive the solution of the Equations:

$$\frac{\partial \emptyset_1}{\partial t} + 3\,\emptyset_0 \frac{\partial \emptyset_0}{\partial x} - 3\theta_0 \frac{\partial \varphi_1}{\partial x} - 3\varphi_0 \frac{\partial \theta_0}{\partial x} - \frac{1}{2}\frac{\partial^3 \emptyset_0}{\partial x^3} = 0$$

$$\frac{\partial \varphi_1}{\partial x} - 3\emptyset_0 \frac{\partial \varphi_0}{\partial x} - \frac{\partial^3 \varphi_0}{\partial x^3} = 0,$$

$$\frac{\partial \theta_1}{\partial x} - 3\emptyset_0 \frac{\partial \theta_0}{\partial x} - \frac{\partial^3 \theta_0}{\partial x^3} = 0.$$

$$\frac{\partial \emptyset_1}{\partial t} = -3\,\emptyset_0 \frac{\partial \emptyset_0}{\partial x} - 3\theta_0 \frac{\partial \varphi_1}{\partial x} - 3\varphi_0 \frac{\partial \theta_0}{\partial x} - \frac{1}{2}\frac{\partial^3 \emptyset_0}{\partial x^3} \tag{67}$$

$$\frac{\partial \varphi_1}{\partial x} = 3\emptyset_0 \frac{\partial \varphi_0}{\partial x} - \frac{\partial^3 \varphi_0}{\partial x^3} \tag{68}$$

$$\frac{\partial \theta_1}{\partial x} = 3\emptyset_0 \frac{\partial \theta_0}{\partial x} - \frac{\partial^3 \theta_0}{\partial x^3} \tag{69}$$

Integrals on both sides

$$\emptyset_1 = \int_0^1 \left[-3\,\emptyset_0 \frac{\partial \emptyset_0}{\partial x} - 3\theta_0 \frac{\partial \varphi_1}{\partial x} - 3\varphi_1 \frac{\partial \theta_0}{\partial x} - \frac{1}{2}\frac{\partial^3 \emptyset_0}{\partial x^3}\right] dt. \tag{70}$$

$$\varphi_1 = \int_0^1 \left(3\emptyset_0 \frac{\partial \varphi_1}{\partial x} - \frac{\partial^3 \varphi_0}{\partial x^3} dt \right). \tag{71}$$

$$\theta_1 = \int_0^1 \left(3\emptyset_0 \frac{\partial \theta_0}{\partial x} - \frac{\partial^3 \theta_0}{\partial x^3} dt \right). \tag{72}$$

Using the values of $\emptyset_0, \varphi_0$ and θ_0 in Equations (70)-(72)

$$\emptyset_1 = 4t \sec h^2(x) \tan h\,(x).$$

$$\varphi_1 = t \sec h^2(x).$$

$$\theta_1 = \frac{8}{3} t \sec h^2(x).$$

$$\emptyset_{app}(x,t) = \sum_{i=0}^{6} \emptyset_i =$$

$$= \frac{-1}{3} + \tan h^2(x) + 4t \sec h^2(x) \tan h\,(x) + \cdots, \tag{73}$$

$$\varphi_{app}(x,t) = \sum_{i=0}^{6} \varphi_i$$

$$= \tan Q\,(x) + 4t \sec Q^2(x) + \cdots \tag{74}$$

$$\theta_{app}(x,t) = \sum_{i=0}^{6} \theta_i$$

$$= \frac{8}{3} \tan h\,(x) + \frac{8}{3} t \sec h^2(x) + \cdots. \tag{75}$$

III. Conclusion

From the above discussed example, it has been demonstrated that the perturbation-based technique is a highly efficient method for solving nonlinear partial equations, compared to other methods, we can solve the nonlinear problems very easily and efficiently in small iterations. We can further use this method to solve many other nonlinear complex problems arising in various fields of science and engineering.

Conflict of Interest

It confirms that there are no relevant conflicts of interest concerning this paper.

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