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INTRODUCING THE NEW INTEGRAL TRANSFORMATION TO SOLVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

To differentiate itself from other integral transformations, this article presents a novel integral transformation known as the $\mathcal{A}$ (or Aman) transform. This transformation was inspired by a thorough study of the effectiveness of the Laplace and Sumudu transforms, specifically with regard to fractional differential equations. Applying these transformations might occasionally make processing their inverse transform challenging. This concept encourages us to reconsider and put in more effort to develop fresh, essential transformations that will make difficult problems easier to tackle. The proposed transformation has been successfully used to solve the Riemann-Liouville and Caputo FDE analytically. The outcomes of using this new approach are in perfect harmony with those of using contemporary methods. This demonstrates the $\mathcal{A}$ transform's dependability and efficiency in the analytical resolution of complex mathematical situations.

Keywords: Laplace Transform, Sumudu Transform, Caputo's Fractional Differential Equations, Riemann-Liouville's Fractional Differential Equations.

Nomenclature

LT	Laplace Transform
ST	Sumudu Transform
FC	Fractional Calculus
FDE	Fractional Differential Equations
${}_0^C D_t^\beta$	Caputo Fractional Derivative
${}_0^{RL} D_t^\beta$	Riemann-Liouville's Fractional Derivative

erf Error function

IVP Initial Value Problems

I. Introduction

The fact that not all differential equations have analytical solutions, determining the analytical solution is never an easy process. Numerous analytical methods may be used to solve differential equations. Laplace created the integral transform, or LT [XVIII], to solve differential equations. The LT makes it simple to tackle IVP. Differential equations can also be solved with the help of the Fourier transform [VI] and the Mellin transform [XI]. While the Fourier transform is used in wave analysis and the Mellin transform is used in asymptotic analysis, the Laplace transform is mostly used in time domain issues.

Later, in 1993, Watugala [XXIII] presented the Sumudu Transform, which was developed especially to deal with differential equations and engineering control problems. It's noteworthy to notice that while performing the inverse transform of ST is simple, issues solved by LT are considerably easier for ST to solve. These transformations are essential to the broader area of fractional calculus [XXVII] and have been extended to handle the complexities of FDE, as shown in references [XIV, XXII]. Additionally, these transformations are valuable because they can aid in solving nonlinear differential equations when employed alongside methods such as the variation iteration technique, the homotopy perturbation method, the new iteration method, and the Adomian decomposition method. For further details, refer to [X, XII, XIII, XIX, XX, XXI, XXVI].

The Shehu transformation, a novel transformation that Shehu developed, is a generalization of LT and ST [XXV]. Despite the aforementioned techniques, other transformations are also available in the realm of differential equations. These include the Aboodh transform [IV], ZZ transform [XVI], Mohand transform [II], Mahgoub transform [I], Sawi transform [III], Elzaki transform [IX], and Yang transform [XXIV].

Applying these transformations might be difficult when working with non-integer orders, particularly when solving fractional-order differential equations. This conceptual issue has led to the creation of a unique transformation known as the $\mathcal{A}$ transformation, which is likely also known as the Aman transformation. It offers a very useful tool for solving fractional-order differential equations.

II. Basics of $\mathcal{A}$ Transform

This section defines the $\mathcal{A}$ transform, outlines its fundamental properties, and discusses how it relates to the Sumudu and Laplace transforms.

Definition of $\mathcal{A}$ Transform: Consider the set S as the set of functions as specified in [XXIII]:

$$\mathrm{S} = \{\xi(\tau): \exists \mathcal{M}, \nu_1, \nu_2 > 0, |\xi(\tau)| < \mathcal{M}e^{(|\tau|/\nu_i)}, \text{if}, \tau \in (-1)^i \times [0, \infty)\} \quad (1)$$

Then, there exists an $\mathcal{A}$ transform of $\xi(\tau)$ specified throughout the collection of functions $\mathbb{S}$, having the following definition:

$$\mathcal{A}[\xi(\tau)] = \bar{\xi}(v^2) = \frac{1}{v}\int_0^\infty \xi(v\tau)e^{\frac{-\tau}{v}}dt\,, v \in (-\nu_1, \nu_2). \tag{2}$$

The inverse of $\mathcal{A}$ transform:

The definition of the $\mathcal{A}$ transform's inverse is:

$$\mathcal{A}^{-1}\left(\bar{\xi}(v^2)\right) = \xi(\tau), for\ \tau \geq 0.$$

Or

$$\xi(\tau) = \mathcal{A}^{-1}\left[\bar{\xi}(v^2)\right] = \frac{1}{2\pi\iota}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}e^{\frac{\tau}{u}}\xi(v^2)dv. \tag{3}$$

In the complex plane i.e., (u = x+iy), this integral is taken along $v = \alpha^2$. The real α constants and the $\mathcal{A}$ Transform variables v^2 are used here.

The relationship between Sumudu and $\mathcal{A}$ Transform

First of all, write the ST [XXIII] of $\xi(\tau) \in \mathbb{S}$ as given below:

$$S[\xi(\tau)] = G(v) = \int_0^\infty \xi(v\tau)e^{-\tau}d\tau = \frac{1}{v}\int_0^\infty \xi(\tau)e^{-\frac{\tau}{u}}dt\,,\ v \in (-\nu_1, \nu_2) \tag{4}$$

Lemma 1: If G(v) and $\bar{\xi}(v^2)$ is the ST and $\mathcal{A}$ transform of g(τ), respectively, then $\bar{\xi}(v^2) = G(v^2)$.

Proof. The $\mathcal{A}$ transform of $\xi(\tau) \in \mathbb{S}$ is

$$\mathcal{A}[\xi(\tau)] = \bar{\xi}(v^2) = \frac{1}{v}\int_0^\infty \xi(v\tau)e^{\frac{-\tau}{v}}dt,$$

Put t = vw in the above equation, and we get

$$\mathcal{A}[\xi(\tau)] = \bar{\xi}(v^2) = \frac{1}{u}\int_0^\infty \xi(v^2w)e^{-v}vdw$$

$$= \int_0^\infty \xi(v^2w)e^{-w}\,dw = G(v^2)$$

Hence proved.

The relationship between Laplace and $\mathcal{A}$ Transform

As we know, the LT of the function $\xi(\tau)$ is [XVIII]:

$$\mathcal{L}[\xi(\tau)] = F(s) = \int_0^\infty \xi(\tau)e^{-s\tau}d\tau.$$

Lemma 2: If $\bar{\xi}(v^2)$ and F(s) is the $\mathcal{A}$ transform and LT of $\xi(\tau)$ respectively then $\bar{\xi}(v^2) = \frac{1}{v^2}F\left(\frac{1}{v^2}\right)\ or\ F(s) = \frac{1}{s}\bar{\xi}\left(\frac{1}{s}\right)$.

Proof. The $\mathcal{A}$ transform of $\xi(\tau) \in S$ is

$$\mathcal{A}[\xi(\tau)] = \bar{\bar{\xi}}(v^2) = \frac{1}{v}\int_0^\infty \xi(v\tau)e^{\frac{-\tau}{v}}d\tau,$$

Put v τ =w in the above equation, and we get

$$\mathcal{A}[\xi(\tau)] = \bar{\bar{\xi}}(v^2) = \frac{1}{v}\int_0^\infty \xi(w)e^{\frac{-w}{v^2}}\frac{dw}{v}$$

$$= \frac{1}{v^2}\int_0^\infty \xi(w)e^{\frac{-w}{v^2}}dw$$

$$= \frac{1}{v^2}F\left(\frac{1}{v^2}\right)$$

The reverse relationship is derived with the help of ST (see Lemma 1)

$$\bar{\bar{\xi}}(v^2) = G(v^2)$$

It implies $\bar{\bar{\xi}}(v) = G(v) = \frac{1}{v}\int_0^\infty \xi(\tau)e^{-\frac{\tau}{v}}d\tau$

Now $\bar{g}\left(\frac{1}{s}\right) = s\int_0^\infty \xi(\tau)e^{-s\tau}d\tau$

$$\frac{1}{s}\bar{\bar{\xi}}\left(\frac{1}{s}\right) = \int_0^\infty \xi(\tau)e^{-s\tau}d\tau = F(s)$$

which completes the proof.

$\mathcal{A}$ transform of some basic functions

Theorem 1: The $\mathcal{A}$ transform of τ^n, n ≥ 1 is given as:

$$\mathcal{A}[\tau^n] = n!\,v^{2n} \tag{5}$$

Proof: By definition,

$$\mathcal{A}[\tau^n] = \frac{1}{v}\int_0^\infty (v^n\tau^n)e^{\frac{-\tau}{v}}d\tau$$

Put τ = vw, we get

$$\mathcal{A}[\tau^n] = \frac{1}{v}\int_0^\infty (v^{2n}w^n)e^{-w}v\,dw$$

$$= v^{2n}\int_0^\infty (w^n)e^{-w}dw$$

$$= v^{2n}\int_0^\infty (v^{n+1-1})e^{-w}dw$$

$$= n!\,v^{2n}$$

Corr. $\mathcal{A}[\tau^\alpha] = \Gamma(\alpha+1)v^{2\alpha}, \alpha \succ -1$

Hint. By using the Gamma function.

Table 1: The $\mathcal{A}$ transformation of some basic functions

$\xi(\tau)$	Laplace Transform [XVIII]	Sumudu Transform [XXIII]	$\mathcal{A}$ *Transform*
1	$\frac{1}{s}$	1	1
$\tau^m,\ m \geq 1$	$\frac{m!}{s^{m+1}}$	$m!\,u^m$	$m!\,v^{2m}$
$\tau^\alpha, \alpha \succ 1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	$\Gamma(\alpha+1)u^\alpha$	$\Gamma(\alpha+1)v^{2\alpha}$
$e^{a\tau}$	$\frac{1}{s-a}$	$\frac{1}{1-au}$	$\frac{1}{1-av^2}$
$\frac{\sin(a\tau)}{a}$	$\frac{1}{s^2+a^2}$	$\frac{u}{1+a^2u^2}$	$\frac{v^2}{1+a^2v^4}$
$cos(a\tau)$	$\frac{s}{s^2+a^2}$	$\frac{1}{1+a^2u^2}$	$\frac{1}{1+a^2v^4}$
$\frac{erf(\sqrt{a\tau})}{\sqrt{a}}$	$\frac{1}{s(\sqrt{s+a})}$	$\frac{\sqrt{u}}{\sqrt{1+au}}$	$\frac{v}{\sqrt{1+av^2}}$
$e^{a\tau}\frac{erf(\sqrt{a\tau})}{\sqrt{a}}$	$\frac{1}{\sqrt{s}(s-a)}$	$\frac{\sqrt{u}}{1-au}$	$\frac{v}{1-av^2}$

Some properties of $\mathcal{A}$ transform

Like the Laplace and Sumudu transforms, the $\mathcal{A}$ transform has the following properties (Table 2). Moreover, these properties have been proved in this manuscript or can be easily derived from the relationship of $\mathcal{A}$ transform with the Laplace and the Sumudu transform.

Table 2: Some properties of $\mathcal{A}$ transform

Formula	Property Name
$\mathcal{A}[\xi(\tau)] = \bar{\xi}(v^2) = \frac{1}{v}\int_0^\infty \xi(v\tau)e^{\frac{-\tau}{v}}d\tau$	Definition of $\mathcal{A}$ *transform*
$\mathcal{A}^{-1}[\bar{\xi}(v^2)] = \xi(\tau)$	The inverse of A Transform
$\mathcal{A}[a\xi(\tau)+b\eta(\tau)] = a\mathcal{A}[\xi(\tau)] + b\mathcal{A}[\eta(\tau)]$	Linearity
$\mathcal{A}[e^{a\tau}\xi(\tau)] = \frac{1}{(1-av^2)}\bar{\xi}\left(\frac{v^2}{1-av^2}\right)$	First Shifting Theorem
$\mathcal{A}[\xi^n(\tau)] = \frac{\mathcal{A}(\xi(\tau))}{v^{2n}} - \frac{\xi(0)}{v^{2n}} - \frac{\xi'(0)}{v^{2(n-1)}} - \dots - \frac{\xi^{n-1}(0)}{v^2}$	$\mathcal{A}$ *transform of derivative of function*

$\mathcal{A}\left[\int_0^{\tau}\xi(\mu)d\mu\right] = v^2\mathcal{A}(\xi(\tau)) = v^2\bar{\xi}(v^2)$	$\mathcal{A}\ transform\ of\ inegral\ of\ function$
$\mathcal{A}(\xi * \eta) = v^2\mathcal{A}[\xi(\tau)]\mathcal{A}[\eta(\tau)]$	Convolution
$\mathcal{A}[\xi(a\tau)] = \bar{\xi}(av^2)$	First Scale Preserving Theorem

$\mathcal{A}$ transformation of the derivative of a function

Theorem 2: The $\mathcal{A}$ transform of $\frac{d\xi(\tau)}{d\tau}$ i.e., the first derivative of $\xi(\tau)$ over a set S, is given below

$$\mathcal{A}\left[\frac{d\xi(\tau)}{d\tau}\right] = \frac{\mathcal{A}(\xi(t))}{v^2} - \frac{\xi(0)}{v^2} \tag{6}$$

Proof. $\mathcal{A}\left[\frac{d\xi(\tau)}{d\tau}\right] = \mathcal{A}(v^2) = \frac{1}{v}\int_0^{\infty}\frac{d\xi(v\tau)}{d\tau}e^{\frac{-\tau}{v}}d\tau$

$$= \frac{1}{v}\left[\left(e^{\frac{-\tau}{v}}\frac{\xi(v\tau)}{v}\right)_{\tau=0}^{\infty} - \int_0^{\infty}\frac{-1}{v}e^{\frac{-\tau}{v}}\frac{\xi(v\tau)}{v}\right]$$

$$= \frac{1}{v}\left[\left(-\frac{\xi(0)}{v}\right) + \frac{1}{v}\frac{1}{v}\int_0^{\infty}e^{\frac{-\tau}{v}}\xi(v\tau)\right]$$

$$= \frac{1}{v}\left[\left(-\frac{\xi(0)}{v}\right) + \frac{1}{v}\mathcal{A}(\xi(\tau))\right]$$

Thus, we have, $\mathcal{A}\left[\frac{d\xi(\tau)}{dt}\right] = \frac{\mathcal{A}(\xi(\tau))}{v^2} - \frac{\xi(0)}{v^2}$

Alternative Proof: By using Lemma 2, we can prove it as below:

$$\mathcal{L}\left(\frac{d\xi(\tau)}{d\tau}\right) = s\xi(s) - \xi(0) = F(s)$$

Now,

$$\mathcal{A}\left[\frac{d\xi(\tau)}{d\tau}\right] = \frac{1}{v^2}F\left(\frac{1}{v^2}\right)$$

$$= \frac{1}{v^2}\left(\frac{1}{v^2}\xi\left(\frac{1}{v^2}\right) - \xi(0)\right)$$

$$= \frac{1}{v^2}\left(\left(\bar{\xi}(v^2)\right) - \xi(0)\right)$$

$$= \frac{1}{v^2}\left(\mathcal{A}(\xi(\tau)) - \xi(0)\right)$$

Hence proved.

Theorem 3: The $\mathcal{A}$ transform of $\frac{d^2\xi(\tau)}{d\tau^2}$ i.e., the 2^{nd} derivative of $\xi(\tau)$ is given as

$$\mathcal{A}\left[\frac{d^2\xi(\tau)}{d\tau^2}\right] = \frac{\mathcal{A}(\xi(\tau))}{v^4} - \frac{\xi(0)}{v^4} - \frac{1}{v^2}\frac{d\xi(\tau)}{d\tau}|_{t=0}$$

Proof. Proceed similarly as explained in Theorem 2.

Theorem 4: The $\mathcal{A}$ transform of $\xi^n(t)$ i.e. the n^{th} *derivative*, $n \geq 1$ of $\xi(t)$ is given as:

$$\mathcal{A}[\xi^n(\tau)] = v^{-2n}\left(\mathcal{A}(\xi(\tau)) - \sum_{k=0}^{n-1} v^{2k}\,\xi^k(0^+)\right), -1 < n-1 < \beta \leq n \qquad (7)$$

Proof: Follow the mathematical induction.

Theorem 5: If $\bar{f}(v^2)$ is the $\mathcal{A}$ transform of f(t), then $\int_0^t f(\tau)d\tau$ has an $\mathcal{A}$ transform, which is given by

$$\mathcal{A}\left[\int_0^t f(\tau)d\tau\right] = v^2\mathcal{A}(f(t)) = v^2\bar{f}(v^2)$$

Proof: By definition

$$\mathcal{A}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{v}\int_0^\infty \left(\int_0^t f(v\tau)d\tau\right)e^{\frac{-t}{v}}dt$$

$$= \frac{1}{v}\left[\left(e^{\frac{-t}{v}}(-v)\int_0^t f(v\tau)d\tau\right)|_{t=0}^{t=\infty} + u\int_0^\infty e^{\frac{-t}{v}}f(vt)dt\right]$$

$$= v^2\left(\frac{1}{v}e^{\frac{-t}{v}}f(vt)dt\right)$$

$$= v^2\mathcal{A}(f(t)) = v^2\bar{f}(v^2)$$

Hence proved.

Theorem 6: If $\bar{\xi}(v^2)$, $\bar{\eta}(v^2)$is A transform of f(τ), g(τ) respectively then the Convolution is provided by $\mathcal{A}[(\xi * \eta)] = v^2\mathcal{A}[\xi(\tau)]\mathcal{A}[\eta(\tau)]$.

Proof: From the Convolution property of the LT, we have

$$\mathcal{L}[(f * g)] = F(s) * G(s)$$

From the relationship between $\mathcal{A}$ and Laplace, i.e, Lemma 2, we have

$$\mathcal{A}[(\xi * \eta)] = \frac{1}{v^2}F\left(\frac{1}{v^2}\right) * G\left(\frac{1}{v^2}\right)$$

$$= v^2\frac{1}{v^2}F\left(\frac{1}{v^2}\right)\frac{1}{v^2}G\left(\frac{1}{v^2}\right)$$

$$= v^2\bar{\xi}(v^2)\bar{\eta}(v^2)$$

$$= v^2\mathcal{A}[\xi(\tau)]\mathcal{A}[\eta(\tau)]$$

III. Preliminaries on FC

Definition 1: A real function ξ(τ) τ >0 be in space C_{μ}, $\mu \in \boldsymbol{R}\ if$ there exists a real number n (≥ μ), such that ξ (τ) = $\tau^{n}\xi_{1}(\tau)$, where ξ(τ) ∈ C(0,∞) and belong to space C^{k} if $\xi^{k} \in C_{\mu}$, k ∈ N [XV].

Definition 2: The fractional derivative of ξ(τ) ∈ C_{-1}^{μ}[XV, XVII] of order γ defined in Caputo's sense as:

$$ {}_{0}^{C}D_{t}^{\gamma}\xi(\tau) = \frac{1}{\Gamma(n-\gamma)}\int_{0}^{\tau}(\tau-r)^{n-\gamma-1}\,\xi^{n}(r)dr, n-1<\gamma\le n, n\in N, $$

And in the Riemann-Liouvilles sense [XVII] as:

$$ {}_{0}^{RL}D_{t}^{\gamma}\xi(\tau) = \frac{1}{\Gamma(n-\gamma)}\frac{d^{n}}{dt^{n}}\left(\int_{0}^{\tau}(\tau-r)^{n-\gamma-1}\,\xi(r)dr\right), n-1<\gamma\le n, n\in N $$

The following describes the relationship between Reimann-Liouvilles and Caputo derivatives:

$$ {}_{0}^{RL}D_{t}^{\gamma}\xi(\tau) = {}_{0}^{C}D_{t}^{\gamma}\xi(\tau) + \sum_{k=0}^{n-1}\frac{\tau^{k-\gamma}}{\Gamma(k-\gamma+1)}\,\xi^{k}(0^{+}) $$

Definition 3: The LT of the Caputo derivative of order γ (n-1< ≤ n) of ξ (t) [XIV], is provided as

$$ \mathcal{L}\left[{}_{0}^{C}D_{t}^{\gamma}\xi(\tau)\right] = s^{\gamma}\mathcal{L}\big(\xi(\tau)\big) - \sum_{k=0}^{n-1}s^{\gamma-k-1}\,\xi^{k}(0^{+}), -1<n-1<\gamma\le n $$

For the Reimann-Liouvilles derivative [XIV]

$$ \mathcal{L}\left[{}_{0}^{RL}D_{t}^{\gamma}\xi(\tau)\right] = s^{\gamma}\mathcal{L}\big(\xi(\tau)\big) - \sum_{k=0}^{n-1}s^{\gamma}\,D^{\gamma-k-1}\xi(t)|_{t=0}, -1<n-1<\gamma\le n $$

Definition 4: The definition of the two-parameter Mittag-Leffler function [XVII] is

$$ \mathcal{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, z, \beta \in C, Real(\alpha) > 0. $$

Theorem 7: If $\bar{\xi}(v^{2})$ is $\mathcal{A}$ transform of ξ(t), then the $\mathcal{A}$ transform of the Riemann-Liouville integral, i.e $I_{0,t}^{n-\beta}$(see [VIII]) is given as:

$$ \mathcal{A}\left[I_{0,t}^{n-\beta}\xi(t)\right] = v^{2(n-\beta)}\mathcal{A}\big(\xi(t)\big) = v^{2(n-\beta)}\bar{\xi}(v^{2}), -1<n-1<\beta\le n $$

Proof: Let $I_{0,t}^{n-\beta}\xi(t) = \frac{1}{\Gamma(n-\beta)}\int_{0}^{t}(t-\tau)^{n-\beta-1}\,\xi(t)d\tau$

Applying the $\mathcal{A}$ transform on both sides, we get

$$\mathcal{A}\left[I_{0,t}^{n-\beta}\xi(t)\right] = \mathcal{A}\left(\frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1}\,\xi(t)d\tau\right)$$

$$= \left(\frac{1}{\Gamma(n-\beta)}\right)\mathcal{A}\left[t^{n-\beta-1}\right] * \mathcal{A}\xi(t)$$

$$= \frac{1}{\Gamma(n-\beta)} v^{2(n-\beta-1)}\big(\Gamma(n-\beta)\big)v^2 A(v^2)$$

$$= v^{2(n-\beta)}\bar{\xi}(v^2)$$

which completes the proof.

Theorem 8: The $\mathcal{A}$ transform of the Caputo derivative of ξ(t) is given as:

$$\mathcal{A}\left[{}_0^C D_t^\beta \xi(t)\right] = v^{-2\beta}\left(\mathcal{A}\big(\xi(t)\big) - \textstyle\sum_{k=0}^{n-1} v^{2k}\,\xi^k(0^+)\right), -1 < n-1 < \beta \le n \quad (8)$$

Proof: Using definition 2, we have

$${}_0^C D_t^\beta \xi(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1}\,\xi^n(\tau)d\tau$$

Applying the $\mathcal{A}$ transform, we get

$$\mathcal{A}\left[{}_0^C D_t^\beta \xi(t)\right] = \mathcal{A}\left(\frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1}\,\xi^n(\tau)d\tau\right)$$

$$\mathcal{A}\left[{}_0^C D_t^\beta \xi(t)\right] = \frac{1}{\Gamma(n-\beta)}\mathcal{A}\left(t^{n-\beta-1}\right) * \mathcal{A}\left(\xi^n(t)\right)$$

Now, using **Theorems 3 and 4,** we get the required result.

Theorem 9: The $\mathcal{A}$ transform of Riemann-Liouville's derivative of ξ(t) is given as:

$$\mathcal{A}\left[{}_0^{RL} D_t^\beta \xi(\tau)\right] = v^{-2\beta}\mathcal{A}\big(\xi(\tau)\big) - \sum_{k=0}^{n-1} v^{-2(k+1)}\left[D^{\beta-k-1}\xi(\tau)|_{\tau=0}\right] - 1 < n-1 \le n$$

Proof: By definition 2, we have

$$\left[{}_0^{RL} D_t^\beta \xi(\tau)\right] = \frac{d^n}{dt^n} I^{n-\beta}\xi(\tau)$$

Let $\quad \eta(\tau) = I^{n-\beta}\xi(\tau)$

Now applying A transform to the above, we get

$$\mathcal{A}\left[{}_0^{RL} D_t^\beta \eta(\tau)\right] = \mathcal{A}\left[\frac{d^n}{dt^n}\eta(\tau)\right]$$

$$= \left(\frac{\mathcal{A}(\eta(\tau))}{v^{2n}} - \sum_{k=0}^{n-1} \frac{\eta^k(0^+)}{v^{2(n-k)}} \right)$$

$$= \left(\frac{\mathcal{A}\left(I^{n-\beta}\xi(\tau)\right)}{v^{2n}} - \sum_{k=0}^{n-1} \frac{\eta^{n-k-1}(0^+)}{v^{2(k+1)}} \right)$$

$$= \left(v^{2(n-\beta)} \frac{\mathcal{A}(\xi(\tau))}{v^{2n}} - \sum_{k=0}^{n-1} v^{-2(k+1)}\, D^{n-k-1} I^{n-\beta} \xi(t)|_{t=0} \right)$$

$$= \left(v^{2(n-\beta)} \frac{\mathcal{A}(\xi(\tau))}{v^{2n}} - \sum_{k=0}^{n-1} v^{-2(k+1)}\, D^{\beta-k-1} \xi(t)|_{t=0} \right)$$

$$= v^{-2\beta}\mathcal{A}(\xi(\tau)) - \sum_{k=0}^{n-1} v^{-2(k+1)}\left[D^{\beta-k-1}\xi(t)|_{t=0}\right]$$

Hence proved.

Theorem 10: $\mathcal{A}$ transform of the Mittag-Leffler function, i.e. $\tau^{\beta-1}\mathcal{E}_{\alpha,\beta}(z)(\lambda\tau^{\alpha})$

$$\mathcal{A}\left[\tau^{\beta-1}\mathcal{E}_{\alpha,\beta}(z)(\lambda\tau^{\alpha})\right] = \frac{v^{2(\beta-1)}}{1-\lambda v^{2\alpha}} \qquad (9)$$

Proof: Expanding the Mittag-Leffler function and utilizing **Theorem 3.**

IV. Applications to FDE

Example 1: Consider the following Caputo fractional IVP

$${}_{0}^{C}D_{\tau}^{\beta}\xi(\tau) + a\xi(\tau) = 0, \xi(0) = c, 0 < \beta \leq 1 \qquad (10)$$

Applying $\mathcal{A}$ transform to (10), we get,

$$\mathcal{A}\left[{}_{0}^{C}D_{t}^{\beta}\xi(\tau) + a\xi(\tau)\right] = 0$$

$$\mathcal{A}\left[{}_{0}^{C}D_{t}^{\beta}\xi(\tau)\right] + \mathcal{A}[a\xi(\tau)] = 0$$

$$v^{-2\beta}\left(\mathcal{A}(\xi(\tau)) - \xi(0) + \mathcal{A}[a\xi(\tau)] = 0\right.$$

$$\bar{\xi}(v^2) = c - v^{2\alpha}\bar{\xi}(v^2)$$

$$\bar{\xi}(v^2) = \frac{c}{1 + av^{2\alpha}}$$

Applying the inverse $\mathcal{A}$ transform, we get

$$\mathcal{A}^{-1}\left[\bar{\xi}(v^2)\right] = \mathcal{A}^{-1}\left[\frac{c}{1 + av^{2\alpha}}\right]$$

$$\xi(\tau) = c\mathcal{E}_{\alpha,1}(-a\tau^{\alpha})$$

(By using **Theorem 10.**)

Example 2. Next, consider the nonhomogeneous fractional ordinary differential equation in the Caputo sense as

$$ {}_{0}^{C}D_{t}^{\alpha}\xi(\tau)+\xi(\tau)=\frac{2}{\Gamma(3-\alpha)}\tau^{2-\alpha}-\frac{1}{\Gamma(2-\alpha)}\tau^{1-\alpha}+\tau^{2}-\tau, $$

$$ \xi(0)=0,\tau>0,0<\alpha\leq 1 \tag{11} $$

Applying the $\mathcal{A}$ transform to both sides of (11), we get

$$ \mathcal{A}\left[{}_{0}^{C}D_{t}^{\alpha}\xi(\tau)+\xi(\tau)\right]=\mathcal{A}\left[\frac{2}{\Gamma(3-\alpha)}\tau^{2-\alpha}\right]-\mathcal{A}\left[\frac{1}{\Gamma(2-\alpha)}\tau^{1-\alpha}\right]+\mathcal{A}[\tau^{2}]-\mathcal{A}[\tau] $$

$$ v^{-2\alpha}\left(\left(\bar{\xi}(v^{2})\right)-\xi(0)\right)+\bar{\xi}(v^{2})=2v^{4-2\alpha}-v^{2-2\alpha}+2v^{4}-v^{2} $$

$$ (1+v^{2\alpha})\bar{\xi}(v^{2})=2v^{4}-v^{2}+2v^{4+2\alpha}-v^{2+2\alpha} $$

$$ \bar{\xi}(v^{2})=\frac{2v^{4}-v^{2}+2v^{4+2\alpha}-v^{2+2\alpha}}{(1+v^{2\alpha})} $$

$$ =2v^{4}-v^{2} $$

After using the inverse $\mathcal{A}$ transform, the outcome is as follows:

$$ \xi(\tau)=\tau^{2}-\tau $$

This matches the solution obtained in [VII].

Example 3: Consider the following Riemann-Liouville's FDE

$$ {}_{0}^{RL}D_{t}^{\frac{1}{2}}\xi(\tau)+a\xi(\tau)=0,\ \mathcal{D}_{\tau}^{-\frac{1}{2}}\xi(\tau)|_{\tau=0}=c \tag{12} $$

Applying $\mathcal{A}$ transform, we get,

$$ \mathcal{A}\left[{}_{0}^{RL}D_{t}^{\frac{1}{2}}\xi(\tau)+a\xi(\tau)\right]=0 $$

$$ \mathcal{A}\left[{}_{0}^{RL}D_{t}^{\frac{1}{2}}\xi(\tau)\right]+\mathcal{A}[a\xi(\tau)]=0 $$

$$ v^{-1}(\mathcal{A}(\xi(\tau))\ -\ v^{-2}\mathcal{D}_{t}^{-\frac{1}{2}}\xi(0)\ +\ \mathcal{A}[a\xi(\tau)]=0 $$

$$ vv^{-1}\bar{\xi}(v^{2})-v^{-2}c+a\bar{\xi}(v^{2})=0 $$

$$ (1+av)\bar{\xi}(v^{2})=\frac{c}{v} $$

$$ \bar{\xi}(v^{2})=\frac{c}{v(1+av)} $$

Applying the Inverse $\mathcal{A}$ transform, we get

$$\mathcal{A}^{-1}[\bar{\xi}(v^2)] = \mathcal{A}^{-1}\left[\frac{c}{1+av^{2\alpha}}\right]$$

$$\xi(\tau) = c\tau^{-\frac{1}{2}}\mathcal{E}_{\frac{1}{2},\frac{1}{2}}\left(-a\tau^{-\frac{1}{2}}\right)$$

(By using **Theorem 10.**)

Example 4. Next, consider the Bagley-Torvik equation [V] as follows

$$\mathcal{D}_\tau^2\xi(\tau) + \mathcal{D}_\tau^{\frac{3}{2}}\xi(\tau) + \xi(\tau) = \tau + 1, \xi(0) = \xi(0)' = 1, \tau > 0, \tag{13}$$

Applying the $\mathcal{A}$ transform to both sides of (13), we get

$$\mathcal{A}\left[\mathcal{D}_\tau^2\xi(\tau) + \mathcal{A}\left[\mathcal{D}_\tau^{\frac{3}{2}}\xi(\tau)\right] + \mathcal{A}\xi(\tau)\right] = \mathcal{A}[\tau] + \mathcal{A}[1]$$

$$\left(\frac{\bar{g}(v^2)}{v^4} - \frac{g(0)}{v^4} - \frac{\xi'(0)}{v^2}\right) + \left(\frac{\bar{g}(v^2)}{v^3} - \frac{g(0)}{v^3} - \frac{\xi'(0)}{v}\right) + \bar{\xi}(v^2) = v^2 + 1$$

$$\bar{\xi}(v^2) - 1 - v^2 + v\bar{\xi}(v^2) - v - v^3 + v^4\bar{\xi}(v^2) = v^6 + v^4$$

$$\bar{\xi}(v^2)(1 + v + v^4) = v^6 + v^4 + 1 + v + v^2 + v^3$$

$$\bar{\xi}(v^2) = \frac{v^6 + v^4 + 1 + v + v^2 + v^3}{(1 + v + v^4)}$$

$$= v^2 + 1$$

On applying the inverse $\mathcal{A}$ transform, we get the solution as

$$\xi(\tau) = \tau + 1$$

which is the exact solution and matches the solution of the same problem obtained by the Shehu transform [V].

Example 5. Next, consider the following equation [XIV]

$${}^{RL}_{0}D_\tau^{\frac{1}{2}}\xi(\tau) - \xi(\tau) = -1,\ \mathcal{D}_\tau^{-\frac{1}{2}}\xi(0) = 0,\ \tau > 0, \tag{14}$$

Applying $\mathcal{A}$ transform to both sides of (14), we get,

$$\mathcal{A}\left[{}^{RL}_{0}D_\tau^{\frac{1}{2}}\xi(\tau)\right] - \mathcal{A}[\xi(\tau)] = -\mathcal{A}(-1)$$

$$\left(\frac{\bar{\xi}(v^2)}{v} - \mathcal{D}_\tau^{-\frac{1}{2}}\xi(0)\right) - \bar{\xi}(v^2) = -1$$

$$\bar{\xi}(v^2)(1 - v) = -v$$

$$\bar{\xi}(v^2) = \frac{-v}{(1 - v)}$$

$$= \frac{v}{(v-1)}$$

$$= \frac{v(1+v)}{(v^2-1)}$$

$$= \frac{(v+1-1+v^2)}{(v^2-1)}$$

$$= 1 - \frac{1}{(1-v^2)} - \frac{v}{(1-v^2)}$$

After applying the inverse $\mathcal{A}$ transform, we get the solution as

$$\xi(\tau) = 1 - e^{\tau} - e^{\tau}\, erf\sqrt{\tau}$$

Which is the exact solution to the given equation and matches the solution obtained in [XIV]; here, erf is the error function.

V. Conclusion and Future Work

This paper concentrates on the $\mathcal{A}$ transform as an effective method for analytically resolving fractional differential equations. This transformative technique is very successful and particularly developed for simplifying the solution process for specific types of problems. Analytical solutions for fractional differential operators of the Caputo and Riemann-Liouville categories are first shown in this paper. The $\mathcal{A}$ transform performs admirably in the domain of FDE. Its primary advantage over other transformations is that it handles fractional-order operators with such skill. This innovation's transformational power is demonstrated by the several challenges it has successfully addressed, including the challenging Bagley-Torvik equation. Looking ahead, it is envisaged that the direction of future research efforts will further explore the intricacies of the $\mathcal{A}$ transform. Furthermore, this new method's revolutionary power will likely allow it to solve differential equations with integer and non-integer orders. To solve particularly nonlinear differential equations of both integer and non-integer orders, this transformation can be used with iteration techniques like ADM, VIM, NIM, and HPM.

Conflict of Interest:

Regarding this paper, there were no relevant conflicts of interest.

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