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# SEMI-ANALYTICAL METHOD FOR SOLVING ONE DIMENSIONAL HEAT EQUATION

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**Abstract**

*The Laplace Transform method and variational iterative approach are combined to create a new semi-analytical methodology that is used in this research to solve one-dimensional heat equations. To illustrate the effectiveness and precision of the suggested approach, numerical results are provided.*

**Keywords:** Variational iterative method, one-dimensional Heat equation, Numerical examples, Laplace transform.

## I. Introduction

Joseph Fourier initially presented the idea of heat equations in 1822. It occurs in many scientific and technical applications when we need to describe the flow of a quantity, like heat, through a certain area. Here is a description of the 1-D heat equation:

$$\frac{\partial u(\alpha,t)}{\partial t} = c^2 \frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}$$

Several applications in the sciences and engineering may be resolved using the Laplace transform approach. The variational iterative method is also a widely used numerical technique to resolve differential equations. Differential equations may have an accurate solution, which can be obtained using variational iterations.

A variety of mathematical techniques were developed to solve heat equations in one dimension. The finite difference method has been used in [II] to solve heat equations in 2-D. The 2-D heat equations' Chebyshev series solution was presented in [IX]. In [VII], a method combining the Finite Difference approach to solve 2-D heat equations with a collocation method was devised. Heat equations in two dimensions are solved using the radial basis function approach in [VIII]. The approach of variational iteration was presented in [IV] to solve nonlinear equations. This paper discusses a few more approaches to solving Heat Equations [VI-XII]. [I, III, V, VI, X, XI, XII].

## II. Linearity Property of the Laplace Transform Method

Let v(t), w(t) be the functions of t which are defined for all positive t values. Then

$$L\{a.v(t) + b.w(t)\} = a.L\{v(t)\} + b.L\{w(t)\}$$

where a, b are arbitrary constants.

## III. Laplace transform for differentiation

Assume that for any positive value of t, u(t) is a function of t. Consequently, the nth derivative of the function u(t), the Laplace transform is

$$L\left[\frac{D^N(U(T))}{DT^N}\right] = P^N\bar{U}(P) - P^{N-1}U(0) - P^{N-2}U'(0) - P^{N-3}U''^{(0)} - \cdots - PU^{(N-2)}(0) - U^{(N-1)}(0)$$

where $\bar{u}(p) = L\{u(t)\}$.

## IV. Linearity Property of Inverse Laplace Transform

For any positive value of t, let v(t), w(t) be two functions of t. Let $\bar{v}(p)$ and $\bar{w}(p)$ be functions of s such that $\bar{v}(p) = L\{v(t)\}$ and $\bar{w}(p) = L\{w(t)\}$. Then

$$L^{-1}\{c.\bar{v}(p) + d.\bar{w}(p)\} = c.L^{-1}\{\bar{v}(p)\} + d.L\{\bar{w}(p)\} = c.v(t) + d.w(t)$$

where the constants c and d are arbitrary.

**Numerical Examples**
**Example 1:**

This section provides examples to illustrate the efficacy of the proposed semi-analytical method.

**Example 1:** Consider the following 1-D heat equation

$$\frac{\partial u(\alpha,t)}{\partial t} = \frac{\partial^2 u(\alpha,t)}{\partial \alpha^2} \tag{1}$$

the initial conditions

$$u(\alpha, 0) = \sin\alpha$$

By L.T. of (1),

$$L\left\{\frac{\partial u(\alpha,t)}{\partial t}\right\} = L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\} \tag{2}$$

This implies

$$pL\{u(\alpha, t)\} - u(\alpha, 0) = L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}$$

By initial conditions, we have

$$pL\{u(\alpha, t)\} = \sin\alpha + L\{\frac{\partial^2 u(\alpha, t)}{\partial \alpha^2}\}$$

$$L\{u(\alpha,t)\} = \frac{\sin\alpha}{p} + \frac{1}{p}L\{\frac{\partial^2 u(\alpha,t)}{\partial\alpha^2}\} \quad (3)$$

By inverse L.T. of (3),

$$u = \sin\alpha \ + L^{-1}\left[\frac{1}{p}L\{\frac{\partial^2 u(\alpha,t)}{\partial\alpha^2}\}\ \right] \quad (4)$$

Using the iteration method, from (4), we obtain

$$u_{m+1} = \sin\alpha + L^{-1}\left[\frac{1}{p}L\left\{L\{\frac{\partial^2 u_m}{\partial\alpha^2}\}\ \right\}\right] \quad (5)$$

From (5), we obtain

$$u_0 = \sin\alpha$$
$$u_1 = \sin\alpha\,(1-t)$$
$$u_2 = \sin\alpha\,\left(1 - t + \frac{(t)^2}{2!}\right)$$
$$u_3 = \sin\alpha\,\left(1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!}\right)$$

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$$u_m = \sin\alpha\left(1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!} + \cdots + \frac{(-1)^m (t)^m}{m!}\right)$$

The solution is

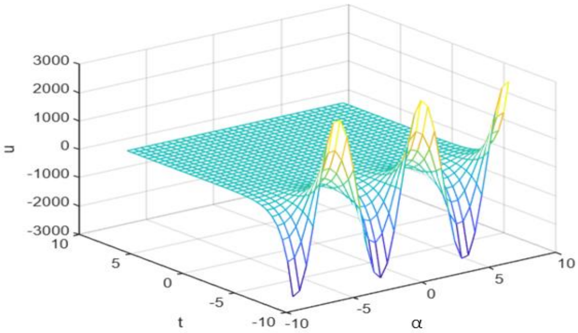
$$u = \lim_{n\to\infty} u_m$$

After simplification, we obtain

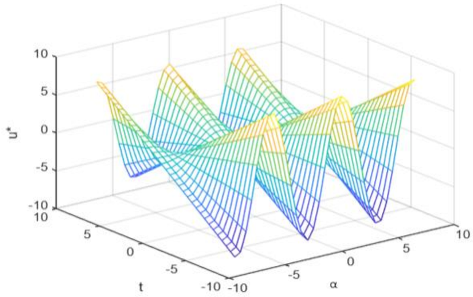
$$u = \sin\alpha\,\left(1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!} \ldots\right)$$
$$u = \sin\alpha\, e^{-t} \quad (6)$$

**Table 1: Absolute Error of Exact, Approximate solution at α = 0.5 (up to 3rd iteration)**

| t | u | u* | \|u-u*\| |
|---|---|---|---|
| 0 | 0.479425538604203 | 0.479425538604203 | 0 |
| 0.2 | 0.392520432266236 | 0.392489707603974 | 3.0725e-05 |
| 0.4 | 0.321368549107831 | 0.320895493839080 | 4.7306e-04 |
| 0.6 | 0.263114314226633 | 0.260807493000686 | 2.3068e-03 |
| 0.8 | 0.215419780632368 | 0.208390300779960 | 7.0295e-03 |
| 1.0 | 0.176370799225032 | 0.159808512868068 | 1.6562e-02 |
| 1.2 | 0.144400197270476 | 0.111226724956175 | 3.3173e-02 |
| 1.4 | 0.118224882255866 | 0.058809532735449 | 5.9415e-02 |
| 1.6 | 0.096794346881901 | -0.001278468102945 | 9.8073e-02 |
| 1.8 | 0.079248508516310 | -0.072872681867839 | 1.5212e-01 |
| 2.0 | 0.064883191057865 | -0.159808512868068 | 2.2469e-01 |




**Pic. 1.** Graph of Exact Solution u




**Pic. 2.** Graph of Approximate solution u* (up to 3rd iteration)

**Example 2:** Consider the one-dimensional Heat equation

$$\frac{\partial u(\alpha,t)}{\partial t} = \frac{\partial^2 u(\alpha,t)}{\partial \alpha^2} \tag{7}$$

where

$u(\alpha, 0) = e^{\alpha}$

By L.T. of (7),

$$L\left\{\frac{\partial u(\alpha,t)}{\partial t}\right\} = L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}$$

This implies

$$pL\{u(\alpha,t)\} - u(\alpha,0) = L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}$$

by initial conditions,

$$pL\{u(\alpha,t)\} = e^{\chi} + L\{\nabla^2 u(\alpha,t)\}$$

Divide by p, we obtain

$$L\{u(\alpha,t)\} = \frac{e^{\alpha}}{p} + \frac{1}{p}L\{\nabla^2 u(\alpha,t)\} \tag{8}$$

By inverse L.T. of (8),

$$u = e^{\alpha} + L^{-1}\left[\frac{1}{p}L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}\ \right] \tag{9}$$

Using the iteration method, from (9), we obtain

$$u_{m+1} = e^{\alpha} + L^{-1}\left[\frac{1}{p} L\left\{L\{\frac{\partial^2 u_m}{\partial \alpha^2}\} \right\}\right] \tag{10}$$

From (10), we obtain

$$u_0 = e^{\alpha}$$
$$u_1 = e^{\alpha}(1+t)$$
$$u_2 = e^{\alpha}\left(1+t+\frac{(t)^2}{2!}\right)$$
$$u_3 = e^{\alpha}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}\right)$$
$$u_4 = e^{\alpha}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\frac{(t)^4}{4!}\right)$$
$$u_5 = e^{\alpha}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\frac{(t)^4}{4!}+\frac{(t)^5}{5!}\right)$$

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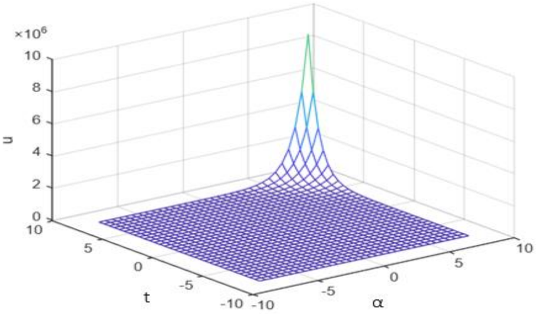
$$u_m = e^{\alpha}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots+\frac{(t)^m}{m!}\right)$$

The solution is

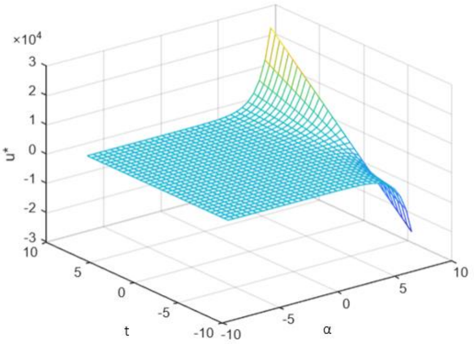
$$u = \lim_{n\to\infty} u_m$$
$$u = e^{\alpha}\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots\right)$$
$$u = e^{\alpha}(e^{t})$$
$$u = e^{\alpha+t}$$

**Table 2: Absolute Error of Exact and Approximate solution at α = 0.5 (up to 5th iteration)**

| t | u | u* | \|u-u*\| |
|---|---|---|---|
| 0 | 1.648721270700128 | 1.648721270700128 | 0 |
| 0.2 | 2.013752707470477 | 2.013752556623192 | 1.5085e-07 |
| 0.4 | 2.459603111156950 | 2.459593167760838 | 9.9434e-06 |
| 0.6 | 3.004166023946433 | 3.004049293836627 | 1.1673e-04 |
| 0.8 | 3.669296667619244 | 3.668620260220490 | 6.7641e-04 |
| 1.0 | 4.481689070338065 | 4.479026118735348 | 2.6630e-03 |
| 1.2 | 5.473947391727199 | 5.465735238463740 | 8.2122e-03 |
| 1.4 | 6.685894442279269 | 6.664491896554444 | 2.1403e-02 |
| 1.6 | 8.166169912567650 | 8.116843869029104 | 4.9326e-02 |
| 1.8 | 9.974182454814718 | 9.870670021588852 | 1.0351e-01 |
| 2.0 | 12.182493960703473 | 11.980707900420931 | 2.0179e-01 |




**Pic. 3.** Behaviour of Exact solution u




**Pic. 4.** Behaviour of Approximate sol u* to 5th iteration

***Example 3:*** Consider the one-dimensional Heat equation

$$\frac{\partial u(\alpha,t)}{\partial t} = \frac{\partial^2 u(\alpha,t)}{\partial \alpha^2} \tag{11}$$

where

$$u(\alpha,0) = sinh\alpha$$

By L.T. of (11),

$$L\left\{\frac{\partial u(\alpha,t)}{\partial t}\right\} = L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}$$

This implies

$$pL\{u(\alpha,t)\} - u(\alpha,0) = L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}$$

Applying initial conditions,

$$pL\{u(\alpha,t)\} = sinh\alpha + L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}$$

$$L\{u(\alpha,t)\} = \frac{sinh\alpha}{p} + \frac{1}{p}L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\} \tag{12}$$

Applying inverse T. of (12),

$$u = \sinh\alpha + L^{-1}\left[\frac{1}{p}L\{\frac{\partial^2 u(\alpha,t)}{\partial \alpha^2}\}\ \right] \tag{13}$$

By the iteration method, from (13),

$$u_{m+1} = \sinh\alpha + L^{-1}\left[\frac{1}{p} L\left\{L\{\frac{\partial^2 u_m}{\partial \alpha^2}\}\right\}\right] \tag{14}$$

From (14), we obtain

$$u_0 = sinh\alpha$$

$$u_1 = sinh\alpha(1+t)$$

$$u_2 = sinh\alpha\left(1+t+\frac{(t)^2}{2!}\right)$$

$$u_3 = sinh\alpha\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}\right)$$

$$u_4 = \sinh\alpha\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\frac{(t)^4}{4!}\right)$$

$$u_5 = \sinh\alpha\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\frac{(t)^4}{4!}+\frac{(t)^5}{5!}\right)$$

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$$u_m = sinh\alpha\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots+\frac{(t)^m}{m!}\right)$$

The solution is obtained as

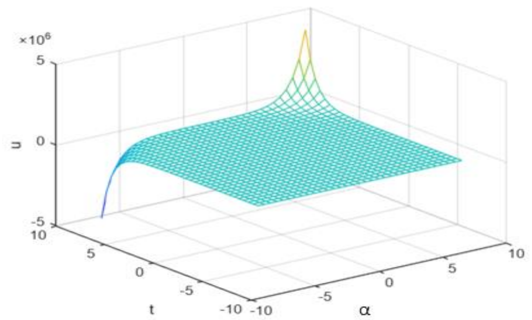
$$u = \lim_{n\to\infty} u_m$$

$$u = sinh\alpha\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots\right)$$

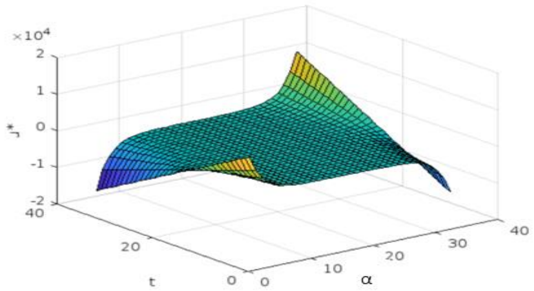
$$u = \sinh\alpha\,(e^t)$$

**Table 3: Absolute Error of Exact and Approximate solution at α = 0.5 (up to 5th iteration)**

| t | u | u* | \|u-u*\| |
|---|---|---|---|
| 0 | 0.521095305493747 | 0.521095305493747 | 0 |
| 0.2 | 0.636467243394379 | 0.636467195717544 | 4.7677e-08 |
| 0.4 | 0.777382846560495 | 0.777379703847942 | 3.1427e-06 |
| 0.6 | 0.949497552935393 | 0.949460659184271 | 3.6894e-05 |
| 0.8 | 1.159718930021621 | 1.159505144510173 | 2.1379e-04 |
| 1.0 | 1.416483899818968 | 1.415642246591347 | 8.4165e-04 |
| 1.2 | 1.730097342128362 | 1.727501806673320 | 2.5955e-03 |
| 1.4 | 2.113145665561160 | 2.106381170979195 | 6.7645e-03 |
| 1.6 | 2.581001944310608 | 2.565411941207415 | 1.5590e-02 |
| 1.8 | 3.152442893597739 | 3.119726725029518 | 3.2716e-02 |
| 2.0 | 3.850402445182704 | 3.786625886587898 | 6.3777e-02 |




**Pic. 5.** Behaviour of Exact Solution u




**Pic. 6.** Behaviour of Approximate sol. u*

## V. Conclusion

The results of the solved numerical examples show that the novel semi-analytic approach, which comprises the Laplace transform with a modified variational iterative approach, is an efficient mathematical approach for solving one-dimensional heat equations. In the future, linear and non-linear heat equations in two and three dimensions may be solved using the suggested mathematical approach.

## Conflict of Interest

The authors declare that there is no conflict of interest regarding this article.

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