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# NUMERICAL & ANALYTICAL SOLUTION OF (2+1)-DIMENSIONAL WAVE EQUATION BY NEW LAPLACE VARIATIONAL ITERATION METHOD

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## Abstract

*This study demonstrates a semi-analytic method for solving two-dimensional wave equations that arise in several scientific and engineering fields by combining the Laplace Transform with a corrected variational iteration technique. A few numerical examples are provided to illustrate the correctness of the suggested method.*

**Keywords:** Variational Iterative method, Laplace Transform, Numerical Examples, Two-dimensional wave equation

## I. Introduction

In classical physics, wave phenomena, including mechanical waves, water waves, sound waves, and light waves, are described by the wave equation, a basic second-order hyperbolic PDE. The equation for waves in two dimensions is:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2\nabla^2 u(x,y,t)$$

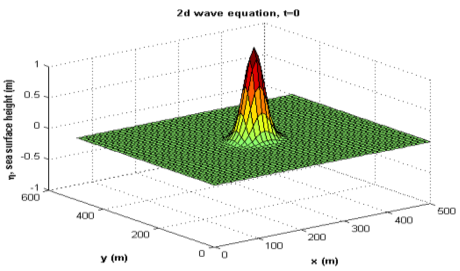


**Fig. 1.** Physical representation of the wave equation at t = 0

In disciplines like fluid dynamics, electromagnetics, and acoustics, such types of equations are commonly encountered.

L.T. can be used to solve P.D.E. mathematically and effectively by converting them into algebraic equations, where convolutions become simple multiplications. Numerous scientific and technical fields have made substantial use of the L.T. approach. The Laplace Transform method has extensive applications for a variety of technical and scientific fields. Similarly, VIM is a widely accepted and effective method for P.D.E. solutions that frequently arise in these fields. This method is capable of handling both linear and nonlinear problems by generating successive approximations that converge to the actual solution if there is one. There are several methods for solving two-dimensional wave equations. For instance, the power series expansion technique has been applied to this class of equations [I]. The Optimal Homotopy Asymptotic Method has also been developed to address such problems [XIX], as well as the New Nonlinear 2-D wave equations have been resolved by utilizing the Homotopy Perturbation Method [II]. Furthermore, delay Differential equations are successfully solved using the variational iteration method [VIII], nonlinear equations [IX], and autonomous ordinary differential equations [VIII]. Other approaches have also been explored to solve two-dimensional wave equations [I, III-VI, XI-XIII, XV-XVIII].

**II. Linearity Property of the Laplace transform:**

For any positive value of t, f(t) and g(t) have been defined. Then

$$L\{a.f(t) + b.g(t)\} = a.L\{f(t)\} + b.L\{g(t)\}$$

a, b are arbitrary constants.

**III. Laplace transform for differentiation:**

Assume that two functions of t, f(t), g(t), have been defined for every positive value of t. Then, L.T. of the $n^{th}$ derivative of f(t) is

$$L\left[\frac{D^N(F(T))}{DT^N}\right] = P^N\overline{F}(P) - P^{N-1}F(0) - P^{N-2}F'(0) - P^{N-3}F''^{(0)} - \cdots - PF^{(N-2)}(0) - F^{(N-1)}(0)$$

where $\overline{f}(p) = L\{f(t)\}$.

**IV. Linearity Property of the Inverse Laplace transform:**

Assume that two functions of t, f(t), g(t), have been defined for every positive value of t. Since $\overline{f}(p)=L\{f(t)\}$ and $\overline{g}(p)=L\{g(t)\}$, let $\overline{f}(p)$, $\overline{g}(p)$ be functions of s.Then

$$L^{-1}\{c.\overline{f}(p) + d.\overline{g}(p)\} = c.L^{-1}\{\overline{f}(p)\} + d.L\{\overline{g}(p)\} = c.f(t) + d.g(t)$$

where c and d are arbitrary constants.

## V. Variational iterative method (VIM)

Multiple problems that arise in various engineering and research areas can be resolved using the well-known VIM technique. The variational iteration approach is useful for handling the nonlinear terms. Consider the differential equations.

$$\mathbf{l}u(\alpha, \beta, t) + \mathbf{n}u(\alpha, \beta, t) = \mathbf{g}(x, \beta, t) \tag{1}$$

Subject to initial conditions

$$u(\alpha, \beta, 0) = \mathbf{h}(\alpha, \beta) \tag{2}$$

where g is a nonhomogeneous term, n is a nonlinear operator, and l is a first-order linear operator. Create a correction functional using the VI approach as

$$u_{m+1} = u_m + \int_0^t \lambda[\mathbf{l}u_m(\alpha, \beta, p) + \mathbf{n}\tilde{u}_m(\alpha, \beta, p) - \mathbf{g}(\alpha, \beta, p)]dp \tag{3}$$

where m stands for the mth approximation and $\lambda$ is a Lagrange's multiplier; $\tilde{u}_m$ is a limited function, meaning that $\delta\tilde{u}_m = 0$. With $\lambda$ and $u_0$, the successive approximation $u_{m+1}$ of the answer u will be found. The solution is

$$u = \lim_{m\to\infty} u_m$$

## VI. New Laplace Variational Iterative Method for Solving Two-Dimensional Wave Equations

Two widely used methods are combined for solving P.D.E. problems: VIM, the Laplace Transform. The following section describes the procedure to solve PDE utilizing the combination of the variational iterative approach, the Laplace transform, as shown below.

Considering l to be a first-order operator $\partial/\partial t$, equation (1) may be rewritten as follows:

$$\frac{\partial}{\partial t}u(\alpha, \beta, t) + \mathbf{n}u(\alpha, \beta, t) = \mathbf{g}(\alpha, \beta, t) \tag{4}$$

Using the Laplace transform on each side of equation (4), we obtain:

$$L\left\{\frac{\partial}{\partial t}u(\alpha, \beta, t)\right\} + L\{\mathbf{n}u(\alpha, \beta, t)\} = L\{\mathbf{g}(\alpha, \beta, t)\} \tag{5}$$

$$pL\{u(\alpha, \beta, t)\} - \mathbf{h}(\alpha, \beta) = L\{\mathbf{g}(\alpha, \beta, t)\} - L\{\mathbf{n}u(\alpha, \beta, t)\} \tag{6}$$

Inverse L.T. is applied to equation (6), then

$$u(\alpha, \beta, t) = \mathbf{G}(\alpha, \beta, t) - L^{-1}\left[\frac{1}{p}L\{\mathbf{n}u(\alpha, \beta, t)\}\right] \tag{7}$$

G is a term that is derived from the source function, initial condition.

Using the corrective functional from VIM,

$$u_{m+1}(\alpha, \beta, t) = \mathbf{G}(\alpha, \beta, t) - L^{-1}\left[\frac{1}{p}L\{\mathbf{n}u_m(\alpha, \beta, t)\}\right] \tag{8}$$

The recently updated correction functional of the L.T. of the variational iteration technique is represented by equation (8), and its solution may be determined by

$$u(\alpha,\beta,t) = \lim_{m\to\infty} u_m(\alpha,\beta,t)$$

## VII. Numerical Examples

This section offers several examples to illustrate the precision and effectiveness of the recommended methodology.

**Example 1:** Examine the following wave equation in two dimensions, based on initial conditions:

$$\frac{\partial^2 u(\alpha,\beta,t)}{\partial t^2} = \nabla^2 u(\alpha,\beta,t) \tag{9}$$

with the initial conditions:

$$u(\alpha,\beta,0) = \sin\pi\alpha \sin\pi\beta \ \text{ and } u_t(\alpha,\beta,0) = 0$$

The following outcome is obtained by applying L. T. to equation (9).

$$L\left\{\frac{\partial^2 u(\alpha,\beta,t)}{\partial t^2}\right\} = L\{\nabla^2 u(\alpha,\beta,t)\} \tag{10}$$

This implies

$$p^2 L\{u(\alpha,\beta,t)\} - s u(\alpha,\beta,0) - u_t(\alpha,\beta,0) = L\{\nabla^2 u(\alpha,\beta,t)\}$$

Applying initial conditions, we obtain

$$p^2 L\{u(\alpha,\beta,t)\} = p(\sin\pi\alpha \sin\pi\beta) + L\{\nabla^2 u(\alpha,\beta,t)\}$$

Divide by $p^2$, we obtain

$$L\{u(\alpha,\beta,t)\} = \frac{\sin\pi\alpha \sin\pi\beta}{p} + \frac{1}{p^2} L\{\nabla^2 u(\alpha,\beta,t)\} \tag{11}$$

we get when we apply L.T. to equation (11).

$$u = \sin\pi\alpha \sin\pi\beta + L^{-1}\left[\frac{1}{p^2} L\{\nabla^2 u(\alpha,\beta,t)\}\right] \tag{12}$$

Applying the iteration method, from (12)

$$u_{m+1} = \sin\pi\alpha \sin\pi\beta + L^{-1}\left[\frac{1}{p^2} L\{\nabla^2 u_m\}\right] \tag{13}$$

From (13),

$$u_0 = \sin\pi\alpha \sin\pi\beta$$

$$u_1 = \sin\pi\alpha \sin\pi\beta\left(1 - \frac{(\sqrt{2}\pi t)^2}{2!}\right)$$

$$u_2 = \sin\pi\alpha \sin\pi\beta\left(1 - \frac{(\sqrt{2}\pi t)^2}{2!} + \frac{(\sqrt{2}\pi t)^4}{4!}\right)$$

$$u_3 = \sin\pi\alpha \sin\pi\beta\left(1 - \frac{(\sqrt{2}\pi t)^2}{2!} + \frac{(\sqrt{2}\pi t)^4}{4!} - \frac{(\sqrt{2}\pi t)^6}{6!}\right)$$

and so on.
The solution is

$$u = \lim_{n \to \infty} u_m$$

After simplification, we get

$$u = \sin\pi\alpha \, \sin\pi\beta \left( 1 - \frac{(\sqrt{2}\pi t)^2}{2!} + \frac{(\sqrt{2}\pi t)^4}{4!} - \frac{(\sqrt{2}\pi t)^6}{6!} + \cdots \right)$$

$$u = \sin\pi\alpha \, \sin\pi\beta \cos(\sqrt{2}\pi t) \tag{14}$$

**Table 1: Exact and Approximate values of u at α = 0.2, β = 0.2**

| t | u | u* | \|u-u*\| |
|---|---|---|---|
| 0 | 0.345491502812526 | 0.345491502812526 | 0 |
| 0.1 | 0.311950080871110 | 0.311950067890842 | 1.2980e-08 |
| 0.2 | 0.217838432443780 | 0.217835131244059 | 3.3012e-06 |
| 0.3 | 0.081429878198664 | 0.081346190037885 | 8.3688e-05 |
| 0.4 | -0.070789622950009 | -0.071612951081848 | 8.2333e-04 |
| 0.5 | -0.209264157330407 | -0.214077215934705 | 4.8131e-03 |
| 0.6 | -0.307106622087704 | -0.327320507129378 | 2.0214e-02 |
| 0.7 | -0.345319297350374 | -0.412813170765623 | 6.7494e-02 |
| 0.8 | -0.316482565246130 | -0.506836675514405 | 1.9035e-01 |
| 0.9 | -0.226195550448350 | -0.697755506076258 | 4.7156e-01 |
| 1.0 | -0.091988958253752 | -1.145946271017842 | 1.0540e+00 |

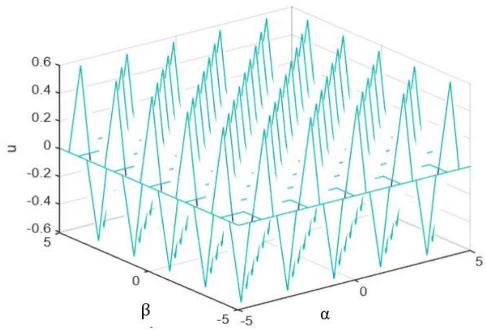


**Fig. 2.** Physical behavior of the solution at t = 0.5

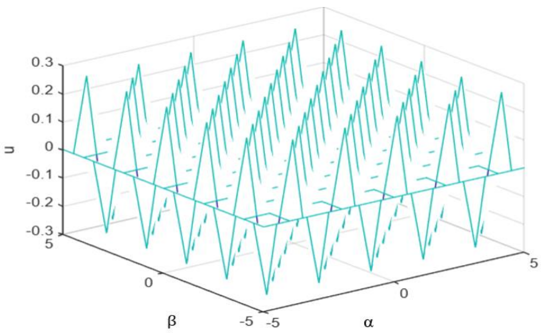


**Fig. 3.** Physical behavior of the solution at t = 1

**Example 2:** Examine the following (2+1)-dimensional wave equation.

$$\frac{\partial^2 u(\alpha,\beta,t)}{\partial t^2} = \lambda^2\, \nabla^2 u(\alpha,\beta,t) \quad (15)$$

where $u(\alpha,\beta,z,0) = x + \beta$ and $u_t(\alpha,\beta,z,0) = \alpha^2 + \beta^2$

Applying L.T. on both sides of (15), we obtain

$$L\left\{\frac{\partial^2 u(\alpha,\beta,t)}{\partial t^2}\right\} = \lambda^2\, L\{\nabla^2 u(\alpha,\beta,t)\}$$

This implies

$$p^2 L\{u(\alpha,\beta,t)\} - pu(\alpha,\beta,0) - u_t(\alpha,\beta,0) = \lambda^2 L\{\nabla^2 u(\alpha,\beta,t)\}$$

Applying initial conditions, we obtain

$$p^2 L\{u(\alpha,\beta,t)\} = p(\alpha+\beta) + (\alpha^2+\beta^2) + \lambda^2 L\{\nabla^2 u(\alpha,\beta,t)\}$$

Divide by $p^2$, we obtain

$$L\{u(\alpha,\beta,t)\} = \frac{\alpha+\beta}{p} + \frac{(\alpha^2+\beta^2)}{p^2} + \frac{\lambda^2}{p^2} L\{\nabla^2 u(\alpha,\beta,t)\} \quad (16)$$

By performing L.T. on both sides of equation (16).

$$u = (\alpha+\beta) + (\alpha^2+\beta^2)t + \lambda^2 L^{-1}\left[\frac{1}{p^2} L\{\nabla^2 u(\alpha,\beta,t)\}\right] \quad (17)$$

By applying the iteration method, from (17)

$$u_{m+1} = (\alpha+\beta) + (\alpha^2+\beta^2)t + \lambda^2 L^{-1}\left[\frac{1}{p^2} L\{\nabla^2 u_m\}\right] \quad (18)$$

From (18),

$$u_0 = (\alpha+\beta)$$
$$u_1 = (\alpha+\beta) + (\alpha^2+\beta^2)t$$
$$u_2 = (\alpha+\beta) + (\alpha^2+\beta^2)t + \frac{2}{3}\lambda^2 t^3$$
$$u_3 = (\alpha+\beta) + (\alpha^2+\beta^2)t + \frac{2}{3}\lambda^2 t^3$$
$$u_4 = (\alpha+\beta) + (\alpha^2+\beta^2)t + \frac{2}{3}\lambda^2 t^3$$

and so on. The solution is obtained as

$$u = \lim_{n\to\infty} u_m$$
$$u = (\alpha+\beta) + (\alpha^2+\beta^2)t + \frac{2}{3}\lambda^2 t^3$$

**Table 2: Numerical values of u at α = 0.5, β = 0.5, for λ = 1 and 2**

| t | u (λ=1) | u (λ=2) |
|---|---|---|
| 0 | 1.000000000000000 | 1.000000000000000 |
| 0.1 | 1.050666666666667 | 1.052666666666667 |
| 0.2 | 1.105333333333333 | 1.121333333333333 |
| 0.3 | 1.168000000000000 | 1.222000000000000 |
| 0.4 | 1.242666666666667 | 1.370666666666667 |
| 0.5 | 1.333333333333333 | 1.583333333333333 |

| | | |
|---|---|---|
| 0.6 | 1.444000000000000 | 1.876000000000000 |
| 0.7 | 1.578666666666667 | 2.264666666666667 |
| 0.8 | 1.741333333333333 | 2.765333333333333 |
| 0.9 | 1.936000000000000 | 3.394000000000000 |
| 1.0 | 2.166666666666667 | 4.166666666666667 |

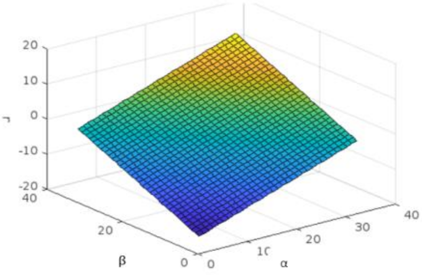


**Fig. 4.** (a) Plot at λ = 1 and t = 0.5

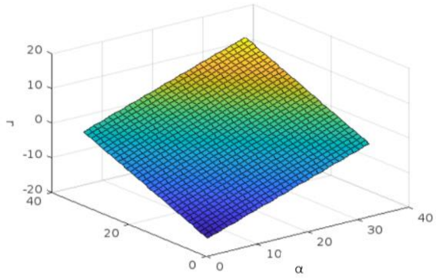


**Fig. 4.** (b) Plot at λ = 2 and t = 0.5

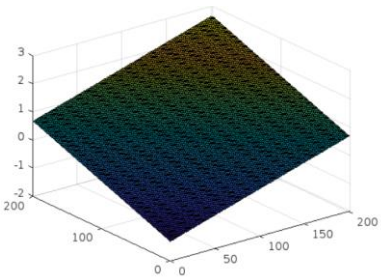


**Fig. 5.** (a) Plot at λ = 1 and t = 1

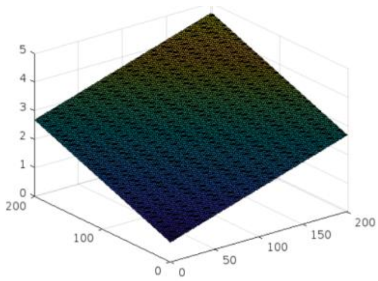


**Fig. 5.** (b) Plot at λ = 2 and t = 1

## VIII. Conclusion

It is observed from the above numerical examples that the combined Laplace transform, modified variational iterative method, provides a strong and efficient mathematical strategy for resolving two-dimensional wave equations. Future use of this integrated approach to solve non-linear and linear wave equations in three dimensions is also possible.

## Conflict of Interest

The authors declare that there is no conflict of interest regarding this article.

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