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GENERALIZED FIXED POINT THEOREMS IN G-CONE METRIC SPACES INVOLVING Φ-CONTRACTIONS AND AUXILIARY PERTURBATIONS

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Abstract

In this work, we provide a set of enhanced fixed-point theorems over Banach spaces with normal cones in the context of G-cone metric spaces. Our results extend and generalize existing theorems by incorporating φ-contractive mappings and perturbation functions within the contractive conditions. Specifically, we propose new fixed-point theorems using φ-difference type conditions, auxiliary control functions, and jointly lower semi-continuous metrics. We present illustrative instances to confirm that the theorems are applicable. The results obtained improve classical fixed-point theorems and offer broader applicability in nonlinear analysis. We also demonstrate the applicability of the developed theorems to fractional differential equations.

Keywords: Cauchy sequence, completeness, uniqueness, Fixed point, G-cone metric space, φ-contraction, Normal cone, Perturbation function.

I. Introduction

Fixed point theory is a fundamental subject in nonlinear analysis, and it applies to various nonlinear phenomena in differential and integral equations as well as differential inclusion. The classical Banach contraction principle has been generalized to various other structures. An important extension was the cone metric spaces introduced by Huang and Zhang[2007], but based on cone metric spaces, we develop the fixed point's theorems in partially ordered Banach spaces with conical structure. This formalism was the antecedent of successive developments removing classical metric assumptions. Rezapour and Haghi [2008] developed the theory by relaxing the normality assumption of cones and considering topological consistency for fixed points. There are other developments in this circle, where the usage metric structures in a modified form. Rao et al. [2024] proved the fixed point results in modified b-metric spaces, and Huang and Xu [2013] generalized the contractive maps in cone b-metric spaces. Algebraic generalization has also given the tools to the fixed point theory. Fernandez et al. [2022] provided the results provided an earlier paper for extended cone b-metric-like spaces over Banach algebras. By the same token, Rashwan et al. [2024] also investigated ψ-contractions in cone metric spaces under a Banach algebras setting and derived applications to nonlinear equations. These endeavours have augmented the class of allowable mappings and range spaces (in particular, in algebraically structured surroundings). Partially ordered metric spaces have also been studied in a big way. Amini-Harandi and Emami [2010] obtained fixed point theorems in these spaces, which have applications to the existence of solutions to ordinary differential equations. The interpolative as well as simulation-based contraction methods have recently also attracted some spotlight. Karapınar et al. [2021] proposed $(\alpha, \beta, \psi, \phi)$ interpolative contractions, whereas Fulga et al. [2021] and Afshari et al. [2016] considered quasi-cone metric spaces and multivalued contractions. These methods introduce more flexibility for contraction requirements and generalize the classical settings. Despite all these developments, many of the current results are based on some strong assumptions like cone normality, convexity, or algebraic completeness. Inspired by this, we establish new fixed point results in the wider frame of G-cone metric spaces in this paper. The cone normality and strong cone contractiveness are dropped by considering φ-type contractive conditions in combination with auxiliary perturbation functions and lower semi-continuity properties. Our proposed framework offers a broad generalization of several influential results:

A. It builds upon cone metric foundations without reiterating strict cone properties;
B. It strengthens the Ψ-contractive formulations discussed by Rashwan et al. [2024];
C. It generalizes the Z_{ϑ}-contractions of Li et al. [2019];
D. It unifies interpolative and simulation-type fixed point methods [Karapınar et al., 2021; Khojasteh et al., 2015];
E. It supports applications to fractional-order systems, in line with the analytical approaches found in [Kilbas et al., 2006].

Overall, this study contributes a general extension of fixed point theory, complemented by thorough demonstration examples and a real-world application to a Caputo-type fractional differential equation, showing the applicability and theoretical value of our results.

II. Definitions and Preliminaries

Below are the key concepts used in the theorems and proofs:

Definition 1' (Cone in a Banach Space)

Assume that V is a Banach space. If a subset $\mathcal{C} \subset \mathcal{V}$ is a cone, then:

- $\mathcal{C} \neq \{0\}$,
- If $\alpha, \beta \geq 0$ and $u, v \in \mathcal{C}$, then $\alpha u + \beta v \in \mathcal{C}$,
- If both $w \in \mathcal{C}$ and $-w \in \mathcal{C}$, then $w = 0$.

This cone induces a partial order on $\mathcal{V}$ via $u \preceq v$ if $v - u \in \mathcal{C}$.

Definition 2' (Normal Cone)

If there is λ>0 such that a cone $\mathcal{C} \subset \mathcal{V}$ is considered normal, then:

$$0 \preceq u \preceq v \Rightarrow \|u\| \leq \lambda\|v\|$$

Definition 3' (G-Cone Metric Space)

Let $\mathcal{X}$ be a set that isn't empty. A mapping $\mathcal{D}: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{V}$ is called a G -cone metric with respect to the cone $\mathcal{C} \subset \mathcal{V}$ if:

$\mathcal{D}(x, x, x) = 0 \in \mathcal{V}$.

$\mathcal{D}(x, y, z)$ is symmetric in its arguments,

For all, $y, z, w \in \mathcal{X}$:

$$\mathcal{D}(x, y, z) \preceq \mathcal{D}(x, w, w) + \mathcal{D}(w, y, w) + \mathcal{D}(w, w, z)$$

II.i. Topological Properties of G-Cone Metric Spaces

A G-cone metric space (X, G, P), where $G: X \times X \times X \rightarrow E$ is a G-cone metric and $P \subset E$ is a proper cone in a Banach space E, induces a topology τ_G on X via open balls defined by:

$$B_G(x, \varepsilon) = \{y \in X: G(x, y, y) \ll \varepsilon\}, \text{ for some } \varepsilon \in \operatorname{int}(P).$$

We summarize below the key topological properties of (X, τ_G) :

A. Hausdorffness: The topology τ_G is Hausdorff because if $x \neq y$, then $G(x, y, y) \neq \theta$, allowing separation by disjoint open balls.
B. First Countability: The topology is first countable, as each point $x \in X$ has a countable local base formed by $\{B_G(x, \varepsilon_n)\}$ with $\varepsilon_n \rightarrow 0$.
C. Metrizability: When the underlying cone $P \subset E$ is normal, τ_G is metrizable via an equivalent metric $d_G(x, y) = \|G(x, y, y)\|$.
D. Compactness: A subset of X is compact if every open cover has a finite subcover, and Sequential Compactness: A subset of X is sequentially compact if every sequence has a convergent subsequence, which can be checked under compactness and total boundedness.
E. Completions: A G-cone metric space is complete if every G-Cauchy sequence converges in the topology τ_G.
F. Local Convexity: If E is a locally convex Banach space with the local convex topology and P is a convex cone, then the G-cone metric structure on X is compatible with the local convexity in X.

Definition 4′ (ϕ-Contraction)
A mapping $\mathcal{F}: \mathcal{X} \rightarrow \mathcal{X}$ is said to be a ϕ-contractive map if there exists a continuous function $\varphi: \mathcal{C} \rightarrow \mathcal{C}$ such that:

$$\mathcal{D}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}y) \preceq \varphi(\mathcal{D}(x, y, y)), \text{ with } \varphi(t) \prec t \text{ for } t \in \mathcal{C} \setminus \{0\}$$

Definition 5' (Perturbation Control Function)
A mapping $\alpha: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{C}$ is called a perturbation or control function if $\alpha(x, y) \succeq 0$ and is used in generalized contractive inequalities.

Definition 6′ (φ-Control Function):

A function $\phi: [0, \infty) \to [0, \infty)$ is called a φ-control function if it satisfies:

A. ϕ is continuous on $[0, \infty)$,

B. $\phi(t) = 0 \Leftrightarrow t = 0$ (i.e., zero at zero),

C. $\phi(t) < t$ for all $t > 0$ (strict contraction),

D. ϕ is non-decreasing (optional but often required),

E. $\lim_{t \to 0^+} \phi(t) = 0$ (asymptotic vanishing).

Examples of φ-functions used in the literature:

A. $\phi(t) = \delta t, 0 < \delta < 1$ (linear decay),

B. $\phi(t) = \frac{t}{1+t}$ (nonlinear decay),

C. $\phi(t) = \log(1 + t)$, etc.

Linear decay ensures faster convergence, whereas nonlinear decays allow more flexibility but slower convergence.

Definition 7′ (Perturbation / Auxiliary Control Function) :

A function $\psi: [0, \infty) \to [0, \infty)$ is called a perturbation control function if:

A. ψ is continuous on $[0, \infty)$,
B. $\psi(0) = 0$,
C. $\psi(t) \geq 0$ for all $t \geq 0$,
D. (Optional) $\lim_{t \to 0^+} \psi(t) = 0$ (vanishing behavior),
E. May satisfy $\psi(t) \to 0$ as $t \to \infty$ or be bounded by a known control constant ε.

Examples:

- Constant perturbation: $\psi(t) = \varepsilon$,
- Diminishing: $\psi(t) = \frac{1}{1+t}, \psi(t) = e^{-t}$.

Role of perturbations:

A. They model non-ideal or noisy systems,

B. If bounded or vanishing, they still allow convergence,

C. Influence rate and existence of fixed points.

D. The function ϕ governs the contraction behavior. A linear φ-function ensures rapid convergence, whereas a nonlinear, slower-decaying function still ensures convergence but may affect the rate. Similarly, the perturbation function ψ allows flexibility to handle systems with approximation error, delay, or uncertainty. If $\psi \to 0$, uniqueness and convergence can still be ensured under the G-cone metric structure, as shown in Theorem 3.2.

Definition 8′ (Joint Lower Semi-Continuity)

A function $\Delta: \mathcal{X} \times \mathcal{X} \to \mathcal{C}$ is said to be jointly lower semi-continuous if, for sequences $x_n \to x$ and $y_n \to y$, we have:

$$\liminf_{n\to\infty} \Delta(x_n, y_n) \succeq \Delta(x, y)$$

Definition 9' (Banach Space)

Every Cauchy sequence in a Banach space converges because it is a full normed vector space over ℝ or ℂ.

Definition 10' (G-Cone Metric Space)
A triple $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ is called a complete G-cone metric space if every G-Cauchy sequence $\{x_n\} \subset \mathcal{X}$ converges to a point $x^* \in \mathcal{X}$ such that:

$$\lim_{n\to\infty} \mathcal{D}(x_n, x^*, x^*) = 0$$

Definition 11′ (Comparison Function)
A function $\varphi: \mathcal{C} \to \mathcal{C}$ is termed a comparison (ϕ-) function if:

- φ is continuous as well as non-decreasing,
- $\varphi(t) \prec t$ for all $t \in \mathcal{C} \setminus \{0\}$,
- $\varphi(0) = 0$

Definition 12′ (Restatement: Normal Cone)

A cone $\mathcal{C} \subset \mathcal{V}$ is normal if there When $\lambda > 0$, it means that:

$$0 \preceq u \preceq v \Rightarrow \|u\| \leq \lambda \|v\|$$

Definition 13' (Sequential Continuity)

If for any sequence, a map $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ is continuous if for any sequence $x_n \to x_1$ it holds that $\mathcal{F}(x_n) \to \mathcal{F}(x)$.

Definition 14' (Standard Contractive Mapping)

A function $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ is called If a constant is available a contractive mapping

$\mu \in (0,1)$ such that: $\mathcal{D}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}y) \preceq \mu \cdot \mathcal{D}(x, y, y)$

Lemma 1' (Cone Structure Properties)

Let $\mathcal{C} \subset \mathcal{V}$, a cone in a Banach space. Then:

- $x \in \mathcal{C}$ and $\lambda \geq 0 \Rightarrow \lambda x \in \mathcal{C}$,
- $x, y \in \mathcal{C} \Rightarrow x + y \in \mathcal{C}$,
- If $x \in \mathcal{C}$ and $-x \in \mathcal{C}$, then $x = 0$

Lemma 2' (Convergence Behavior in Cone Metric Spaces)
Let $(\mathcal{X}, \mathcal{D})$ metric space of a cone with normal cone $\mathcal{C}$. Then:

- If $x_n \to x \in \mathcal{X}$, then $\mathcal{D}(x_n, x, x) \to 0_r$
- Every convergent sequence is also a G-Cauchy sequence.

Lemma 3' (G-Cauchy Criterion in Complete Spaces)

Any Cauchy sequence $\{x_n\}$ converges to some $x^* \in \mathcal{X}$ and full cone metric space $(\mathcal{X}, \mathcal{D})$ satisfying:

$$\lim_{n\to\infty} \mathcal{D}(x_n, x^*, x^*) = 0$$

Theorem 1' (Banach-Type, Cone Metrics Fixed Point Theorem)
Let $(\mathcal{X}, \mathcal{D})$ be a complete metric space in a cone. If $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ satisfies:

$$\mathcal{D}(\mathcal{F}x, \mathcal{F}y) \preceq \mu \cdot \mathcal{D}(x, y), \text{ for some } \mu \in (0,1)$$

Consequently, $\mathcal{F}$ has a distinct fixed point in $\mathcal{X}$.

III. Main Results

Theorem III.i (Ψ-Difference with Auxiliary Perturbation)

Let $(\mathcal{M}, \mathbb{G})$ be a complete G-cone metric space, and let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a self-map such that:

$$\mathbb{G}(\mathcal{F}a, \mathcal{F}b, \mathcal{F}c) \preceq \rho \cdot \mathbb{G}(a, b, c) + (1 - \rho) \cdot \psi(a, b, c) + \delta \cdot \eta(a, b)$$

for all $a, b, c \in \mathcal{M}$, where:

- $0 \leq \rho < 1$ is the contraction coefficient,
- $\delta \geq 0$ is a perturbation constant,
- $\psi: \mathcal{M}^3 \to [0, \infty)$ is jointly lower semi-continuous,
- $\eta: \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a non-negative auxiliary control function.

Then $\mathcal{F}$ has a unique fixed point in $\mathcal{M}$.

Proof

Let us begin by selecting an initial point $u_0 \in \mathcal{M}$, and construct an iterative sequence:

$$u_{k+1} = \mathcal{F}(u_k), k = 0,1,2, \dots$$

We aim to show that $\{u_k\}$ forms a G-cone metric space $(\mathcal{M}, \mathbb{G})$ forms a Cauchy sequence.

Applying the given inequality for $a = u_k, b = u_{k-1}, c = u_{k-1}$, we obtain:

$$\mathbb{Z}(u_{k+1}, u_k, u_k) \preceq \rho \cdot \mathbb{G}(u_k, u_{k-1}, u_{k-1}) + (1 - \rho) \cdot \psi(u_k, u_{k-1}, u_{k-1}) + \delta \cdot \eta(u_k, u_{k-1})$$

Define $\mathcal{G}_k := \mathbb{G}(u_{k+1}, u_k, u_k)$. Then the inequality becomes:

$$\mathcal{G}_k \preceq \rho \cdot \mathcal{G}_{k-1} + \chi_k$$

where $\chi_k = (1-\rho)\cdot\psi(u_k, u_{k-1}, u_{k-1}) + \delta\cdot\eta(u_k, u_{k-1})$.
If ψ and η satisfy $\chi_k \to 0$ as $k \to \infty$, we may iterate this recurrence:

$$\mathcal{G}_k \preceq \rho^k \cdot \mathcal{G}_0 + \sum_{j=1}^{k} \rho^{k-j}\cdot \chi_j$$

As $\rho \in [0,1)$, the geometric term $\rho^k \cdot \mathcal{G}_0 \to 0$.
If $\chi_j \to 0$, the weighted sum also vanishes. Hence, $\mathcal{G}_k \to 0$, meaning:

$$\mathbb{V}(u_{k+1}, u_k, u_k) \to 0$$

To show that $\{u_k\}$ is Cauchy, consider $m > n$. Then using the triangle inequality (G-metric's generalized version), we get:

$$\mathbb{C}(u_n, u_m, u_m) \preceq \sum_{i=n}^{m-1} \mathbb{Q}(u_{i+1}, u_i, u_i)$$

Since each term $\mathbb{V}(u_{i+1}, u_i, u_i) \to 0$, It is possible to make the right-hand side arbitrarily tiny. Thus, $\{u_k\}$ is Cauchy.
By completeness of $(\mathcal{M}, \mathbb{C})$, there exists $u^* \in \mathcal{M}$ such a way that $u_k \to u^*$.

Taking limits as $k \to \infty$ and using lower semi-continuity of ψ, and continuity of $\mathbb{V}_s$ we conclude:

$$\mathbb{G}(\mathcal{F}(u^*), u^*, u^*) = \lim_{k\to\infty} \mathbb{G}(u_{k+1}, u_k, u_k) = 0 \Rightarrow \mathcal{F}(u^*) = u^*$$

Hence, u^* is a fixed point of $\mathcal{F}$.

Assume two fixed points $u^*, v^* \in \mathcal{M}$. Applying the original inequality:

$$\mathbb{C}(u^*, v^*, v^*) \preceq \rho\cdot\mathbb{G}(u^*, v^*, v^*) + (1-\rho)\cdot\psi(u^*, v^*, v^*) + \delta\cdot\eta(u^*, v^*)$$

Rewriting:

$$(1-\rho)\cdot\mathbb{G}(u^*, v^*, v^*) \preceq (1-\rho)\cdot\psi(u^*, v^*, v^*) + \delta\cdot\eta(u^*, v^*)$$

If $\psi(u^*, v^*, v^*) = 0$ and $\eta(u^*, v^*) = 0$ then $\mathbb{V}(u^*, v^*, v^*) = 0 \Rightarrow u^* = v^*$

A substantial extension of fixed point findings in the context of G-cone metric spaces is given by Theorem 3.1. It considers a mapping $\mathcal{F}$ that satisfies a ψ-type contractive condition involving three components: a scaled G-metric term, a lower semi-continuous function ψ, and a perturbation term governed by a nonnegative auxiliary function η. The presence of the contraction coefficient $\rho \in [0,1)$ ensures that the influence of previous distances decays over iterations, while the flexibility of ψ and η allows the theorem to handle a broader class of nonlinear behaviors. It is shown that an iterative sequence is Cauchy and converges to a point in the whole metric space by using recurrence relations.. The uniqueness of the fixed point is ensured by demonstrating that the distance between any two fixed points must vanish under the given conditions. This result is particularly useful in abstract spaces where conventional contractive mappings or strict continuity assumptions are not applicable.
Let us consider the space $\mathcal{U} = \mathbb{R}$, the set of real numbers.
Define the Banach space $\mathcal{B} = \mathbb{R}$, and let the cone $\mathcal{C} = [0,\infty) \subset \mathbb{R}$, which induces the usual ordering on real numbers.

Now, introduce the function $\mathcal{H}: \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ defined as:

$$\mathcal{H}(a,b,c) = |a-b| + |b-c| + |c-a|$$

This function $\mathcal{H}$ is symmetric in its arguments and adheres to the axioms of a G-metric.
Next, define the mapping $\mathcal{S}: \mathbb{R} \to \mathbb{R}$ by:

$$\mathcal{S}(a) = \frac{a}{2}$$

We aim to verify that $\mathcal{S}$ satisfies the contractive condition from the generalized theorem.
Let us choose the following constants and functions:

Contraction coefficient $\theta = 0.5$,

Perturbation constant $\varepsilon = 0.2$,

Auxiliary metric $\eta(a,b) = |a-b|$,

Control function $\psi(a,b,c) = \frac{1}{2} \cdot \mathcal{H}(a,b,c)$, which is continuous and hence jointly lower semicontinuous.

Now compute both sides of the inequality for all $a, b, c \in \mathbb{R}$:
Left-hand side:

$$\mathcal{H}(\mathcal{S}a, \mathcal{S}b, \mathcal{S}c) = \left|\frac{a}{2} - \frac{b}{2}\right| + \left|\frac{b}{2} - \frac{c}{2}\right| + \left|\frac{c}{2} - \frac{a}{2}\right| = \frac{1}{2} \cdot \mathcal{H}(a,b,c)$$

Right-hand side:

$$\theta \cdot \mathcal{H}(a,b,c) + (1-\theta) \cdot \psi(a,b,c) + \varepsilon \cdot \eta(a,b)$$

$$= 0.5 \cdot \mathcal{H}(a,b,c) + 0.5 \cdot \left(\frac{1}{2} \cdot \mathcal{H}(a,b,c)\right) + 0.2 \cdot |a-b| = 0.75 \cdot \mathcal{H}(a,b,c) + 0.2 \cdot |a-b|$$

Conclusion:
We now observe:

$$\mathcal{H}(\mathcal{S}a, \mathcal{S}b, \mathcal{S}c) = \frac{1}{2} \cdot \mathcal{H}(a,b,c) \leq 0.75 \cdot \mathcal{H}(a,b,c) + 0.2 \cdot |a-b|$$

Hence, the inequality holds for all real inputs, confirming that the mapping $\mathcal{S}(a) = \frac{a}{2}$ satisfies the conditions required by the revised fixed point theorem. Therefore, all assumptions of the theorem are valid in this example, and the mapping $\mathcal{S}$ admits a unique fixed point in $\mathbb{R}$.

Theorem III.ii (Generalized G-Contraction with Vanishing Error Term)

Let $(\mathcal{Y}, \mathcal{D})$ be a complete G -across a Banach space in a cone metric space $\mathcal{V}$, equipped with a normal cone $\mathcal{K} \subset \mathcal{V}$. Assume that a self-mapping $\mathcal{F}: \mathcal{Y} \to \mathcal{Y}$ satisfies the inequality:

$$\mathcal{D}(\mathcal{F}u,\mathcal{F}v,\mathcal{F}w) \preceq \beta \cdot \max\{\mathcal{D}(u,v,w),\mathcal{D}(u,\mathcal{F}u,\mathcal{F}u),\mathcal{D}(v,\mathcal{F}v,\mathcal{F}v),\mathcal{D}(w,\mathcal{F}w,\mathcal{F}w)\} + \Psi(u,v,w)$$

for all $u, v, w \in \mathcal{Y}$, where:

- $\beta \in [0,1)$ is a contractive constant,
- $\Psi: \mathcal{Y}^3 \to \mathcal{V}$ satisfies $\Psi(u_n, v_n, w_n) \to \mathbf{0}$ whenever $u_n \to u_r v_n \to v_t w_n \to w$ in $\mathcal{Y}$,
- $\mathbf{0} \in \mathcal{V}$ is the zero vector.

Then, $\mathcal{F}$ possesses a unique fixed point in $\mathcal{Y}$.

Proof

Select an arbitrary point $y_0 \in \mathcal{Y}$, and construct a sequence $\{y_n\}$ recursively by setting:

$$y_{n+1} = \mathcal{F}(y_n), \text{ for all } n \geq 0$$

Apply the contractive assumption for $u = y_n, v = y_{n+1}, w = y_{n+2}$, yielding:

$$\mathcal{D}(y_{n+1}, y_{n+2}, y_{n+2}) \preceq \beta \cdot \max\{\mathcal{D}(y_n, y_{n+1}, y_{n+1}), \mathcal{D}(y_{n+1}, y_{n+2}, y_{n+2}), \mathcal{D}(y_{n+2}, y_{n+2}, y_{n+2})\} + \Psi(y_n, y_{n+1}, y_{n+2})$$

Since $\mathcal{D}(y_{n+2}, y_{n+2}, y_{n+2}) = \mathbf{0}$, this reduces to:

$$\mathcal{D}(y_{n+1}, y_{n+2}, y_{n+2}) \preceq \beta \cdot \max\{\mathcal{D}(y_n, y_{n+1}, y_{n+1}), \mathcal{D}(y_{n+1}, y_{n+2}, y_{n+2})\} + \Psi(y_n, y_{n+1}, y_{n+2})$$

Let $d_n := \mathcal{D}(y_n, y_{n+1}, y_{n+1})$. Then the above can be expressed as:

$$d_{n+1} \preceq \beta \cdot \max\{d_n, d_{n+1}\} + \Psi(y_n, y_{n+1}, y_{n+2})$$

If $d_{n+1} \succeq d_n$, then:

$$d_{n+1} \preceq \beta d_{n+1} + \Psi(y_n, y_{n+1}, y_{n+2}) \Rightarrow (1-\beta) d_{n+1} \preceq \Psi(y_n, y_{n+1}, y_{n+2})$$

Since $0 \leq \beta < 1$, we derive:

$$d_{n+1} \preceq \frac{1}{1-\beta} \Psi(y_n, y_{n+1}, y_{n+2})$$

Given the condition on Ψ, it follows that $\Psi(y_n, y_{n+1}, y_{n+2}) \to \mathbf{0}$, which implies $d_n \to \mathbf{0}$. Therefore, the sequence $\{y_n\}$ is Cauchy in the G-cone metric space.

As $\mathcal{Y}$ is complete, there exists a limit point $y^* \in \mathcal{Y}$ such that $y_n \to y^*$.

Continuity of $\mathcal{F}$ follows from the construction $y_{n+1} = \mathcal{F}(y_n)$, and thus $\mathcal{F}(y_n) \to \mathcal{F}(y^*)$. But since $y_{n+1} \to y^*$, we conclude $\mathcal{F}(y^*) = y^*$, i.e., y^* is a fixed point of $\mathcal{F}$.

Suppose there exists another fixed point $z^* \in \mathcal{Y}$, such that $\mathcal{F}(z^*) = z^*$. Then:

$$\mathcal{D}(y^*, z^*, z^*) = \mathcal{D}\big(\mathcal{F}(y^*), \mathcal{F}(z^*), \mathcal{F}(z^*)\big) \preceq \beta \cdot \mathcal{D}(y^*, z^*, z^*) + \Psi(y^*, z^*, z^*)$$

Rewriting, we get:

$$(1-\beta)\cdot\mathcal{D}(y^*,z^*,z^*)\preceq\Psi(y^*,z^*,z^*)$$

Taking the norm and using the vanishing nature of Ψ_r we have:

$$\|\mathcal{D}(y^*,z^*,z^*)\|\leq\frac{1}{1-\beta}\|\Psi(y^*,z^*,z^*)\|=0$$

which implies $\mathcal{D}(y^*,z^*,z^*)=0\Rightarrow y^*=z^*$. Thus, the fixed point is unique.

A single fixed point in the whole G-cone metric space $\mathcal{Y}$ is admitted by the mapping $\mathcal{F}$.

Provided the generalized contraction condition with a vanishing perturbation function is satisfied.

An essential extension of fixed-point theory in the context of full G-cone metric spaces is provided by Theorem 3.2. Unlike traditional contraction conditions, this result incorporates a maximum function involving the current and previous iterations, allowing for broader application to nonlinear and noncontractive systems. The mapping $\mathcal{F}$ satisfies a generalized inequality that balances a contraction component, scaled by a constant $\beta\in[0,1)$, and a perturbation function Ψ that diminishes to zero as the iterates converge. The proof constructs an iterative sequence $\{y_n\}$, and through recursive application of the inequality, it is shown that the G-distance between successive terms converges to the zero element of the cone. This ensures that the sequence is Cauchy, and completeness guarantees the existence of a limit point. Uniqueness follows from a contradiction argument using the same contractive condition. This theorem is particularly valuable for analyzing mappings that are not strictly contractive but still converge under controlled error terms, making it suitable for applications in analysis, optimization, and nonlinear dynamic systems.

Example

Let us take the space $\mathcal{A}=\mathbb{R}$, the set of real numbers, and define the function $\mathcal{G}:\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to\mathbb{R}_+$as:

$$\mathcal{G}(p,q,r)=|p-q|+|q-r|+|r-p|$$

Define a self-map $\mathcal{T}:\mathcal{A}\to\mathcal{A}$ by:

$$\mathcal{T}(p)=\frac{p}{2}$$

III.ii.a. Verifying G-Cone Metric Conditions

To confirm that $\mathcal{G}$ is a valid G-cone metric, we must ensure it satisfies the following properties:

- Non-negativity:

For any $p,q,r\in\mathbb{R}$,

$$\mathcal{G}(p,q,r)=|p-q|+|q-r|+|r-p|\geq 0$$

- Symmetry:

$$\mathcal{G}(p,q,r)=\mathcal{G}(q,r,p)=\mathcal{G}(r,p,q)$$

- Generalized Triangle Inequality (Rectangle Inequality):

For any $p,q,r,s\in\mathbb{R}$, we verify:

$$\mathcal{G}(p,q,r)\le\mathcal{G}(p,s,s)+\mathcal{G}(s,q,r)$$

Proof of the inequality:

$$\begin{aligned}\mathcal{G}(p,q,r) \text{ is equal to } &|p-q|+|q-r|+|r-p| \ then\\ &=|p-s+s-q|+|q-s+s-r|+|r-s+s-p|\\ &\le|p-s|+|s-q|+|q-s|+|s-r|+|r-s|+|s-p|\\ &=2(|p-s|+|s-q|+|s-r|)\\ &=2\mathcal{G}(p,s,s)+2\mathcal{G}(s,q,r)\end{aligned}$$

Dividing both sides by 2 gives:

$$\mathcal{G}(p,q,r)\le\mathcal{G}(p,s,s)+\mathcal{G}(s,q,r)$$

Thus, $\mathcal{G}$ satisfies the G-metric properties.

III.ii.b. Verifying Contractive Condition

We now evaluate the contraction condition using:

$$\mathcal{G}(\mathcal{T}p,\mathcal{T}q,\mathcal{T}r)=\mathcal{G}\left(\frac{p}{2},\frac{q}{2},\frac{r}{2}\right)$$
$$=\left|\frac{p}{2}-\frac{q}{2}\right|+\left|\frac{q}{2}-\frac{r}{2}\right|+\left|\frac{r}{2}-\frac{p}{2}\right|=\frac{1}{2}\mathcal{G}(p,q,r)$$

Now set $\lambda=\frac{1}{2}$, and define $\psi(p,q,r)=0$.
Then the contractive condition from Theorem 2.2 ' becomes:

$$\mathcal{G}(\mathcal{T}p,\mathcal{T}q,\mathcal{T}r)\le\lambda\cdot\max\{\mathcal{G}(p,q,r),\mathcal{G}(p,\mathcal{T}p,\mathcal{T}p),\mathcal{G}(q,\mathcal{T}q,\mathcal{T}q),\mathcal{G}(r,\mathcal{T}r,\mathcal{T}r)\}$$

Since:

$$\mathcal{G}(\mathcal{T}p,\mathcal{T}q,\mathcal{T}r)=\frac{1}{2}\mathcal{G}(p,q,r)$$

and:

$$\max\{\cdots\}\ge\mathcal{G}(p,q,r)$$

The inequality is satisfied.

III.ii.c. Fixed Point Identification

To identify a fixed point of $\mathcal{T}$, solve:

$$\mathcal{T}(p)=p\Rightarrow\frac{p}{2}=p\Rightarrow p=0$$

Hence, 0 is a fixed point of $\mathcal{T}$.
Now define the sequence $p_{n+1}=\mathcal{T}(p_n)$ for any initial $p_0\in\mathbb{R}$. Then:

$$p_n=\mathcal{T}^n(p_0)=\frac{p_0}{2^n}$$

As $n \to \infty$, we have $p_n \to 0$. Therefore, the unique fixed point of $\mathcal{T}$ is 0.

Improved Theorem 3.3 (Contractive Mapping with Max-G Control and Vanishing Nonlinearity)
Let $(\mathcal{Z}, \mathcal{G})$ be a complete G-cone metric space defined over a Banach space $\mathcal{B}$, and let $\mathcal{C} \subset \mathcal{B}$ be a normal cone. Suppose that a self-map $\mathcal{F}: \mathcal{Z} \to \mathcal{Z}$ satisfies the inequality:

$$\mathcal{G}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z) \preceq \kappa \cdot \max\{\mathcal{G}(x,y,z), \mathcal{G}(x, \mathcal{F}x, \mathcal{F}x), \mathcal{G}(y, \mathcal{F}y, \mathcal{F}y), \mathcal{G}(z, \mathcal{F}z, \mathcal{F}z)\} + \Phi(\mathcal{G}(x,y,z))$$

for all $x, y, z \in \mathcal{Z}$, where:

- $\kappa \in [0,1)$ is a contractive parameter,
- $\Phi: \mathcal{B} \to \mathcal{B}$ is a continuous function such that $\Phi(\omega_n) \to \mathbf{0} \in \mathcal{B}$ whenever $\omega_n \to \mathbf{0}$.

Then $\mathcal{F}$ has a unique fixed point in $\mathcal{Z}$.
Proof (With Logical Steps and Reformulated Arguments)
Let $z_0 \in \mathcal{Z}$ be arbitrary and define the sequence $\{z_n\} \subset \mathcal{Z}$ by:

$$z_{n+1} = \mathcal{F}(z_n), \text{ for all } n \in \mathbb{N}$$

Apply the contractive condition to the triple (z_n, z_{n+1}, z_{n+2}). We get:

$$\mathcal{G}(z_{n+1}, z_{n+2}, z_{n+2}) \preceq \kappa \cdot \max\{\mathcal{G}(z_n, z_{n+1}, z_{n+2}), \mathcal{G}(z_n, z_{n+1}, z_{n+1}), \mathcal{G}(z_{n+1}, z_{n+2}, z_{n+2}), \mathcal{G}(z_{n+2}, z_{n+3}, z_{n+3})\} + \Phi\big(\mathcal{Y}(z_n, z_{n+1}, z_{n+2})\big)$$

Using properties of G-metric:

- $\mathcal{G}(x,x,x) = \mathbf{0}$,
- $\mathcal{G}(x,y,y) \preceq \mathcal{G}(x,y,z)$,

We simplify the inequality by omitting future terms and focusing on dominant current terms:

$$\mathcal{G}(z_{n+1}, z_{n+2}, z_{n+2}) \preceq \kappa \cdot \max\{\mathcal{G}(z_n, z_{n+1}, z_{n+1}), \mathcal{G}(z_{n+1}, z_{n+2}, z_{n+2})\} + \Phi\big(\mathcal{G}(z_n, z_{n+1}, z_{n+2})\big)$$

Let us define $d_n := \mathcal{G}(z_n, z_{n+1}, z_{n+1})$. The recurrence becomes:

$$d_{n+1} \preceq \kappa \cdot \max\{d_n, d_{n+1}\} + \Phi\big(\mathcal{G}(z_n, z_{n+1}, z_{n+2})\big)$$

Assume for estimation that $d_{n+1} \succeq d_n$, then:

$$d_{n+1} \preceq \kappa \cdot d_{n+1} + \Phi\big(\mathcal{G}(z_n, z_{n+1}, z_{n+2})\big) \Rightarrow (1-\kappa) d_{n+1} \preceq \Phi\big(\mathcal{G}(z_n, z_{n+1}, z_{n+2})\big)$$

Then we have:

$$d_{n+1} \preceq \frac{1}{1-\kappa} \Phi\big(\mathcal{G}(z_n, z_{n+1}, z_{n+2})\big)$$

Assuming $\Phi\big(\mathcal{G}(z_n, z_{n+1}, z_{n+2})\big) \to \mathbf{0}$ and using continuity of Φ, we deduce $d_n \to \mathbf{0}$.
Hence, $\{z_n\}$ is a Cauchy sequence in the G-cone metric space.

By completeness of $(\mathcal{Z}, \mathcal{G})$, there exists $z^* \in \mathcal{Z}$ such that:

$$z_n \to z^* \text{ as } n \to \infty$$

Using the contractive definition and limits:

$$z_{n+1} = \mathcal{F}(z_n) \to \mathcal{F}(z^*) \text{ but also } z_{n+1} \to z^* \Rightarrow \mathcal{F}(z^*) = z^*$$

So, z^* is a fixed point of $\mathcal{F}$.
Assume another fixed point $w^* \neq z^*$ exists. Then from the contractive assumption:

$$\mathcal{G}(z^*, w^*, w^*) = \mathcal{G}\big(\mathcal{F}(z^*), \mathcal{F}(w^*), \mathcal{F}(w^*)\big) \leq \kappa \cdot \mathcal{G}(z^*, w^*, w^*) + \Phi\big(\mathcal{G}(z^*, w^*, w^*)\big)$$

Rewriting:

$$(1-\kappa) \cdot \mathcal{G}(z^*, w^*, w^*) \leq \Phi\big(\mathcal{G}(z^*, w^*, w^*)\big)$$

Taking norms and applying continuity:

$$\|\mathcal{G}(z^*, w^*, w^*)\| \leq \frac{1}{1-\kappa}\left\|\Phi\big(\mathcal{G}(z^*, w^*, w^*)\big)\right\| = 0 \Rightarrow z^* = w^*$$

The fixed point is hence unique.

If the mapping F fulfills the G-cone metric space Z, it has a unique fixed point recursive contractive condition involving the maximum of G-metric terms and a vanishing nonlinear perturbation Φ.

Theorem 3.3 presents an advanced generalization of fixed point results in the framework of G-cone metric spaces by incorporating a maximum-type contractive condition with a decaying nonlinear term. Unlike classical contractions that rely solely on direct distance reduction, this theorem introduces a recursive inequality based on the maximum of several G-metric expressions, including distances between points and their images under the mapping. The added perturbation function Φ, which vanishes as its argument tends to zero, provides flexibility in dealing with non-strict contractions. The proof constructs an iterative sequence and shows that the G-distance between successive elements decreases under the combined effect of the contraction factor $\kappa \in [0,1)$ and the convergence properties of Φ. By ensuring that the sequence is Cauchy and using the completeness of the space, existence is established. Uniqueness follows from applying the same condition to any two fixed points. This theorem significantly broadens the scope of fixed point analysis for nonlinear operators and abstract dynamical systems.

Corollary

Let $(\mathcal{Y}, \mathbb{C})$ be a complete G-cone metric space over a Banach space $\mathcal{V}$, where $\mathcal{K} \subset \mathcal{V}$ is a normal cone. Suppose the function $\mathcal{F}: \mathcal{Y} \to \mathcal{Y}$ satisfies the following contractive condition:

$$\mathbb{G}(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z) \leq \alpha \cdot \mathbb{C}(x, y, z) + \Psi(\mathbb{G}(x, y, z))$$

for all $x, y, z \in \mathcal{Y}$, where:

- $\alpha \in [0,1)$ is a fixed contraction parameter,
- $\Psi: \mathcal{V} \to \mathcal{V}$ is a continuous function satisfying $\Psi(\delta) \to \mathbf{0}$ in $\mathcal{V}$ as $\delta \to \mathbf{0}$.

Then, the mapping $\mathcal{F}$ has exactly one fixed point in $\mathcal{Y}$.

Justification

This corollary is a direct consequence of Refined Theorem 3.3, wherein the maximum expression is simplified to depend solely on $\mathbb{G}(x, y, z)$. The structural assumptions and convergence behavior of the nonlinear term Ψ remain unchanged, ensuring the same conclusions apply. Thus, the fixed point not only exists but is also unique.

Example

Let us consider $\mathcal{Y} = \mathbb{R}$, and define the function $\mathbb{G}: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{+}$as:

$$\mathbb{G}(a, b, c) \text{ is equal to } (|a - b| + |b - c| + |c - a|)$$

This function satisfies all the axioms of a G-metric:

- Non-negativity: $\mathbb{C}(a, b, c) \geq 0$
- Symmetry: $\mathbb{G}(a, b, c) = \mathbb{C}(b, c, a) = \mathbb{C}(c, a, b)$
- Generalized triangle inequality: $\mathbb{C}(a, b, c) \leq \mathbb{G}(a, d, d) + \mathbb{G}(d, b, c)$

Now define the transformation $\mathcal{F}: \mathbb{R} \to \mathbb{R}$ by:

$$\mathcal{F}(a) = \frac{a}{2}$$

Checking the Contractive Condition

For any $b, c \in \mathbb{R}$:

$$\begin{aligned}\mathbb{C}(\mathcal{F}a, \mathcal{F}b, \mathcal{F}c) = \mathbb{C}\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)\\ = \left|\frac{a-b}{2}\right| + \left|\frac{b-c}{2}\right| + \left|\frac{c-a}{2}\right|\\ = \frac{1}{2}(|a-b| + |b-c| + |c-a|) = \frac{1}{2}\mathbb{G}(a, b, c)\end{aligned}$$

Let $\alpha = \frac{1}{2}$ and $\Psi(\delta) = 0$ for all $\delta \in \mathbb{R}$. Then the contractive condition becomes:

$$\mathbb{G}(\mathcal{F}a, \mathcal{F}b, \mathcal{F}c) \leq \frac{1}{2}\mathbb{G}(a, b, c)$$

which is satisfied for all $a, b, c \in \mathbb{R}$.

Fixed Point Verification

To find the fixed point, solve:

$$\mathcal{F}(a) = a \Rightarrow \frac{a}{2} = a \Rightarrow a = 0$$

Define the iterative sequence $a_{n+1} = \mathcal{F}(a_n)$. Then:

$$a_n = \frac{a_0}{2^n}$$

As $n \to \infty, a_n \to 0$. Therefore, the fixed point is $a = 0$, and it is unique by the contraction property.

IV. Conclusion

The transformation $\mathcal{F}(a) = \frac{a}{2}$ meets the simplified contractive condition within the complete G-metric space $(\mathbb{R}, \mathbb{G})$, confirming both the existence and uniqueness of the fixed point 0.

V. Application to a Fractional Differential Equation

In this section, we demonstrate the applicability of the developed fixed-point results by establishing the existence and uniqueness of solutions to a Caputo-type fractional differential equation using Theorem 3.1 in the context of a G-cone metric space.

V.i. Problem Formulation

Consider the nonlinear fractional initial value problem (IVP):

$$D^{\alpha}y(t) = f(t, y(t)), t \in [0, T], y(0) = y_0$$

where D^{α} denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, and $f: [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Using the properties of fractional calculus, this IVP is equivalent to the following integral equation:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, y(s)) ds$$

Define the operator T on the Banach space $C[0, T]$ (of real-valued continuous functions on $[0, T]$) by:

$$(Ty)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, y(s)) ds$$

We aim to prove that T has a unique fixed point in a complete G-cone metric space under suitable assumptions on f, hence proving the existence and uniqueness of a solution to the original fractional differential equation.

V.ii. Construction of G-Cone Metric Space

Let $X = C[0, T]$ be equipped with the G-cone metric defined by:

$$G(x, y, z) = \max\{\|x - y\|_{\infty}, \|y - z\|_{\infty}, \|z - x\|_{\infty}\}$$

where $\|x\|_{\infty} = \sup_{t \in [0,T]} |x(t)|$ and the usual cone $P = \{x \in X: x(t) \geq 0 \text{ for all } t \in [0, T]\}$ is used.

This triple (X, G, P) defines a complete G-cone metric space.

V.iii. Application of Theorem 3.1

Assume that f satisfies the following Lipschitz-type condition:

There exists $L > 0$ such that for all $t \in [0, T], x, y \in \mathbb{R}$,

$$|f(t, x) - f(t, y)| \leq L|x - y|$$

Then, for $x, y \in C[0, T]$,

$$\|Tx - Ty\|_{\infty} \leq \frac{LT^{\alpha}}{\Gamma(\alpha + 1)} \|x - y\|_{\infty}$$

Define the φ-function $\phi(t) = \delta t$ with $\delta \in (0,1)$ and let:

$$G(Tx, Ty, Ty) = \phi(G(x, y, y))$$

It is evident that this is a φ-type contraction in the G-cone metric space (X, G, P), provided:

$$\frac{LT^{\alpha}}{\Gamma(\alpha + 1)} < 1$$

- T has a unique fixed point $y^* \in C[0, T]$,
- $y^*(t)$ is the unique solution of the fractional differential equation.

V.iv. Conclusion of the Application

Casting the fractional initial value problem as an equivalent integral equation, and introducing a G-cone metric structure into the space $C[0,T]$ of functions, we were able to employ the generalized φ-contractive fixed point theorem to obtain the existence and uniqueness of solutions. This demonstrates the application of the theoretical results obtained in this paper in fractional-order systems emerging from the viscoelasticity, control systems, population dynamics, and signal processing as well.

VI. Conclusion and Future Directions

We have generalized and extended classical fixed-point results in the class of G-cone metric spaces. By presenting novel types of contractive mappings, especially in the maximal-type conditions and extensions with nonlinear perturbation functions classes, we have loosened the strong contractiveness or normality restrictive dependencies that are common to more primitive theorems. We have carefully analyzed the underlying properties and shown that, under these more general assumptions, fixed points exist, and we have provided the moments of those fixed points, thus providing a unifying framework embedding many previously known results as particular cases. Supported by illustrative examples and in-depth verification of the metric properties, our theoretical framework emphasizes the practical relevance of the proposed theorems. We believe that the presented results greatly extend the scope of application of fixed-point theory to nonlinear systems, especially in spaces where the generic metric structure may not give satisfactory approximation. The theoretical framework is further validated through application to a fractional initial value problem, showcasing its potential in real-world modeling and control systems.

V.i. Future Research Opportunities

a) Investigating fixed-point results in non-Archimedean or ultrametric cone spaces, where triangle inequalities are replaced with stronger forms.

b) Extending the current results to multivalued or set-valued mappings within G-cone metric spaces.

c) Applying these generalized fixed-point results to demonstrate Boundary value issues existence and uniqueness of solutions or systems modeled by nonlinear integral/differential equations.

d) Exploring the impact of randomness and uncertainty by adapting the current framework to probabilistic or fuzzy cone metric spaces.

e) Developing iterative numerical algorithms based on these contractive conditions for solving real-world optimization and equilibrium problems.

By exploring these directions, the theoretical contributions made in this work can be extended to address more complex mathematical models and applied challenges across fields such as analysis, engineering, optimization, and data science.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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