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A MODIFIED CLOSED-TYPE HYBRID QUADRATURE FOR THE NUMERICAL SOLUTION OF SINGULAR COMPLEX-VALUED INTEGRALS

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Abstract

A novel closed-type modified anti-Gaussian 4-point transformed rule has been developed for solving Cauchy principal value complex integrals. Furthermore, a more precise mixed quadrature rule MQ(f), has been created by combining the closed-type modified quadrature rule with the Gauss-Legendre 2-point transformed technique. Theoretical analysis of errors confirms the enhanced performance of the newly proposed quadrature rule. Numerical computation of various sample integrals is performed. The numerical calculations demonstrate the superiority of the new rule among others.

Keywords: Cauchy principal value integrals, Gauss-Legendre transformed rule, closed-type anti-Gaussian transformed rule, mixed rule, singularity.

I. Introduction

In complex analysis, Cauchy principal value problems commonly appear when working with complex functions. A special type of Cauchy principal value (CPV) integral is given by

$$I(f(z)) = \int_{z_0-h}^{z_0+h} \frac{f(z)}{z-z_0}dz \tag{1}$$

where $f(z)$ is analytic in simply connected domain $\Omega = \{z \in C\colon |z - z_0| \le \rho = r|h|\colon r > 1\}$ containing the line segment $z = z_0 + ht;\ -1 \le t \le 1$.

These types of integrals show up in various fields like signal processing, potential theory, and solving boundary value problems. However, evaluating these integrals directly can be difficult because the singularities make them undefined or divergent. Therefore, a numerical approach can be employed by transforming standard quadrature for real integrals, adapting it to effectively solve the CPV integral. In 1979, Acharya and Das [I] developed a transformed rule by utilizing the pair of rules originally formulated by Price [VIII]. Many authors [VI, XIII] also successfully constructed rules for the numerical solution of CPV integrals. The anti-quadrature rule was introduced by D.P. Laurie in 1996 [IV]. He developed a suboptimal anti-Gaussian quadrature rule by using the Gaussian quadrature rule. This approach offered a different way to numerically evaluate definite integrals of analytic functions over an interval $[-1,1]$.

In literature review, methods such as Richardson extrapolation and Kronrod extension [IX, VII] are known to improve the accuracy of certain mathematical rules, but these methods can be quite complicated. To simplify this, in 1996, R.N. Das and G. Pradhan introduced a more straightforward approach called the mixed quadrature rule, as discussed in [XII]. Very recently, in 2025, Tusar Singh et al [XVIII] worked on a mixed quadrature technique of higher precision. Further studies in [III, XI, II] have also successfully improved accuracy by applying a combination of simpler quadrature rules. S.K.Mohanty and R.B.Dash [XV] in 2022 generalised the idea of mixed quadrature rule in their paper.

In this paper, getting inspiration from Laurie, a closed-type anti-Gaussian 4-point rule $AG_4(f)$ has been constructed by adopting the Gauss-Legendre 2-point rule, which is then utilised to construct an anti-Gaussian 4-point transformed rule for solving Cauchy principal value integrals involving complex-valued functions. The error associated with the rule is thoroughly analysed, and a hybrid quadrature rule is constructed via blending anti-Gaussian 4-point and Gauss-Legendre 2-point transformed rules. Furthermore, the theoretical predictions of the rule are validated numerically using test integrals.

II. Formulation of Closed Type Anti-Gaussian 4-point Transformed Rule

Making partial modifications to D. P. Laurie's principle, a closed-type anti-Gaussian 4-point rule, denoted as $AG_4(f)$ is developed utilizing the following characteristics:

- The nodes -1 and 1 are fixed as pre-assigned endpoints.
- Error related to $AG_4(f)$ is $-\frac{1}{2}$ times the error of the Gauss-Legendre 2-point rule when applied to integrate polynomials of degree up to 5. This relationship can be expressed mathematically as:

$$AG_4(f) = \omega_1 f(-1) + \omega_2 f(\xi_1) + \omega_3 f(\xi_2) + \omega_4 f(1) \tag{2}$$

such that

$$I(f) - AG_4(f) = -\frac{[I(f) - G_2(f)]}{2}$$

$$Or, \quad AG_4(f) = \frac{3I(f) - G_2(f)}{2} \tag{3}$$

$$where \quad G_2(f) = f\left(\sqrt{\frac{1}{3}}\right) + f\left(-\sqrt{\frac{1}{3}}\right) \tag{4}$$

Choosing the monic polynomials $1, \xi, \xi^2, \xi^3, \xi^4, \xi^5$ and using them in (3), we get the following equations.

$$\omega_1 + \omega_2 + \omega_3 + \omega_4 = 2$$

$$-\omega_1 + \omega_2\xi_1 + \omega_3\xi_2 + \omega_4 = 0$$

$$\omega_1 + \omega_2\xi_1^2 + \omega_3\xi_2^2 + \omega_4 = \frac{2}{3}$$

$$-\omega_1 + \omega_2\xi_1^3 + \omega_3\xi_2^3 + \omega_4 = 0$$

$$\omega_1 + \omega_2\xi_1^4 + \omega_3\xi_2^4 + \omega_4 = \frac{22}{45}$$

$$-\omega_1 + \omega_2\xi_1^5 + \omega_3\xi_2^5 + \omega_4 = 0$$

Solving the above system of equations, we have

$$\xi_1 = \sqrt{\frac{2}{15}}, \xi_2 = -\sqrt{\frac{2}{15}}, \omega_2 = \frac{20}{26} = \omega_3, \omega_1 = \frac{6}{26} = \omega_4$$

By putting the values of ξ_i's and ω_i's in Equation(2), we get

$$AG_4(f) = \frac{6}{26}f(-1) + \frac{20}{26}f\left(\sqrt{\frac{2}{15}}\right) + \frac{20}{26}f\left(-\sqrt{\frac{2}{15}}\right) + \frac{6}{26}f(1) \tag{5}$$

Considering the suggestion to transform the integral employing a Lather [V] transformation, Equation (1) can be rewritten as:

$$I(f(z)) = \int_{-1}^{1} \frac{f(z_0 + ht)}{t} dt \qquad -1 \leq t \leq 1 \tag{6}$$

Substituting equation (5) in (6), the closed-type anti-Gaussian 4-point transformed rule is formulated as follows:

$$AG_4(f(z)) = \frac{6}{20}[f(z_0 + h) - f(z_0 - h)] + \frac{5\sqrt{30}}{13}\left[f\left(z_0 + h\sqrt{\frac{2}{15}}\right) - f\left(z_0 - h\sqrt{\frac{2}{15}}\right)\right] \tag{7}$$

II.i. Error Analysis

Theorem 1

Assuming an analytic function $f(z)$ over interval$[z_0 - h, z_0 + h]$. Then, error associated with $AG_4(f(z))$ is given by

$$EAG_4(f(z)) = -\frac{1}{1350}h^5 f^v(z_0) - \frac{19529}{309582000}h^7 f^{vii}(z_0) - \ldots$$

Proof:

The error corresponding to the rule defined in (7) is given by

$$EAG_4(f(z)) = I(f(z)) - AG_4(f(z)) \tag{8}$$

Taylor series expansion of f(z) about z_0 is given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \cdots \tag{9}$$

Using equation (9) in (8), we have

$$EAG_4(f(z)) = -\frac{1}{1350}h^5 f^v(z_0) - \frac{19529}{309582000}h^7 f^{vii}(z_0) - \cdots \tag{10}$$

Degree of precision of $AG_4(f(z))$ is 4.

III. Formulation of Gauss-Legendre 2-point Transformed Rule:

A Gauss-Legendre 2-point [XIV, X] transformed rule, by utilizing (4), is constructed for numerical approximation of the Cauchy principal value integral defined in (1). The rule is given by

$$G_2(f(z)) = \sqrt{3}\left[f(z_0 + \frac{h}{\sqrt{3}}) - f(z_0 - \frac{h}{\sqrt{3}})\right] \tag{11}$$

III.i. Error Analysis

Theorem 2

Assuming an analytic function $f(z)$ over interval $[z_0 - h, z_0 + h]$. Then error associated with $G_2(f(z))$ is given by

$$EG_2(f(z)) = \frac{1}{675}h^5 f^v(z_0) + \frac{1}{23814}h^7 f^{vii}(z_0) + \ldots \tag{12}$$

Proof:

Use Taylor series expansion about z_0.

The aforementioned theorem indicates that the level of precision is four.

IV. Construction of Mixed Quadrature Rule

Combined integration rule of enhanced degree is formulated by using $AG_4(f(z))$ and $G_2(f(z))$ transformed rules. By [XV, XVI, XVII], a mixed quadrature rule can be defined by

$$MQ(f(z)) = \alpha_1 AG_4(f(z)) + \alpha_2 G_2(f(z)) \tag{13}$$

Such that $\alpha_1 + \alpha_2 = 1$

Multiplying α_1 and α_2 with equations (10) and (12) respectively, and adding them, we have

$$EMQ(f(z)) = \left(-\frac{\alpha_1}{1350} + \frac{\alpha_2}{675}\right) h^5 f^v(z_0) + \left(-\frac{19529\cdot\alpha_1}{309582000} + \frac{\alpha_2}{23815}\right) h^7 f^{vii}(z_0) + \cdots \quad (14)$$

where $EMQ(f(z)) = I(f(z)) - MQ(f(z))$, error associated with the mixed rule.

Restricting the degree of precision of $MQ(f(z))$ to 6 ,we have

$$\alpha_1 + \alpha_2 = 1$$

$$-\frac{\alpha_1}{1350} + \frac{\alpha_2}{675} = 0$$

Solving the above equations, we obtain

$$\alpha_1 = \frac{2}{3}$$

$$\alpha_2 = \frac{1}{3}$$

Putting the above values in (13), the mixed rule will become

$$MQ(f(z)) = \frac{2}{3} AG_4(f(z)) + \frac{1}{3} G_2(f(z))$$

Which has a degree of precision of 6.

IV.i. Error Analysis

Theorem 3

Assuming an analytic function $f(z)$ in domain Ω along line L with end points $z_0 - h$ and $z_0 + h$ .Then error associated with MQ(f(z)) is given by

$$\mathrm{EMQ}\big(\mathrm{f(z)}\big) = \frac{-8686}{61425 \times 7!} \mathrm{h}^7 \mathrm{f}^{\mathrm{vii}}(\mathrm{z}_0) - \frac{73578}{394875 \times 9!} \mathrm{h}^9 \mathrm{f}^{\mathrm{ix}}(\mathrm{z}_0) \ldots$$

V. Results and Discussion

Numerical evaluations of test integrals have been conducted to verify the theoretical predictions, as demonstrated through tables and figures. The results, presented in these tables and figures, include a comparison of our method with the 4-point rule R(f) from [VI], highlighting the differences in accuracy and efficiency between the two methods.

The results of the test integrals due to $G_2(f(z)), R(f), AG_4(f(z)), MQ(f(z))$ are reflected in the **Table-1,2,3,4**. The modulus of truncation errors E_1 , E_2 , E_3 , E_4 associated with these rules, respectively, are also provided in the **Table-1,2,3,4**.

Table 1: $I_1 = \int_{-i}^{i} \frac{e^z}{z} dz$

Exact Value	1.892166140734366i
$G_2(f(z))$	1.8907261113408342i
$\|E_1\|$	0.001440029393531717
$R(f)$	1.905428775506396i
$\|E_2\|$	0.013262634772030157
$AG_4(f(z))$	1.8928719260409321i
$\|E_3\|$	0.000705785306566176
$MQ(f(z))$	1.8921566544742325i
$\|E_4\|$	0.000009486260133418

Table 2: $I_2 = \int_{-i}^{i} \frac{1+z\cos z}{z} dz$

Exact Value	2.3504023872876i
$G_2(f(z))$	2.3426960879097303i
$\|E_1\|$	0.007706299377869819
$R(f)$	2.417642383665751i
$\|E_2\|$	0.067239996378150924
$AG_4(f(z))$	2.354361381358771i
$\|E_3\|$	0.003958994071171063
$MQ(f(z))$	2.350472950209091i
$\|E_4\|$	0.000070562921490769

Table 3: $I_3 = \int_{\frac{-(i+1)}{\sqrt{2}}}^{\frac{(i+1)}{\sqrt{2}}} \frac{z^2 e^z}{z} dz$

Exact Value	-0.5168305647486302+0.4226120102352736i
$G_2(f(z))$	-0.4971537270819438+0.4447823752407798i
$\|E_1\|$	0.029642925379228392
$R(f)$	-0.7028991466676475+0.23205575087880048i
$\|E_2\|$	0.266332884108000323
$AG_4(f(z))$	-0.5262155849127564+0.41111299553928676i
$\|E_3\|$	0.014842706709343066
$MQ(f(z))$	-0.5165282989691522+0.42233612210645105i
$\|E_4\|$	0.000409241812463809

Table 4: $I_4 = \int_{-i}^{i} \frac{(1+z)e^z}{z} dz$

Exact Value	3.5751081103501594i
$G_2(f(z))$	3.5665497667308204i
$\|E_1\|$	0.008558343619339048
$R(f)$	3.654552273207412i
$\|E_2\|$	0.07944162857252643
$AG_4(f(z))$	3.5792742052415036i
$\|E_3\|$	0.004166094891344230
$MQ(f(z))$	3.5750327257379424i
$\|E_4\|$	0.000075384612217011

The figure provided below shows the behaviour of the integrand in I_1.

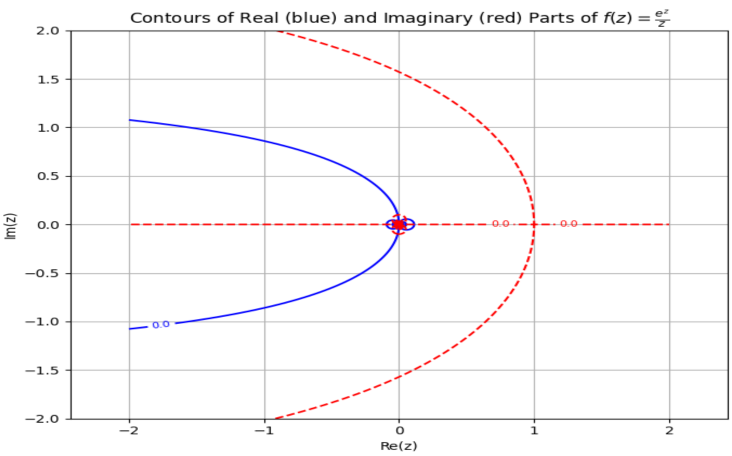


Fig. 1.

The following figure represents the behaviour of the integrand of I_2.

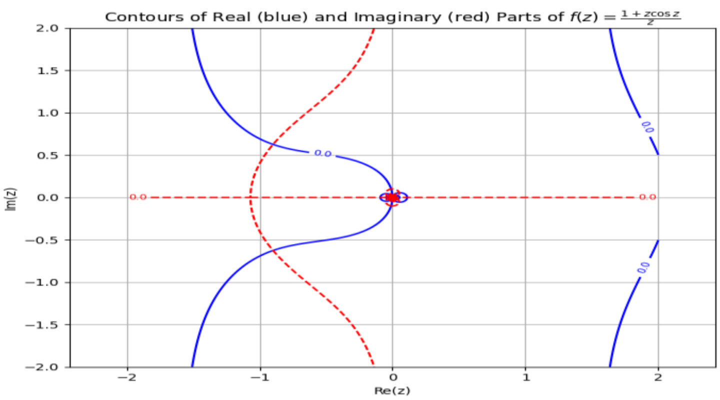


Fig. 2.

The figure depicted below demonstrates the behaviour of the integrand in I_3.

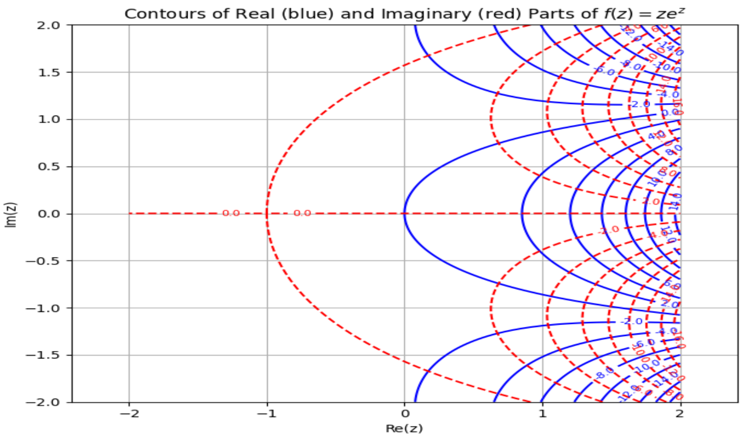


Fig. 3.

The following figure represents the behaviour of the integrand in I_4.

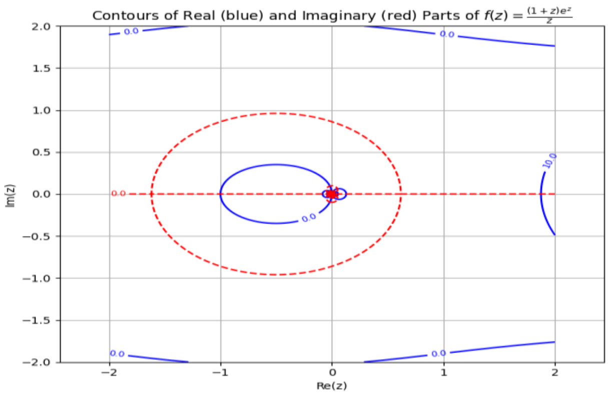
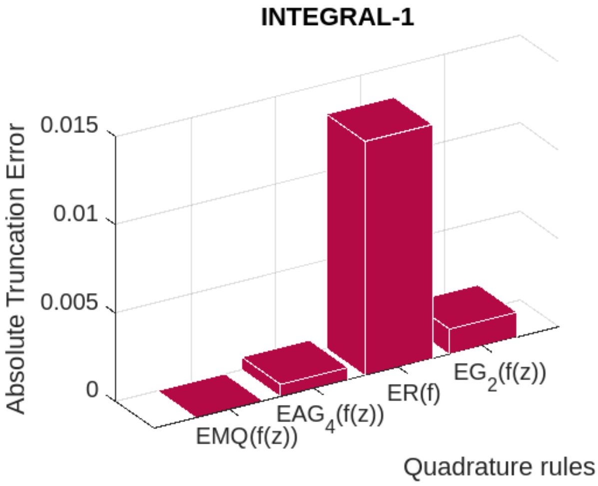
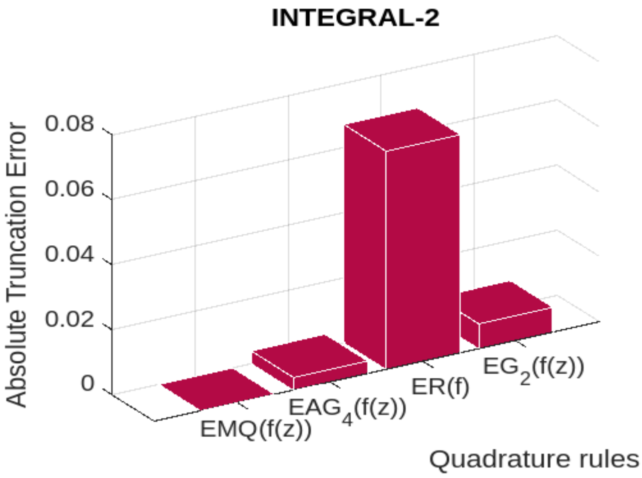
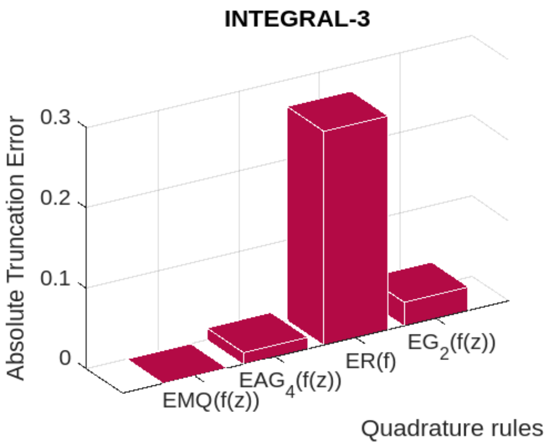


Fig. 4.

To provide a better comparison of the constructed rule with both its base rules and the approach presented in [VI], the absolute values of the truncation errors E_1, E_2, E_3, E_4 obtained by the four quadrature rules $G_2(f(z)), R(f), AG_4(f(z)), MQ(f(z))$, respectively, when applied to the test integrals I_1 to I_4, are shown in **Figure-5**.


INTEGRAL-1
Absolute Truncation Error
0.015
0.01
0.005
0
EMQ(f(z))
EAG4(f(z))
ER(f)
EG2(f(z))
Quadrature rules


INTEGRAL-2
Absolute Truncation Error
0.08
0.06
0.04
0.02
0
EMQ(f(z))
EAG4(f(z))
ER(f)
EG2(f(z))
Quadrature rules


INTEGRAL-3
Absolute Truncation Error
0.3
0.2
0.1
0
EMQ(f(z))
EAG4(f(z))
ER(f)
EG2(f(z))
Quadrature rules

INTEGRAL-4

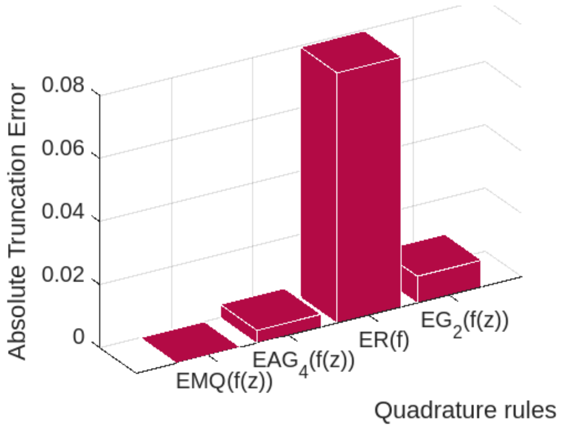


Fig. 5.

Observations:

The following observations are derived from the Table-1,2,3,4 and Figure 5:

• For integral-1, we observed that the absolute value of truncation error of $MQ(f)$ is significantly less than those of $AG_4(f)$ and $G_2(f)$. $MQ(f)$ gives an accurate result up to 5 significant places, whereas $AG_4(f)$ gives accuracy up to 4 significant places. $G_2(f)$ is accurate up to 3 significant places, while $R(f)$ has the highest error, retaining accuracy up to only 1 significant place. From the error comparison figure, it can be seen that $MQ(f)$ provides a better result than its base rule and $R(f)$, further confirming its superior accuracy.

• For integral-2, we observed that the absolute value of truncation error of $MQ(f)$ is significantly less than those of $AG_4(f)$ and $G_2(f)$. $MQ(f)$ gives an accurate result up to 5 significant places, whereas $AG_4(f)$ gives accuracy up to 3 significant places. $G_2(f)$ is accurate up to 2 significant places, while $R(f)$ has the highest error, retaining accuracy up to only 1 significant place. From the error comparison figure, it can be seen that $MQ(f)$ provides a better result than its base rule and $R(f)$, further confirming its superior accuracy.

• For integral-3 real part of $MQ(f)$ gives an accurate result up to 3 significant places, whereas the imaginary part of $MQ(f)$ also gives an accurate result up to 3 significant places. $AG_4(f)$ gives the real part accurate up to 1 significant place and the imaginary part up to 1 significant place as well. $G_2(f)$ gives the imaginary part up to 1 significant place. $R(f)$ has the highest error and does not provide a correct result up to any significant place. From the error comparison, it can be seen that $MQ(f)$ provides a better result than its base rule and $R(f)$.

• For integral-4, we observed that the absolute value of truncation error of $MQ(f)$ is significantly less than those of $AG_4(f)$ and $G_2(f)$. $MQ(f)$ gives an accurate result up to 4 significant places, whereas $AG_4(f)$ gives accuracy up to 3 significant places. $G_2(f)$ is accurate up to 2 significant places, while $R(f)$ has the highest error, retaining accuracy up to only 1 significant place. From the error comparison figure, it can be seen that $MQ(f)$ provides a better result than its base rule and $R(f)$, further confirming its superior accuracy.

VI. Conclusions

The tables and graphs suggest that the developed closed-type mixed rule generally provides more accurate, favorable results compared to its base rule and the alternative proposed by [VI]. The degree of precision of the final quadrature rule is 6 where whereas the degrees of precision of other rules are less. Also, the error value of the final quadrature rule is very minimal compared to other quadrature rules. The error value of quadrature rules varies inversely with their corresponding degree of precision. Hence newly constructed quadrature rule $MQ(f)$ is the most efficient as compared to other methods. In the future, this work can be further improved using splines and various interpolation formulas. It can be further extended to multidimensional quadrature rules.

Conflict of Interest:

This paper has no conflict of interest.

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