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EXTRACTION OF NEW EXACT TRAVELING WAVE SOLUTIONS OF THE $(3 + 1)$-DIMENSIONAL GENG EQUATION BY EMPLOYING TWO EXPANSION STRATEGIES IN MATHEMATICAL PHYSICS

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Abstract

We inspect a nonlinear partial differential equation, known as the Geng equation, which captures the behavior of systems such as shallow water wave dynamics and quantum field interactions and has notable applications in the areas of engineering sciences, mechanics, and quantum mechanics in the present research work. Multiple exact wave solutions are determined for the Geng equation by utilizing two effective strategies, namely, $(G'/(G'+G+A))$*-expansion and two variables* $(G'/G\,,1/G)$*-expansion strategies. The solutions derived are formulated through elementary functions having rational, hyperbolic, exponential, and trigonometric forms. With specific values of chosen constants, the graphic representations of the obtained exact wave solutions are depicted using density, contour, 2D, and 3D plots to illuminate the inherent structure of the phenomenon. Additionally, we obtained kink-shaped, anti-kink-shaped, compacton, and singular-periodic-shaped solitons. The findings demonstrate that the mentioned strategies serve as influential mathematical tools and are shown to be highly efficient, computationally adaptable, and easily manageable for exploring solutions of nonlinear partial differential equations in mathematical physics.*

Keywords: The (3+1) - dimensional Geng equation; the exact travelling wave solutions; the $(G'/(G'+G+A))$-expansion strategy; the two variables $(G'/G\,,1/G)$-expansion strategy.

I. Introduction

Nonlinear partial differential equations (NLPDEs) stand as the vital tool for the analysis of nonlinear phenomena that are currently being studied in numerous scientific disciplines, for instance, mathematical physics, quantum mechanics, nonlinear dynamics, neural networks, epidemiology, thermodynamics, biology, as well as medicine [XLIII]. NLPDEs represent physical issues in engineering sciences, meteorology, fluid mechanics, plasma physics, diseases, nonlinear optics, and the aerospace industry [XXXIX]. A particular type of NLPDEs is the nonlinear evolution equations (NLEEs), which govern the behavior of systems evolving over continuous time. It serves an important function in representing complex systems and modeling numerous problems in the universe. Nonlinear wave equations and the concept of solitons have introduced remarkable achievements in the field of applied sciences. The phenomenon of the solitary wave was first observed in early 1834 by British scientist John Scott Russell. Solitary waves are localized waves traveling with constant speeds and shape, asymptotically approaching zero at large distances [XXVI]. Solitons are solitary waves that remain unchanged after interacting with other solitons. The exploration of completely integrable nonlinear evolution equations is advancing rapidly, engaging physicists and mathematicians alike, because these equations encapsulate the true nature of phenomena present in numerous scientific, technological, and engineering applications [XXVIII]. Mathematicians have concentrated on formulating and utilizing advanced techniques to solve integrable equations.

The determination of exact solutions for NLEEs is challenging due to the lack of universal methods applicable to all such equations, often requiring individual analysis. However, recent advancements have resulted in the creation of several reliable and efficient strategies, including the $\left(\frac{G'}{G}\right)$-expansion scheme [XIV], [XV], [XL], the extended $\left(\frac{G'}{G^2}\right)$-expansion process [III], [XXIX], the Sardar sub-equation scheme [IX], [XII], [XXXVII], the modified exp-function technique [XXXVIII], the logistic process [XVI], the inverse scattering technique [XXVII], the homogeneous balance scheme [XXXIV], the modified auxiliary equation scheme [IV], the tanh-coth expansion approach [VI], [XXX], the improved tanh $\left(\frac{\emptyset}{2}\right)$ procedure [XXXV], [XXXVI], the Lie symmetry approach [XXII], [XXIII], the sine-Gordon procedure [XXI], the generalized Kudryashov technique [XIX], the two variables $(G'/G\,,1/G)$-expansion approach [V], the generalized exponential rational function (GERF) scheme [XXIV], the modified sub-equation process [XXXIII], and there are many more. Miah et al. [XXXII] applied the $(G'/G\,,1/G)$-expansion strategy for exact traveling wave solutions of NLPDEs, and obtained kink, periodic, singular periodic, anti-bell, and many other shaped solitons. Recently, some scholars also utilized this expansion technique and discovered closed-form wave solutions [VII], [XI], [XVIII], [XXXI], [XLII]. In [XVII], [XX] researchers operated the novel $(G'/(G'+G+A))$-

expansion process to attain the solutions of NLEEs. Very lately, Borhan et al. [X] executed this new $(G'/(G'+G+A))$ -expansion scheme for two NLEEs and established various analytic solutions.

The $(3+1)$-dimensional Geng equation (GE) [I], [XIII],

$$3u_{xz}-(2u_t+u_{xxx}-2uu_x)_y+2\left(u_x\partial_x^{-1}u_y\right)_x=0, \tag{1.1}$$

was originally introduced in [XIII] as part of the investigation of the algebraic geometrical solutions. The GE contains the nonlinear effect $u_x\partial_x^{-1}u_y$ and the dispersion effect u_{xz} and for this, the equation represents a nonlinear, nonlocal wave equation arising in integrable systems, optical physics, and fluid mechanics. This equation is significant because it illustrates nonlinear wave phenomena that are essential in many areas of physics and applied sciences, including nonlinear optics, acoustics, fluid dynamics, high-energy physics and field theory models, plasma physics, geophysics, and astrophysics. A limited number of mathematicians have researched Eq. (1.1); for example, Geng obtained the algebraic geometrical solution of multi-dimensional NLEEs [XIII]. In [I], Ahmed et al. applied the homoclinic breathers approach for the nonlinear (3+1)-dimensional GE and constructed solutions having kink, periodic cross-rational, in addition, M-shape solitons. Li et al. [XXV] considered the GE, besides finding the hybrid soliton as well as breather waves of this model. Lately, dipole, damped periodic, breather, and kink solitons have been observed for the mentioned equation [VIII]. Recently, Ahmed et al. [II] probed the mentioned GE and uncovered distinct outcomes like X-waves, bright, dark lump waves, butterfly waves, etc. While studies exist on the Geng model and related solution methods, a literature review suggests that the nominated model has not yet been considered using both variables $(G'/G\,,1/G)$-expansion strategy and the novel $(G'/(G'+G+A)$ -expansion technique. From this motivation, we analyze the selected equation through these new techniques.

The objectives of our article are to extract the exact solutions of the mentioned equation by engaging two distinct approaches, the $(G'/(G'+G+A))$-expansion technique and the two variables $(G'/G\,,1/G)$-expansion strategy. Using these techniques, we derive exact solutions to the GE, including trigonometric, hyperbolic, rational, and exponential solutions. Moreover, to visualize the exact solutions, we generated the 3D and 2D plots, as well as contour and density plots. Furthermore, studying the different wave behaviors could help gather further information about the mentioned GE.

Our article is structured as follows: Section I presents the introduction. Segment II presents the main procedures of the proposed strategies. Segment III demonstrates the application of these strategies to evaluate the solutions of the considered equation. In segment IV, the consequences are graphically presented and discussed, followed by the conclusions and Acknowledgements in Sections V and VI, respectively.

II. Clarification of the proposed Strategies

This section of the research work details the comprehensive methodology employed to achieve traveling wave solutions to the NLEEs. The NLEEs under consideration are,

$$H\left(u, u_x, u_t, u_y, u_z, u_{xx}, u_{xt}, u_{zt}, u_{xxx}, u_{ytt}, \dots \dots \dots\right) = 0, \tag{2.1}$$

Here, H denotes a polynomial expression involving $u = u(x, y, z, t)$ and its multiple partial derivatives up to a specified order; the dependent variable $u = u(x, y, z, t)$ depends on the self-regulating variables x, y, z, t.

At first, define a new variable μ which represents a combination of all the monitored variables $x, y, z,$ and t,

$$u(x, y, z, t) = v(\mu),\ \mu = x + y + z - st, \tag{2.2}$$

at which s represents a constant wave velocity.

The ordinary differential equation (ODE) can be obtained after utilizing Eq. (2.2) and Eq. (2.1),

$$L(v, v', -sv', v'', -sv'', v''', s^2v''', \dots \dots ..) = 0, \tag{2.3}$$

wherein L is a polynomial of v, and both it and its several ordinary derivatives are functions of the independent variable μ, the primes ($'$) indicate differentiation for μ.

The $(\boldsymbol{G'/(G' + G + A)})$***- expansion Strategy***

NLPDEs are notoriously difficult to solve. A brief summary of the $(G'/(G' + G + A))$- expansion strategy is presented here, aimed at finding precise solutions to NLEEs. For analytical purposes, the outcome of Eq. (2.3),

$$v(\mu) = \sum_{p=0}^{q} b_p \left(\frac{G\prime}{G'+G+A}\right)^p, \tag{2.1.1}$$

Herein, q signifies the degree of the polynomial, which is evaluated by the balancing process, and $G(\mu)$ obeys Eq. (2.1.2),

$$G'' + EG' + FG + AF = 0, \tag{2.1.2}$$

additionally, the coefficients for $\left(\frac{G\prime}{G'+G+A}\right)^p$ $(p = 0,1,2,3, \dots .., q)$ are calculable. The terms b_p $(p = 0,1,2,3, \dots \dots, q), A, E,$ and F represent constants, while G is derived from $A, E, F,$ and Eq. (2.1.2).

Solving Eq. (2.1.2) yielded two possible outcomes, which are,

Outcome 1: For $D = E^2 - 4F > 0$,

$$\left(\frac{G\prime}{G'+G+A}\right) = \frac{C_1\left(E+\sqrt{D}\right)+C_2\left(E-\sqrt{D}\right)e^{\sqrt{D}\mu}}{C_1\left(E+\sqrt{D}-2\right)+C_2\left(E-\sqrt{D}-2\right)e^{\sqrt{D}\mu}}. \tag{2.1.3}$$

Outcome 2: For $D = E^2 - 4F < 0$,

$$\left(\frac{G\prime}{G'+G+A}\right) = \frac{\sin\left(\frac{\sqrt{-D}\mu}{2}\right)\left(EC_2+C_1\sqrt{-D}\right)+\cos\left(\frac{\sqrt{-D}\mu}{2}\right)\left(EC_1-C_2\sqrt{-D}\right)}{\sin\left(\frac{\sqrt{-D}\mu}{2}\right)\left((E-2)C_2+C_1\sqrt{-D}\right)+\cos\left(\frac{\sqrt{-D}\mu}{2}\right)\left((E-2)C_1-C_2\sqrt{-D}\right)}, \tag{2.1.4}$$

wherein, C_1 and C_2 are constants.

The resulting algebraic equations involving $E, F,$ and b_p $(p = 0,1,2,3, \ldots\ldots, q)$ are obtained by equating the coefficients of $\left(\frac{G\prime}{G'+G+A}\right)^p$ $(p = 0,1,2,3, \ldots\ldots, q)$ to zero.

Following these algebraic steps, the desired solution of the given NLEEs can be quickly obtained by inserting the values found for the b_p's and s.

The two-variable $(G'/G\,, 1/G)$ -expansion strategy

Many real-world phenomena, especially in engineering, quantum mechanics, fluid dynamics, and plasma physics, are governed by nonlinear PDEs. The strategy is specifically tailored to handle the complexities introduced by nonlinearity. This section succinctly describes the fundamental procedures of the two variables $(G'/G\,, 1/G)$-expansion strategy for generating new exact wave solutions related to NLEEs. The associated auxiliary linear ordinary differential equation (LODE) is given by,

$$G''(\mu) + \lambda G(\mu) = \tau. \tag{2.2.1}$$

For $T = \frac{G\prime}{G}, M = \frac{1}{G}$This equation yields the two relations shown below,

$$T' = -T^2 + \tau M - \lambda, \quad M' = -TM. \tag{2.2.2}$$

Three possibilities are identified by the solution to the previous Eq. (2.2.1), which is dependent on λ,

Scenario 1: For $\lambda < 0$, we have,

$$G(\mu) = K_1\, sinh\left(\sqrt{-\lambda}\mu\right) + K_2 cosh\left(\sqrt{-\lambda}\mu\right) + \frac{\tau}{\lambda}, \tag{2.2.3}$$

K_1 and K_2 are two constants chosen arbitrarily. As a consequence, we obtain

$$M^2 = \frac{-\lambda(T^2 - 2\tau M + \lambda)}{\lambda^2 \rho_1 + \tau^2} \text{ and } \rho_1 = {K_1}^2 - {K_2}^2. \tag{2.2.4}$$

Scenario 2: For $\lambda > 0$, we have,

$$G(\mu) = K_1\, sin\left(\sqrt{\lambda}\mu\right) + K_2 cos\left(\sqrt{\lambda}\mu\right) + \frac{\tau}{\lambda}, \tag{2.2.5}$$

and therefore,

$$M^2 = \frac{\lambda(T^2 - 2\tau M + \lambda)}{\lambda^2 \rho_2 - \tau^2} \text{ and } \rho_2 = {K_1}^2 + {K_2}^2. \tag{2.2.6}$$

Scenario 3: For $\lambda = 0$ we have,

$$G(\mu) = \frac{\tau}{2}\mu^2 + K_1 \mu + K_2, \tag{2.2.7}$$

and hence,

$$M^2 = \frac{(T^2 - 2\tau M)}{{K_1}^2 - 2\tau K_2}. \tag{2.2.8}$$

By applying the $(G'/G\,,1/G)$-expansion strategy, the consequence of Eq. (2.1.3) takes the form,

$$v(\mu) = c_0 + \sum_{i=1}^{q} c_i T^i(\mu) + \sum_{i=1}^{q} d_i T^{i-1}(\mu) M(\mu), \tag{2.2.9}$$

herein, $c_i{}^2 + d_i{}^2 \neq 0$ $(i = 1,2,3, \ldots \ldots q)$, with constants $\lambda, s,\ c_0, c_i, d_i$ $(\ \forall\ i)$ and q is the positive integer that represents the homogeneous balance number. Now, these constants are to be determined using the two variables $(G'/G\,,1/G)$-expansion strategy, with additional explanations available in references [XLI].

III. Applications of the projected approaches

The current section presents the precise, sophisticated, advanced, and exceptionally practical outcomes of the GE, derived using projected approaches. The exact solution for the GE in Eq. (1.1) will be examined in this subsection. Appling the transformation $\boldsymbol{u(x,y,z,t) = w_x(x,y,z,t)}$ in Eq. (1.1), we obtain,

$$\boldsymbol{3w_{xxz} - 2w_{xty} - w_{xxxxy} + 2(w_x w_{xx})_y + 2\left(w_{xx} w_y\right)_x = 0}. \tag{3.1}$$

Currently, Eq. (1.1) is transformed into the resulting ODE by adjusting the formulas in Eq. (2.2) and (3.1),

$$\boldsymbol{(3 + 2s)v''' - v^{(v)} + 4(v'')^2 + 4v'v''' = 0} \quad . \tag{3.2}$$

Utilizing the homogeneous balancing approach, one might have $\boldsymbol{q = 1}$.

Application of $\boldsymbol{(G'/(G' + G + A))}$***-expansion strategy***

The outcomes of Eq. (3.2),

$$\boldsymbol{v(\mu) = b_0 + b_1\left(\frac{G'}{G' + G + A}\right)}, \tag{3.1.1}$$

herein, $\boldsymbol{b_0}$ and $\boldsymbol{b_1}$ are constants.

We have the constants,

$$\boldsymbol{b_0 = b_0,\quad b_1 = 3(-1 + E - F),\ \ s = \frac{1}{2}\left(-3 + E^2 - 4F\right)}.$$

Using the above values, the wave solutions of Eq. (1.1).

Case –I: For $\boldsymbol{D = E^2 - 4F > 0}$,

$$\boldsymbol{v(\mu) = b_0 + 3(-1 + E - F)\,\frac{C_1(E+\sqrt{D}) + C_2(E-\sqrt{D})e^{\sqrt{D}\mu}}{C_1(E+\sqrt{D}-2) + C_2(E-\sqrt{D}-2)e^{\sqrt{D}\mu}}}. \tag{3.1.2}$$

And hence, the exact wave solution of the nonlinear GE,

$$\boldsymbol{u(x,y,z,t) = b_0 + 3(-1 + E - F)\,\frac{C_1(E+\sqrt{D}) + C_2(E-\sqrt{D})e^{\sqrt{D}\left(x+y+z-(1/2)(-3+E^2-4F)t\right)}}{C_1(E+\sqrt{D}-2) + C_2(E-\sqrt{D}-2)e^{\sqrt{D}\left(x+y+z-(1/2)(-3+E^2-4F)t\right)}}}, \tag{3.1.3}$$

wherein, $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ constants are chosen arbitrarily.

Case –II: For $\boldsymbol{D = E^2 - 4F < 0}$,

$$v(\mu) = b_0 + 3(-1 + E - F)\frac{\sin\left(\frac{\sqrt{-D}\mu}{2}\right)(EC_2 + C_1\sqrt{-D}) + \cos\left(\frac{\sqrt{-D}\mu}{2}\right)(EC_1 - C_2\sqrt{-D})}{\sin\left(\frac{\sqrt{-D}\mu}{2}\right)\left((E-2)C_2 + C_1\sqrt{-D}\right) + \cos\left(\frac{\sqrt{-D}\mu}{2}\right)\left((E-2)C_1 - C_2\sqrt{-D}\right)}. \tag{3.1.4}$$

Accordingly, the precise solution of Eq. (1.1),

$$u(x,y,z,t) = b_0 + 3(-1 + E - F)\frac{\sin\left(\frac{\sqrt{-D}\mu}{2}\right)(EC_2 + C_1\sqrt{-D}) + \cos\left(\frac{\sqrt{-D}\mu}{2}\right)(EC_1 - C_2\sqrt{-D})}{\sin\left(\frac{\sqrt{-D}\mu}{2}\right)\left((E-2)C_2 + C_1\sqrt{-D}\right) + \cos\left(\frac{\sqrt{-D}\mu}{2}\right)\left((E-2)C_1 - C_2\sqrt{-D}\right)}, \tag{3.1.5}$$

wherein, $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are arbitrary constants and $\boldsymbol{\mu = x + y + z - (1/2)\left(-3 + E^2 - 4F\right)t}$.

Now, we implement the two variables $(\boldsymbol{G'/G\,, 1/G})$-expansion strategy to solve the (3+1)-dimensional nonlinear GE in Eq. (3.1).

Application of the two variables $(G'/G\,, 1/G)$-expansion strategy

The (3+1) nonlinear GE in Eq. (1.1) moves to an ODE in Eq. (3.2) by using the transformation $\boldsymbol{u(x,y,z,t) = w_x(x,y,z,t)}$ and the transformations in Eq. (2.2). The outcomes of Eq. (3.2),

$$v(\mu) = c_0 + c_1 T(\mu) + d_1 M(\mu), \tag{3.2.1}$$

Herein, the constants $\boldsymbol{c_0, c_1}$ and $\boldsymbol{d_1}$ will be governed, as well as the next three circumstances, will be examined.

Case I: For $\boldsymbol{\lambda < 0}$,

We have the following results,

$$c_0 = c_0,\ c_1 = -\frac{3}{2},\ d_1 = \pm\frac{3\sqrt{-\tau^2 - \lambda^2\rho_1}}{2\sqrt{\lambda}},\ s = (1/2)\,(-3 - \lambda).$$

Now, the outcome of Eq. (3.2),

$$v(\mu) = c_0 - \frac{3\sqrt{-\lambda}\left(K_1\cosh(\sqrt{-\lambda}\mu) + K_2\sinh(\sqrt{-\lambda}\mu)\right)}{2\left(K_1\sinh(\sqrt{-\lambda}\mu) + K_2\cosh(\sqrt{-\lambda}\mu) + \frac{\tau}{\lambda}\right)} \pm \frac{3\sqrt{-\tau^2 - \lambda^2\rho_1}}{2\sqrt{\lambda}\left(K_1\sinh(\sqrt{-\lambda}\mu) + K_2\cosh(\sqrt{-\lambda}\mu) + \frac{\tau}{\lambda}\right)} \tag{3.2.2}$$

Thus, the outcome of Eq. (1.1),

$$u(x,y,z,t) = c_0 - \frac{3\sqrt{-\lambda}\left(K_1\cosh\left(\sqrt{-\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + K_2\sinh\left(\sqrt{-\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right)\right)}{2\left(K_1\sinh\left(\sqrt{-\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + K_2\cosh\left(\sqrt{-\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + \frac{\tau}{\lambda}\right)} \pm \frac{3\sqrt{\tau^2 + \lambda^2\rho_1}}{2\sqrt{-\lambda}\left(K_1\sinh\left(\sqrt{-\lambda}(x+y+z-1/2(-3-\lambda)t)\right) + K_2\cosh\left(\sqrt{-\lambda}(x+y+z-1/2(-3-\lambda)t)\right) + \frac{\tau}{\lambda}\right)} \tag{3.2.3}$$

wherein, $\boldsymbol{\rho_1 = K_1{}^2 - K_2{}^2}$.

In a special case, if $\boldsymbol{K_1 \neq 0}$, $\boldsymbol{K_2 = 0}$ and $\boldsymbol{\tau = 0}$ in Eq. (3.2.3), the solitary wave solution,

$$\boldsymbol{u(x,y,z,t) = c_0 - \frac{3}{2}\sqrt{-\lambda}\left(coth\left(\sqrt{-\lambda}(x+y+z-1/2\,(-3-\lambda)t)\right) \pm cosech\left(\sqrt{-\lambda}(x+y+z-1/2\,(-3-\lambda)t)\right)\right)}. \tag{3.2.4}$$

Case-II: For $\boldsymbol{\lambda > 0}$,

We have the findings,

$$\boldsymbol{c_0 = c_0,\ c_1 = -\frac{3}{2}, d_1 = \pm\frac{3\sqrt{-\tau^2+\lambda^2\rho_2}}{2\sqrt{\lambda}}, s = 1/2\,(-3-\lambda)}.$$

Now, the following form represents the solution to Eq. (3.2),

$$\boldsymbol{v(\mu) = c_0 - \frac{3\sqrt{\lambda}\left(K_1\,cos(\sqrt{\lambda}\mu) - K_2 sin(\sqrt{\lambda}\mu)\right)}{2\left(K_1\,sin(\sqrt{\lambda}\mu) + K_2 cos(\sqrt{\lambda}\mu) + \frac{\tau}{\lambda}\right)} \pm \frac{3\sqrt{-\tau^2+\lambda^2\rho_2}}{2\sqrt{\lambda}\left(K_1\,sin(\sqrt{\lambda}\mu) + K_2 cos(\sqrt{\lambda}\mu) + \frac{\tau}{\lambda}\right)}}. \tag{3.2.5}$$

Thus, the exact solution of Eq. (1.1),

$$\boldsymbol{u(x,y,z,t) = c_0 - \frac{3\sqrt{\lambda}\left(K_1\,cos\left(\sqrt{\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) - K_2 sin\left(\sqrt{\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right)\right)}{2\left(K_1\,sin\left(\sqrt{\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + K_2 cos\left(\sqrt{\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + \frac{\tau}{\lambda}\right)} \pm \frac{3\sqrt{-\tau^2+\lambda^2\rho_2}}{2\sqrt{\lambda}\left(K_1\,sin\left(\sqrt{\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + K_2 cos\left(\sqrt{\lambda}(x+y+z-(1/2)(-3-\lambda)t)\right) + \frac{\tau}{\lambda}\right)}}, \tag{3.2.6}$$

wherein, $\boldsymbol{\rho_2 = K_1{}^2 + K_2{}^2}$.

In a special case, if $\boldsymbol{K_1 \neq 0}$, $\boldsymbol{K_2 = 0}$ and $\boldsymbol{\tau = 0}$ in Eq. (3.2.6), the solitary wave solution,

$$\boldsymbol{u(x,y,z,t) = c_0 - \frac{3}{2}\sqrt{\lambda}\left(cot\left(\sqrt{\lambda}(x+y+z-(1/2)\,(-3-\lambda)t)\right) \pm cosec\left(\sqrt{\lambda}(x+\ y+z-(1/2)\,(-3-\lambda)t)\right)\right)}, \tag{3.2.7}$$

whereas, if $\boldsymbol{K_1 = 0}$, $\boldsymbol{K_2 \neq 0}$ and $\boldsymbol{\tau = 0}$ in Eq. (3.2.6), the solitary wave solution,

$$\boldsymbol{u(x,y,z,t) = c_0 + \frac{3}{2}\sqrt{\lambda}\left(tan\left(\sqrt{\lambda}(x+y+z-(1/2)\,(-3-\lambda)t)\right) \pm sec\left(\sqrt{\lambda}(x+\ y+z-(1/2)\,(-3-\lambda)t)\right)\right)}. \tag{3.2.8}$$

Case-III: For $\boldsymbol{\lambda = 0}$,

We get the following outcomes,

$$\boldsymbol{c_0 = c_0,\ c_1 = -\frac{3}{2}, d_1 = \pm\frac{3}{2}\sqrt{K_1^2 - 2\tau K_2}, s = -\frac{3}{2}}.$$

Consequently, the consequence of Eq. (3.2),

$$\boldsymbol{v}(\boldsymbol{\mu}) = \boldsymbol{c_0} - \frac{3(\tau\mu + K_1)}{\tau\mu^2 + 2(K_1\mu + K_2)} \pm \frac{3\sqrt{K_1^2 - 2\tau K_2}}{\tau\mu^2 + 2(K_1\mu + K_2)}. \tag{3.2.9}$$

Thus, the solution for Eq. (1.1) is given by,

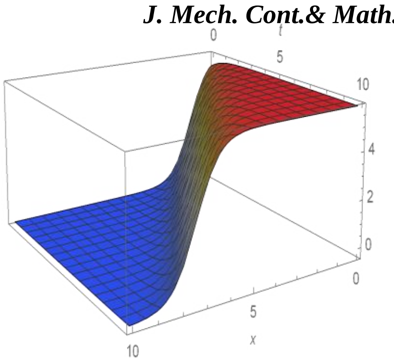
$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{t}) = \boldsymbol{c_0} - \frac{3(\tau(x+y+z+(3/2)t)+K_1)}{\tau(x+y+z+(3/2)t)^2 + 2(K_1(x+y+z+(3/2)t)+K_2)}$$

$$\pm \frac{3\sqrt{K_1^2 - 2\tau K_2}}{\tau(x+y+z+(3/2)t)^2 + 2(K_1(x+y+z+(3/2)t)+K_2)}. \tag{3.2.10}$$

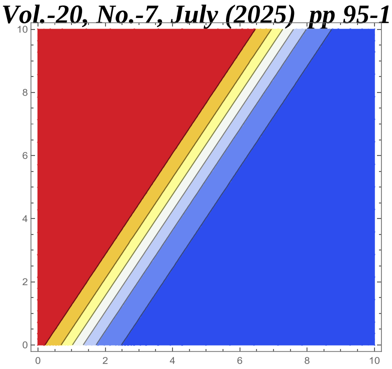
Remark: In this case, by choosing arbitrary values for $\boldsymbol{K_1}$and $\boldsymbol{K_2}$, two additional solitary wave solutions of the GE can be found, but we have excluded them.

IV. Graphical depiction and discussion of results

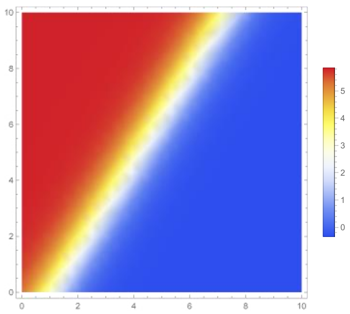
In essence, the selected equation provides a framework for understanding how systems in nature behave when nonlinearity plays a significant role, leading to phenomena such as solitons, wave interactions, and complex dynamics. We illustrate and explain the physical aspects of selected exact wave solutions discussed in our study. Since the functioning of exact solutions is contingent upon their graphical representations, we demonstrate several types of solitons, including kink-shaped, singular anti-kink-shaped, compacton-shaped, and singular periodic shape solitons. In this section, we scrutinize the results obtained, focusing on only five solutions for simplicity. Initially, we display the outcome in Eq. (3.1.3) with four different formats, and acquire the kink-shaped solution within the interval $x, t \in [0, 10]$, which are depicted in Figure 1, including the parameters $b_0 = -1$, $E = 0.5$, $F = -1$, $C_1 = 1$, $C_2 = -1$, $y = -1$, $z = -1$ and $t = 1,2,3$. The solutions in Eq. (3.1.5) and Eq. (3.2.6) reveal singular periodic-shaped solitons, which are given in Figures 2 and 5, respectively, for the parameters $b_0 = -1$, $E = 0.1$, $F = 0.1$, $C_1 = 1$, $C_2 = -1$, $y = -1$, $z = -1$ $t = 1,2,3$ and $c_0 = -0.1$, $\lambda = 1.5$, $\tau = 0.005$, $K_1 = -0.1$, $K_2 = -0.1$, $y = 0.1$, $z = 0.1$ $t = 1,2,3$ within the intervals $-10 < x < 10$, $-10 < t < 10$ and $0 < x < 10$, $0 < t < 10$ respectively. In Figure 3, the compacton-shaped solution for Eq. (3.2.3) is presented within $-10 < x < 0$ and $-10 < t < 0$ for the parameters $c_0 = -1$, $\lambda = -1$, $\tau = 2$, $K_1 = 0.02$, $K_2 = -0.01$, $y = 1$, $z = 1$ and $t = 1,2,3$. In Figure 4, the singular anti-kink-shaped soliton of Eq. (3.2.4) for parameters $c_0 = 1$, $\lambda = -4$, $y = -1$, $z = -1$ and $t = 1,2,3$.



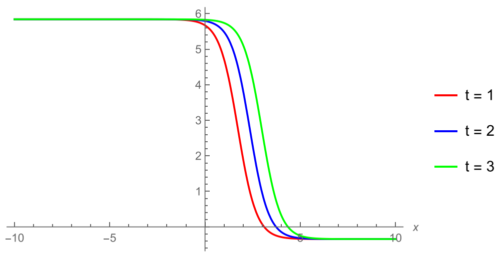
(a) 3D



(b) Contour

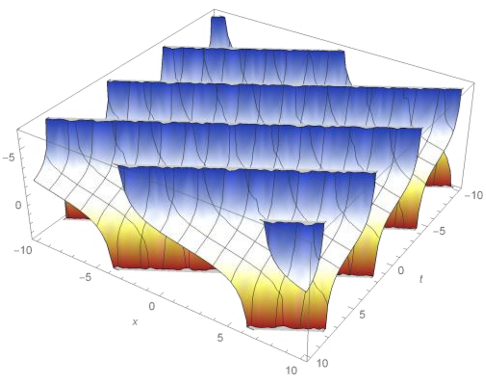


(c) Density

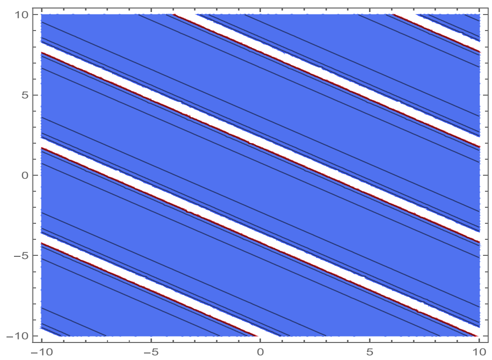


(d) 2D

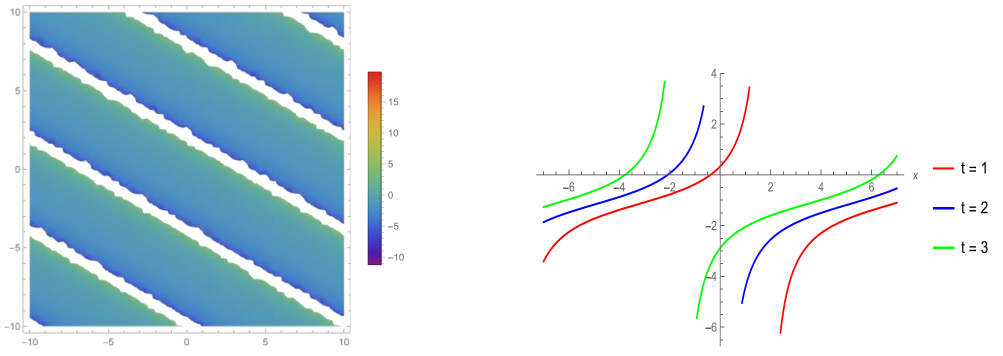
Fig. 1. The kink-shaped soliton of Eq. (3.1.3) together with appropriate values of the parameters.



(a) 3D

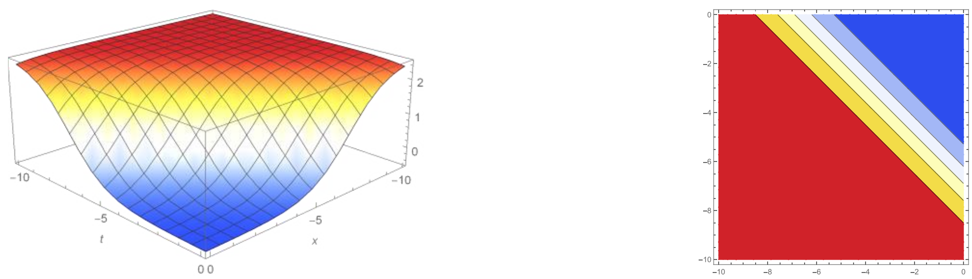


(b) Contour

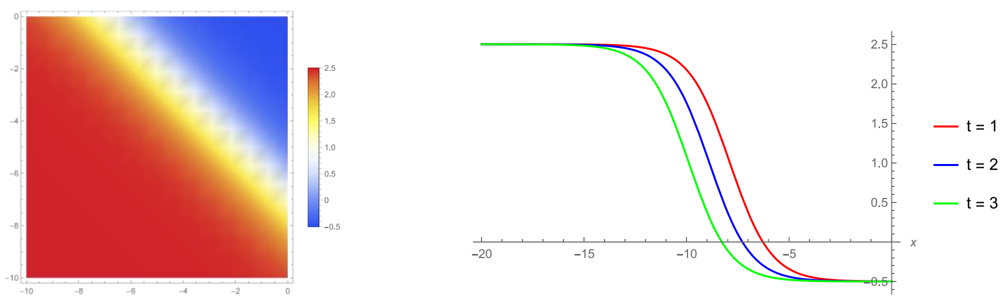


(c) Density (d) 2D

Figure 2. Singular periodic-shaped soliton of Eq. (3.1.5) having proper values of the parameters.

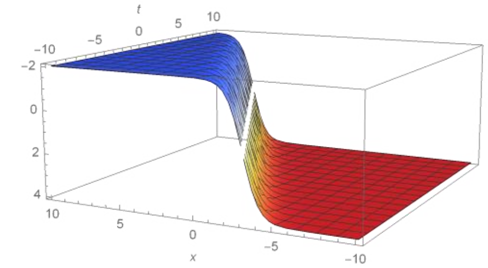


(a) 3D (b) Contour

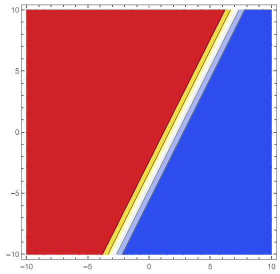


(c) Density (d) 2D

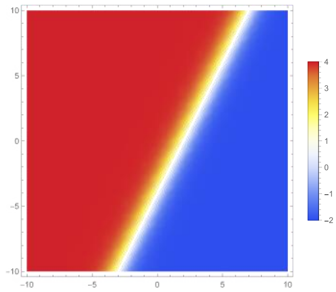
Fig. 3. Compacton-shaped soliton of Eq. (3.2.3) along with appropriate values of the parameters.



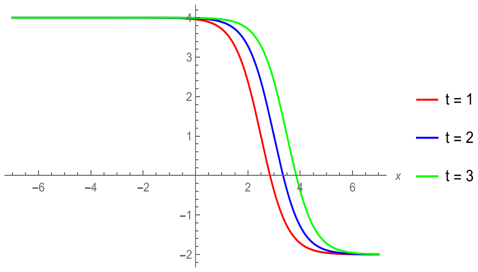
(a) 3D



(b) Contour

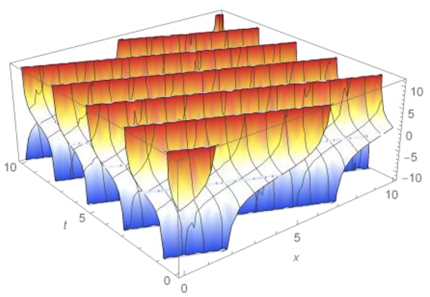


(c) Density

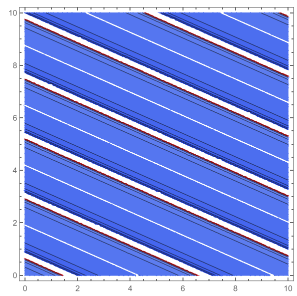


(d) 2D

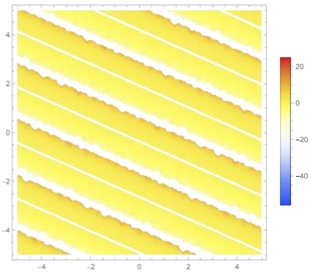
Fig. 4. Singular anti-kink-shaped soliton of Eq. (3.2.4) together with suitable values of the parameters.



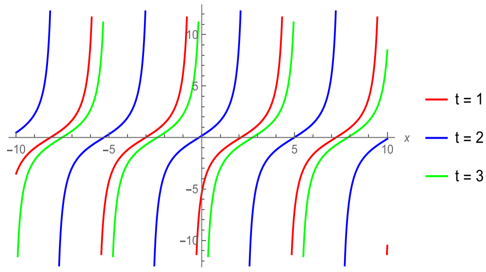
(a) 3D



(b) Contour



(c) Density



(d) 2D

Fig. 5. Singular periodic wave soliton of Eq. (3.2.6) with suitable values of the parameters.

By utilizing contour, density, 3D together with 2D plots of the soliton solutions of GE, we successfully present accurate physical behavior. We have determined that the solutions of this equation are novel compared to those obtained in previous literature.

Results comparison:

Here, we associate our consequences with the existing outcomes of our considered GE in the following part,

Table 1: Comparison of our findings with others' existing outcomes

Author(s)	Method(s)	Outcomes
Ahmed, Sarfaraz, et al.[I]	The symbolic computation with ansatz functions technique, the logarithmic transformation, and the Homocentric breathers approach	periodic cross-rational, kink cross-rational, and M-shape solitons
Ahmed, Sarfaraz, et al. [II]	The Hirota bilinear and the Cole-Hopf transformation techniques	lump, kink, periodic, butterfly, and X-waves
Li, Bang-Qing, et al. [XXV]	The Hirota bilinear method	hybrid soliton, and breather waves
Our research work	the $\left(\frac{G'}{G'+G+A}\right)$-expansion technique, and the two variables $(G'/G\,,1/G)$-expansion strategy	compacton, singular-periodic shaped, kink-shaped, singular anti-kink-shaped

The aforementioned comparison confirms to us that our extracted solutions are novel.

V. Conclusion

The $(3+1)$-dimensional nonlinear GE is noteworthy in numerous physical contexts due to its ability to model complex, nonlinear phenomena in high-dimensional spaces comprising nonlinear optics, acoustics, fluid dynamics, field theory models, geophysics, and astrophysics. In this article, new solutions have been constructed for the Geng model by employing the proposed strategies, including different free parameters. To explain internal behavior, the solitons are visually depicted in 3D, contour, density, and 2D plots. The obtained solutions signify the kink-shaped soliton, singular periodic soliton, compacton-shaped soliton, singular anti-kink-shaped soliton, etc. It is significant to point out that some of the resulting solutions had not been recorded in previous studies. The outcomes demonstrated that the analyzed strategies yield prospective and strong mathematical frameworks that decrease computational difficulty and furnish an effective theoretical approach. These strategies have the potential to examine further NLEEs that recurrently occur in engineering, fluid mechanics, plasma science, and mathematical physics, and this is our next research initiative.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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