JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-20, No.-3, March (2025) pp 198 - 207 ISSN (Print) 0973-8975

COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF QUASI-CONVEX FUNCTIONS ASSOCIATED WITH CARLSON-SHAFFER OPERATOR

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https://doi.org/10.26782/jmcms.2025.03.00013

(Received: December 10, 2024; Revised: February 20, 2025; Accepted: March 3, 2025)

Abstract

Let Υ denote the class of functions $\chi(\xi)$ of the form $\chi(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n$ which are analytic in the open unit disc $\Delta = \{\xi \in \mathbb{C}: |\xi| < 1\}$. In recent times investigating the properties of several existing and new subclasses of quasi-convex functions have gained importance and attracted researchers working in the theory of univalent functions. Using the Carlson-Shaffer operator, we introduce new subclasses of quasi-convex functions. The coefficient bounds for functions belonging to the defined function classes are our main results. Further, we establish various well-known results as corollaries to our main results.

Keywords : Analytic function, quasi-convex, close to convex, close to star, Janowski function, coefficient estimates, Carlson-Shaffer operator.

I. Introduction

Let Υ denote the class of functions

$$\chi(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots \tag{1}$$

analytic in the open unit disc $\Delta = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$ and normalized by $\chi(0) = 0, \chi'(0) = 1$. Let S denote the subclass of Υ which are univalent in Δ. Let ϕ_1 and ϕ_2 be analytic functions in the unit disk Δ. A function ϕ_1 is said to be subordinate to ϕ_2, written as $\phi_1 \prec \phi_2$ if there exists a Schwarz function $w: \Delta \to \Delta$ with $w(0) = 0$ and $|w(\xi)| < 1$ such that $\phi_1(\xi) = \phi_2(w(\xi))$. In particular, if ϕ_2 is univalent in Δ,

$\phi_1(\xi) \prec \phi_2(\xi)$ if and only if $\phi_1(0) = \phi_2(0)$ and $\phi_1(\Delta) \subset \phi_2(\Delta)$.

Let S^* and K denote the subclass of S consisting of functions starlike and convex in Δ respectively. Starlike and convex functions of order γ, $0 \le \gamma < 1$ will be denoted by $S^*(\gamma)$ and $K(\gamma)$ respectively. We observe that $S^*(0) = S^*$, $K(0) = K$. The classes $S^*(\mathcal{R},\mathcal{T})$ and $K(\mathcal{R},\mathcal{T})$ was introduced by Janowski [V] are defined as follows:

$$S^*(\mathcal{R},\mathcal{T}) = \left\{\chi \in \Upsilon : \frac{\xi\, \chi'(\xi)}{\chi(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}\right\},\ K(\mathcal{R},\mathcal{T}) = \left\{\chi \in \Upsilon : 1 + \frac{\xi\, \chi''(\xi)}{\chi'(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}\right\},$$

where $\xi \in \Delta, -1 \le \mathcal{T} < \mathcal{R} \le 1$. We note that $S^*(\gamma) = S^*(1-2\gamma, 1)$, $S^* = S^*(1,-1)$, $K(\gamma) = K(1-2\gamma,1)$ and $K = K(1,-1)$.

A function $\chi \in \Upsilon$ is said to be close-to-convex if and only if there exists $\psi \in K$ such that $\Re\left[\frac{\chi'(\xi)}{\psi'(\xi)}\right] > 0$ for all $\xi \in \Delta$. The class CK of close-to-convex functions was introduced by Kaplan in [VI]. A function $\chi \in \Upsilon$ is said to be close to star if and only if there exists $\psi \in S^*$ such that $\Re\left[\frac{\chi(\xi)}{\psi(\xi)}\right] > 0$ for all $\xi \in \Delta$. The class CS of close-to-star functions was introduced by Reade in [XVI]. The class $CK(\gamma)$ of close-to-convex functions of order γ was introduced by Goodman in [IV]. The class of close-to-star functions of order γ is referred to as $CS(\gamma)$. We notice that $CK = CK(0)$ and $CS(0) = CS$.

A function $\chi \in \Upsilon$ is Janowski-type close-to-convex in Δ [IX], if there is a convex function $\psi(\xi)$ such that

$$\frac{\chi'(\xi)}{\psi'(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi},$$

where $\xi \in \Delta, -1 \le \mathcal{T} < \mathcal{R} \le 1$. This class is denoted by $CK(\mathcal{R},\mathcal{T})$.

A function $\chi \in \Upsilon$ is Janowski-type close-to-star in Δ [X, XI] if there is a starlike function $\psi(\xi)$ such that

$$\frac{\chi(\xi)}{\psi(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi},$$

where $\xi \in \Delta, -1 \le \mathcal{T} < \mathcal{R} \le 1$. We denote this class by $CS(\mathcal{R},\mathcal{T})$.

A function $\chi \in \Upsilon$ is said to be quasi-convex if there exists a convex function $\psi(\xi)$ such that $\Re\left\{\left(\xi\, \chi'(\xi)\right)' / \psi'(\xi)\right\} > 0$ for all $\xi \in \Delta$. Further, every quasi-convex function is univalent. The class of quasi-convex functions is denoted by Q. The class of quasi-convex introduced by Noor and Thomas in [XIII]. Similarly, we denote the class of quasi-convex functions of order γ and type 0 [XIV] by $Q(\gamma)$, where $0 \le \gamma < 1$.

A function $\chi(\xi)$ in the form (1) is Janowski-type quasi-convex in Δ, there is a convex function $\psi(\xi)$ such that

$$\frac{\left(\xi\, \chi'(\xi)\right)'}{\psi'(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi},$$

where $\xi \in \Delta, -1 \leq \mathcal{T} < \mathcal{R} \leq 1$. This class denoted by $QC(\mathcal{R},\mathcal{T})$ was investigated in [XXII].

A function $\chi(\xi)$ in the form (1) is in the class $QC^{CK}(\beta,\mathcal{R},\mathcal{T})$ if there exists a function $\psi(\xi) \in K$ such that

$$\frac{\chi'(\xi) + \beta\, \xi\, \chi''(\xi)}{\psi'(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}, \qquad 0 \leq \beta \leq 1,$$

where $\xi \in \Delta, -1 \leq \mathcal{T} < \mathcal{R} \leq 1$. A function $\chi(\xi)$ in the form (1) is in the class $QC^{CS}(\beta,\mathcal{R},\mathcal{T})$ if there exists a function $\psi(\xi) \in S^*$ such that

$$\frac{(1-\beta)\chi(\xi) + \beta\, \xi\, \chi'(\xi)}{\psi(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}, \qquad 0 \leq \beta \leq 1,$$

where $\xi \in \Delta, -1 \leq \mathcal{T} < \mathcal{R} \leq 1$. The classes $QC(\mathcal{R},\mathcal{T})$, $QC^{CK}(\beta,\mathcal{R},\mathcal{T})$ and $QC^{CS}(\beta,\mathcal{R},\mathcal{T})$ was introduced by Altıntas et al. [I].

Motivated by the works of Altıntas et al. [I] and Karthikeyan et al. [VII, VIII], we introduce two new classes $\mathcal{L}_{u,v}^{CK}(\alpha,\beta,\mathcal{R},\mathcal{T})$ and $\mathcal{L}_{u,v}^{CS}(\alpha,\beta,\mathcal{R},\mathcal{T})$ using Carlson Shaffer operator [II], where the Carlson-Shaffer linear operator $L_{u,v}\colon \Upsilon \longrightarrow \Upsilon$ is given by

$$L_{u,v}\chi(\xi) = \varphi(u,v;\xi) * \chi(\xi) = \xi + \textstyle\sum_{n=2}^{\infty} (k)_{n-1}\, a_n\, \xi^n, \qquad (k)_n = \frac{(u)_n}{(v)_n}. \tag{2}$$

The function $\varphi(u,v;\xi) = \xi + \sum_{n=2}^{\infty} \frac{(u)_{n-1}}{(v)_{n-1}} \xi^n$, $u \in \mathbb{C}, v \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$ is known as the incomplete beta function. The term $(v)_s$ is the Pochhammer symbol that can be expanded in Gamma function as

$$(v)_n = \begin{cases} 1, & (n=0) \\ v(v+1)\dots(v+n-1), & (n \in \mathbb{N}) \end{cases} \tag{3}$$

A function $\chi(\xi) \in \Upsilon$ is said to belong to the class $\mathcal{L}_{u,v}^{CK}(\alpha,\beta,\mathcal{R},\mathcal{T})$ if

$$\frac{\alpha\,[L_{u,v}\,\chi(\xi)]' + \beta\,\xi [L_{u,v}\,\chi(\xi)]''}{\alpha\,\psi'(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}, \tag{4}$$

where $\alpha > 0$, $0 \leq \beta \leq 1$, $-1 \leq \mathcal{T} < \mathcal{R} \leq 1$, $\psi(\xi) \in K$ and $L_{u,v}\chi(\xi)$ is defined by (2).

Also a function $\chi(\xi)$ belonging to Υ is said to be in the class $\mathcal{L}_{u,v}^{CS}(\alpha,\beta,\mathcal{R},\mathcal{T})$ if

$$\frac{\alpha\, L_{u,v}\,\chi(\xi) + \beta\,\xi [L_{u,v}\,\chi(\xi)]'}{(\alpha+\beta)\psi(\xi)} \prec \frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}, \tag{5}$$

where $\alpha > 0$, $0 \leq \beta \leq 1$, $-1 \leq \mathcal{T} < \mathcal{R} \leq 1$, $\psi(\xi) \in S^*$ and $L_{u,v}\chi(\xi)$ is defined by (2).

Remark 1.

(i) $\mathcal{L}_{1,1}^{CK}(1,\beta,\mathcal{R},\mathcal{T})$ is the class $QC^{CK}(\beta,\mathcal{R},\mathcal{T})$ introduced by Altıntas et al. [I].

(ii) $\mathcal{L}_{1,1}^{CK}(1,1,\mathcal{R},\mathcal{T})$ is the Janowski-type quasi-convex class $QC(\mathcal{R},\mathcal{T})$ introduced by Altıntas et al. [I].

(iii) $\mathcal{L}_{1,1}^{CK}(1,1,1-2\gamma,-1)$ is the class $Q(\gamma)$ of quasi-convex functions of order γ and type 0 [XIV].

(iv) $\mathcal{L}_{1,1}^{CK}(1,1,1,-1)$ is the class Q of quasi-convex functions introduced by Noor and Thomas [XIII].

(v) $\mathcal{L}_{1,1}^{CK}(1,0,\mathcal{R},\mathcal{T})$ is the Janowski-type close-to-convex class $CK(\mathcal{R},\mathcal{T})$ studied by Mehrok and Singh [IX].

(vi) $\mathcal{L}_{1,1}^{CK}(1,0,1-2\gamma,1)$ is the class $CK(\gamma)$ of close-to-convex functions of order γ studied by Goodman [IV].

(vii) $\mathcal{L}_{1,1}^{CK}(1,0,1,-1)$ is the class CK of close-to-convex functions introduced by Kaplan [VI].

(viii) $\mathcal{L}_{1,1}^{CK}(1-\beta,\beta,\mathcal{R},\mathcal{T})$ is the class $QC^{CS}(\beta,\mathcal{R},\mathcal{T})$ studied by Altıntas et al. [I].

(ix) $\mathcal{L}_{1,1}^{CK}(1,0,\mathcal{R},\mathcal{T})$ is the class $CS(\mathcal{R},\mathcal{T})$ Janowski-type close-to star functions [X, XI].

(x) $\mathcal{L}_{1,1}^{CK}(1,0,1-2\gamma,1)=CS(0,1-2\gamma,1)$ [XVIII]. Note that $CS(0,1-2\gamma,1)=CS(\gamma)$ namely $\Re\left\{\frac{\chi(\xi)}{\psi(\xi)}\right\}>\gamma,\ 0\leq\gamma<1$.

(xi) $\mathcal{L}_{1,1}^{CK}(1,0,1,1)$ is the class CS of close-to-starlike functions introduced by Reade [XVI].

In proving our results we need the next lemmas.

Lemma 1:- [XIX] If the function $h(\xi)$ of the form $h(\xi)=1+h_1\xi+h_2\xi^2+h_3\xi^3+\cdots$ is analytic in Δ and $h(\xi)\prec\frac{1+\mathcal{R}\,\xi}{1+\mathcal{T}\,\xi}$, then

$$\Re\, h(\xi)>\frac{1-\mathcal{R}}{1-\mathcal{T}} \text{ and } |h_n|\leq\frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}},$$

where $\xi\in\Delta,\ -1\leq\mathcal{T}<\mathcal{R}\leq 1$.

II. Coefficient estimates of functions in $\mathcal{L}_{u,v}^{CK}(\alpha,\beta,\mathcal{R},\mathcal{T})$

We begin with the following.

Theorem 1 If $\chi(\xi)\in\mathcal{L}_{u,v}^{CK}(\alpha,\beta,\mathcal{R},\mathcal{T})$, then we have

$$|a_n|\leq\frac{\alpha}{[\alpha+\beta\,(n-1)]|(k)_{n-1}|}\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right]. \quad (6)$$

The extremal function $\chi(\xi)$ satisfying the inequality (6) is given by

$$\begin{aligned}\alpha\,[L_{u,v}\,\chi(\xi)]' + \beta\,\xi[L_{u,v}\,\chi(\xi)]'' &= \left[1+\frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\left(\frac{\xi}{1-\xi}\right)\right]\frac{\alpha}{(1-z)} \\ &= \sum_{n=2}^{\infty}\alpha n\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right]\xi^{n-1}.\end{aligned}$$

Proof. Suppose that $\chi \in \mathcal{L}_{u,v}^{CK}(\alpha\,,\beta\,,\mathcal{R},\mathcal{T})$. Then, if there exists a function $\psi(\xi) = \xi + \sum_{n=2}^{\infty} b_n\,\xi^n \in\ K$ and $h(\xi) = 1 + \sum_{n=1}^{\infty} h_n\,\xi^n$ as in Lemma 1 such that

$$\frac{\alpha\,[L_{u,v}\,\chi(\xi)]' + \beta\,\xi[L_{u,v}\,\chi(\xi)]''}{\alpha\,\psi'(\xi)} = h(\xi) \qquad (7)$$

for all $\xi\in\Delta$. From (7), we obtain

$$\alpha\,[L_{u,v}\,\chi(\xi)]' + \beta\,\xi[L_{u,v}\,\chi(\xi)]'' = \alpha\,\psi'(\xi)h(\xi)$$

or

$$\alpha+\sum_{n=2}^{\infty} n[\alpha+\beta\,(n-1)](k)_{n-1}\,a_n\xi^{n-1} = \alpha + \alpha\sum_{n=2}^{\infty}\left(\sum_{m=1}^{\infty} b_{n-m+1}h_{m-1}(n-m+1)\right)\xi^{n-1}\,. \qquad (8)$$

Comparing the coefficient of ξ^{n-1} on both sides of above equation, we have

$$a_n = \frac{\alpha\,[nb_n+(n-1)b_{n-1}h_1+(n-2)b_{n-2}h_2+\cdots+h_{n-1}]}{n[\alpha+\beta\,(n-1)](k)_{n-1}}\,. \qquad (9)$$

Since $\psi(\xi)$ is convex in Δ and $|b_n| \leq\ 1$ and $|h_n| \leq \frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}$ in Lemma 1, we have

$$\begin{aligned}|a_n| &\leq \frac{\alpha}{n[\alpha+\beta\,(n-1)](k)_{n-1}}\left[n+\frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\sum_{p=1}^{n-1}p\right] \\ &= \frac{\alpha}{n[\alpha+\beta\,(n-1)](k)_{n-1}}\left[1+\frac{(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}(n-1)\right].\end{aligned}$$

This completes the proof of Theorem 1.

Corollary 1. If $\chi(\xi) \in\ QC^{CK}(\beta,\mathcal{R},\mathcal{T})$, then we have

$$|a_n| \leq \frac{1}{(1+\beta\,(n-1))}\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right].$$

Proof. The desired result can be obtained by taking $\alpha = u = v = 1$ in Theorem 1.

Corollary 2. If $\chi(\xi) \in\ QC(\mathcal{R},\mathcal{T})$, then we have

$$|a_n| \leq \frac{1}{n}\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right].$$

Proof. The desired result can be obtained on taking $\alpha = \beta = u = v = 1$ in Theorem 1.

Corollary 3. If $\chi(\xi) \in Q(\gamma)$, then we have

$$|a_n| \leq \frac{1}{n}\,[n(1-\gamma)+\gamma\,].$$

Proof. The desired result can be obtained by taking $\alpha = \beta = u = v = 1,\ \mathcal{R} = 1-2\gamma,$ $\mathcal{T} = -1$ in Theorem 1.

Corollary 4. If $\chi(\xi) \in Q$, then we have $|a_n| \leq 1$.

Proof. The desired result can be obtained by taking $\alpha = \beta = u = v = 1,\ \mathcal{R} = 1,\ \mathcal{T} = -1$ in Theorem 1.

Corollary 5. If $\chi(\xi) \in CK(\mathcal{R},\mathcal{T})$, then we have

$$|a_n| \leq \left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right].$$

Proof. The desired result can be obtained by taking $\alpha = u = v = 1$ and $\beta = 0$ in Theorem 1.

Corollary 6. If $\chi(\xi) \in CK(\gamma)$, then we have $|a_n| \leq n(1-\gamma)+\gamma$.

Proof. The desired result can be obtained by taking $\alpha = u = v = 1,\ \beta = 0,\ \mathcal{R} = 1-2\gamma,\ \mathcal{T} = -1$ in Theorem 1.

Corollary 7. If $\chi(\xi) \in CK$, then we have $|a_n| \leq n$.

Proof. The desired result can be obtained by taking $\alpha = u = v = 1, \beta = 0,\ \mathcal{R} = 1,$ $\mathcal{T} = -1$ in Theorem 1.

III. Coefficient bounds of functions in $\mathcal{L}_{u,v}^{CS}(\alpha\,,\beta\,,\mathcal{R},\mathcal{T})$

Theorem 2. If $\chi(\xi) \in \mathcal{L}_{u,v}^{CS}(\alpha\,,\beta\,,\mathcal{R},\mathcal{T})$, then we have

$$|a_n| \leq \frac{(\alpha+\beta)n}{[\alpha+\beta\,n](k)_{n-1}}\left[1\ +\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right]. \tag{10}$$

The extremal function $\chi(\xi)$ satisfying the inequality (10) is given by

$$\alpha\, L_{u,v}\,\chi(\xi)+\beta\,\xi\big[L_{u,v}\,\chi(\xi)\big]' = \left[1+\frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\left(\frac{\xi}{1-\xi}\right)\right]\frac{(\alpha+\beta)\xi}{(1-\xi)^2}$$

$$= \sum_{n=1}^{\infty}(\alpha\ +\beta)\,n\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right]\xi^n.$$

Proof. Suppose that $\chi \in \mathcal{L}_{u,v}^{CS}(\alpha\,,\beta,\ \mathcal{R},\mathcal{T})$. Then, if there exists a function $\psi(\xi) = \xi+\sum_{n=2}^{\infty} b_n\xi^n\ \in S^*$ and $h(\xi) = 1+\sum_{n=1}^{\infty} h_n\,\xi^n$ as in Lemma 1 such that

$$\frac{\alpha\, L_{u,v}\,\chi(\xi)+\beta\,\xi\big[L_{u,v}\,\chi(\xi)\big]'}{(\alpha+\beta)\psi(\xi)} = h(\xi) \tag{11}$$

for all $\xi \in \Delta$. From (11), we obtain

$$\alpha\, L_{u,v}\,\chi(\xi)+\beta\,\xi\big[L_{u,v}\,\chi(\xi)\big]' = (\alpha+\beta)\psi(\xi)\mathrm{h}(\xi)$$

or

$$(\alpha+\beta)\xi+\sum_{n=2}^{\infty}(k)_{n-1}\,(\alpha+n\beta)a_n\xi^n=(\alpha+\beta)\left[\xi+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n}b_{n-k+1}h_{k-1}\right)\xi^n\right].$$

Comparing the coefficient of ξ^n on both sides of the above equation, we have

$$a_n=\frac{\alpha+\beta}{[\alpha+\beta\, n](k)_{n-1}}[b_n+b_{n-1}h_1+b_{n-2}h_2+\ldots+h_{n-1}].$$

Since $\psi(\xi)$ is starlike in Δ and $|b_n|\le n$ and $|h_n|\le\frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}$ in Lemma 1, we have

$$|a_n|\le\frac{(\alpha+\beta)}{[\alpha+\beta\, n](k)_{n-1}}\left[n+\frac{2(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\sum_{p=1}^{n-1}p\right]=\frac{(\alpha+\beta)n}{[\alpha+\beta\, n](k)_{n-1}}\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right].$$

This completes the proof of Theorem 2.

Corollary 8 If $\chi(\xi)\in\ QC^{CS}(\beta,\mathcal{R},\mathcal{T})$, then we have

$$|a_n|\le\frac{n}{[1+\beta\,(n-1)]}\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right].$$

Proof. The desired result can be obtained by taking $\alpha=\ 1-\beta$ in Theorem 2.

Corollary 9. If $\chi(\xi)\in\ CS(\mathcal{R},\mathcal{T})$, then we have

$$|a_n|\le\ n\left[1+\frac{(n-1)(\mathcal{R}-\mathcal{T})}{1-\mathcal{T}}\right].$$

Proof. The desired result can be obtained by taking $\alpha=\ 1,\beta=0$ in Theorem 2.

Corollary 10. If $\chi(\xi)\in\ CS(\gamma)$, then we have

$$|a_n|\le\ n^2(1-\gamma)+n\gamma.$$

Proof. The desired result can be obtained on taking $\alpha=\ 1,\beta\ =0,\mathcal{R}=1-2\gamma,\ \mathcal{T}=-1$ in Theorem 2.

Corollary 11. If $\chi(\xi)\in\ CS$, then we have $|a_n|\le\ n^2$.

Proof. The desired result can be obtained by taking $\alpha=\ 1,\beta\ =0,\mathcal{R}=1,\ \mathcal{T}=-1$ in Theorem 2.

IV. Results and Discussion

Vasil'ev in [XX] applied methods of the theory of univalent functions to the problems of fluid mechanics. Further, Vasil'ev and Markina in [XXI] obtained time evolution of the free boundary of a viscous fluid in the non-zero-surface-tension models for planar flows in Hele-Shaw cells under injection. They used methods of univalent function theory to establish various properties such as directional convexity and starlikeness, which are preserved in time. Mukhamadullina et al in [XII] studied the technique of the Riemann problem with displacement to solve the steady two-dimensional problem of freezing of groundwater flow by two freeze pipes.

From the residue theorem, we learn that coefficient estimates play a very important role in establishing the properties of functions of a complex variable. For example, the coefficient of $(z-a)^{-1}$ in the Laurent's series expansion f is nothing but the residue f at the point of singularity a. That is, coefficient estimates help us to obtain various properties like growth, distortion, and area of the functions. Also, plays a vital role in control theory [III, XV], viscous flow [XX], pure and applied mathematics, as well as in engineering [XVII]. Hence, here we have several interesting coefficient bounds which in turn would help establish various geometric properties.

V. Conclusions

It is well-known that various geometrically defined subclasses satisfy analytic characterization which involves one or higher order derivatives. So studying the functions classes involving the Carlson-Shaffer operator would not only unify and generalize the existing studies but would generate new and attractive subclasses. Here in this current paper, we have introduced two new classes $\mathcal{L}_{u,v}^{CK}(\alpha\,,\beta\,,\mathcal{R},\mathcal{T})$ and $\mathcal{L}_{u,v}^{CS}(\alpha\,,\beta\,,\mathcal{R},\mathcal{T})$ for analytic functions related to the Carlson-Shaffer operator. Coefficient estimates of functions belonging to these two classes are obtained. We have also presented the relevant connection with several known classes in the literature already investigated earlier by many authors. For further study, some new classes of analytic functions using q-calculus operators and q-integral operators may be explored.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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