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OBTAINING A UNIQUE SPLINE FUNCTION TO INTERPOLATE A POLYNOMIAL WITH LACUNARY DATA VALUES

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Abstract

The present paper deals with the problem of obtaining a unique spline function for approximating a polynomial function. We have given values of the polynomial; its first derivatives are at the node points and also the third derivatives are given at the knot points of the unit interval I = [0, 1]. The problem is solved majorly in two parts, the first part shows the unique existence of the interpolatory spline function and the second part deals with the convergence theorem and error bounds. Later we discussed its applications for computer-aided design and image processing also.

Keywords: Approximation, Computer-aided design, Image processing, lacunary Interpolation, Modulus of continuity, Spline functions, Taylor's theorem, error bounds.

I. Introduction

Techniques of Lacunary Interpolation [III] were initially studied in the early 20th century as part of problems involving sparse or selectively missing data. It was motivated by practical applications, such as numerical integration and approximation, where full data sets were unavailable. Afterward, splines were introduced in the 1940s by mathematicians like Schoenberg, who formulated spline functions to address problems in approximation theory, especially for lacunary data [V]. Schoenberg (1946) popularized B-splines, which laid the groundwork for modern spline theory. By the mid-20th century, splines became a preferred tool for interpolation due to their ability to minimize oscillations and provide smooth approximations. Efforts were made to adapt spline theory to address the practical need for robust interpolation in engineering and physical sciences. Studies explored continuity and error analysis in splines under lacunary conditions [VII]. Researchers like De Boor [VI], Ahlberg, and Nilson [IV] worked on computational algorithms for spline interpolation.

The idea of this present research work was initiated from the work of P. Turan's (0, 2) interpolation problem [I]. Later Burkett, J. and Verma, A.K. [2] worked on a similar problem and extended the results. It has been observed that the above-mentioned work shows the construction of spline function when only node values are given, but does not deal with the situations for knot points. To fill this research gap, here in this current research we have shown that the construction of a spline is still possible when we have some data at the knots also i.e. other than the node points.

Here in the present paper we work with the lacunary data (0, 1; 3) in which the function values and first derivatives are known at the node points and 3rd derivatives are known at any intermediate point in the unit Interval I = [0,1]. Using these lacunary data and interpolatory conditions we obtain a spline function which exists uniquely and is also shown convergent by finding error bounds. Further, we discussed its applications in image processing and computer-aided designing. We state the problem as follows:

Let $\Delta: 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a partition of unit interval I = [0, 1] with $x_{k+1} - x_k = h_k$, k = 0, ... , n-1. Denote by $S_{n,5}^{(2)}$ the class of quintic splines s(x) satisfying the condition that $s(x) \in C^3(I)$ and is quintic in each subinterval of I. In the past, this class of splines is used by various authors with different different interpolatory conditions. In [II] this class of splines is used to solve the interpolation problem with the following conditions:

$$s_\Delta(x_k) = f_k \quad , \quad k = 0, \ldots, n;$$

$$s'_\Delta(x_k) = f'_k \quad , \quad k = 0, \ldots, n;$$

$$s'''_\Delta(x_{k+1/3}) = f'''_{k+1/3}, \quad k = 0, \ldots, n-1;$$

where

$$x_{k+1/3} = \frac{1}{3}(x_k + x_{k+1})$$

$$s''_\Delta(x_0) = f''_0 \text{ or } s'''_\Delta(x_0) = f'''_0 \; .$$

Some other authors also solved similar problems with other intermediate points. But the interesting thing is that here in this paper we solved a generalized problem when we take $\frac{1}{5}(0 \leq \frac{1}{5} \leq 1)$ as an intermediate point where third derivatives are prescribed.

This paper consists of four sections section 1 gives an introduction to the whole research work and also consists of relevant history. In section 2 we mention the theorems of unique existence and their proofs along with the related sub-theorems and lemmas. Similar work can be seen in [XIV].

In section 3 we obtained the error bounds for showing the convergence of this spline function. In the last, we discuss the future prospects along with the conclusion.

II. Unique Existence Theorem

Theorem 1

Given a partition Δ of the unit interval I = [0,1] and the numbers f_k ,f_k' , k = 0,1, ... , n-1 ;

$f_{k+\lambda}'''(0\le \frac{1}{5}\le 1)$, k = 0, 1, …, n – 1 ; f_0'', f_0''' ; there exists a unique spline $s_\Delta(x) \in S_{n,5}^{(2)}$ such that

$$\begin{cases} s_\Delta(x_k) = f_k , & k = 0,1,\dots,n; \\ s_\Delta'(x_k) = f_k' , & k = 0,1,\dots,n; \\ s_\Delta'''\left(x_{k+\frac{1}{5}}\right) = f_{k+\frac{1}{5}}''' , & k = 0,1,\dots,n-1; \\ s_\Delta''(x_0) = f_0'' \quad \text{or} \quad s_\Delta'''(x_0) = f_0''' . \end{cases} \tag{2.1}$$

Here

$$x_{k+\frac{1}{5}} = \frac{1}{5}(x_k + x_{k+1}) \text{ and } h_k = x_{k+1} - x_k, \ k = 0, 1, \dots\dots, n-1.$$

Proof of Theorem 1

Here we prove the theorem with the initial condition $s_\Delta''(x_0) = f_0''$ only, for the condition $s_\Delta'''(x_0) = f_0'''$ the similar method can be applied.

Let us set

$$s_\Delta(x) = \begin{cases} s_\Delta(x) \quad when\ x_0 \le x \le x_1 \\ s_k(x) \quad when\ x_k \le\le x_{k+1},\ k = 1,2,\dots\dots,n-1. \end{cases} \tag{2.2}$$

$$s_\Delta(x) = f_0 + (x-x_0)f_0' + \frac{(x-x_0)^2}{2!}f_0'' + \frac{(x-x_0)^3}{3!}a_{0,3} + \frac{(x-x_0)^4}{4!}a_{0,4} + \frac{(x-x_0)^5}{5!}a_{0,5} \tag{2.3}$$

$$s_k(x) = f_k + (x-x_k)f_k' + \frac{(x-x_k)^2}{2!}a_{k,2} + \frac{(x-x_k)^3}{3!}a_{k,3} + \frac{(x-x_k)^4}{4!}a_{k,4} + \frac{(x-x_k)^5}{5!}a_{k,5} \tag{2.4}$$

For determining the coefficients we apply the interpolatory condition (2.1) and the continuity requirements that $s_\Delta(x_k) \in C^2(I)$. Then we have

$$\begin{cases} f_1 = f_0 + h_0 f_0' + \frac{(h_0)^2}{2!} f_0'' + \frac{(h_0)^3}{3!} a_{0,3} + \frac{(h_0)^4}{4!} a_{0,4} + \frac{(h_0)^5}{5!} a_{0,5} \\ f_1' = f_0' + h_0 f_0'' + \frac{(h_0)^2}{2!} a_{0,3} + \frac{(h_0)^3}{3!} a_{0,4} + \frac{(h_0)^4}{4!} a_{0,5} \\ f_{\frac{1}{5}}''' = a_{0,3} + 0.5\, h_0 a_{0,4} + \frac{(0.5\, h_0)^2}{2!} a_{0,5} \end{cases} \quad (2.5)$$

$$\begin{cases} f_{k+1} = f_k + h_k f_k' + \frac{(h_k)^2}{2!} a_{k,2} + \frac{(h_0)^3}{3!} a_{k,3} + \frac{(h_0)^4}{4!} a_{k,4} + \frac{(h_0)^5}{5!} a_{k,5} \\ f_{k+1}' = f_k' + h_k a_{k,2} + \frac{(h_k)^2}{2!} a_{k,3} + \frac{(h_0)^3}{3!} a_{k,4} + \frac{(h_0)^4}{4!} a_{k,5} \\ f_{k+\frac{1}{5}}''' = a_{k,3} + 0.5\, h_k a_{k,4} + \frac{(0.5\, h_0)^2}{2!} a_{k,5} \end{cases} \quad (2.6)$$

k = 1,2, ...,n-2

and

$$\begin{cases} a_{k+1} = a_k + h_k a_{k,3} + \frac{(h_k)^2}{2!} a_{k,4} + \frac{(h_k)^3}{3!} a_{k,5} \\ a_{1,2} = f_0'' + h_0 a_{0,3} + \frac{(h_0)^2}{2!} a_{0,4} + \frac{(h_0)^3}{3!} a_{0,5} \end{cases} \quad (2.7)$$

$$a_{0,5} = -0.2\left[-\frac{192}{h_0^5}\left(f_1 - f_0 - h_0 f_0' - \frac{h_0^2}{2!} f_0''\right) - \frac{120}{h_0^4}\left(f_1' - f_0' - h_0 f_0''\right) + \frac{20 f_{\frac{1}{5}}'''}{h_0^2}\right] \quad (2.8)$$

$$a_{0,4} = -0.2\left[\frac{91.2}{h_0^4}\left(f_1 - f_0 - h_0 f_0' - \frac{h_0^2}{2!} f_0''\right) - \frac{14.4}{h_0^3}\left(f_1' - f_0' - h_0 f_0''\right) - \frac{8 f_{\frac{1}{5}}'''}{h_{0,}}\right] \quad (2.9)$$

$$a_{0,3} = -0.2\left[\frac{-72}{h_0^3}\left(f_1 - f_0 - h_0 f_0' - \frac{h_0^2}{2!} f_0''\right) + \frac{12}{h_0^2}\left(f_1' - f_0' - h_0 f_0''\right) + f_{\frac{1}{5}}'''\right] \quad (2.10)$$

From (2.6) we have,

$$a_{k,5} = -1.7\left[\frac{576}{h_k^5}\left(f_{k+1} - f_k - h_k f_k'\right) + \frac{72}{h_k^4}\left(f_{k+1}' - f_k'\right) + \frac{216}{h_k^3} a_{k,2} + \frac{60}{h_k^2} f_{k+\frac{1}{5}}'''\right] \quad (2.11)$$

$$a_{k,4} = -1.7\left[\frac{273.6}{h_k^4}\left(f_{k+1} - f_k - h_k f_k'\right) - \frac{43.2}{h_k^3}\left(f_{k+1}' - f_k'\right) - \frac{93.6}{h_k^2} a_{k,2} - \frac{24}{h_k} f_{k+\frac{1}{5}}'''\right] \quad (2.12)$$

$$a_{k,3} = 1.7\left[-\frac{331.2}{h_k^3}\left(f_{k+1} - f_k - h_k f_k'\right) + \frac{7.2}{h_k^2}\left(f_{k+1}' - f_k'\right) + \frac{7.2}{h_k} a_{k,2} + 3 f_{k+\frac{1}{5}}'''\right] \quad (2.13)$$

Using values of these coefficients in (2.7) we get

$$a_{1,2} = 5\left[f_0'' + \frac{1}{3} h_0\, f_{0.2}''' - \frac{0.8}{h_0^2}\left(f_1 - f_0 - h_0 f_0'\right) - \frac{0.8}{h_0}\left(f_1' - f_0'\right)\right] \quad (2.14)$$

$$a_{k+1,2} + 5\, a_{k,2} = -1.7 \begin{bmatrix} -\frac{146.4}{h_k^2}(f_{k+1} - f_k - h_k f_k') \\ -\frac{2.4}{h_k}(f_{k+1}' - f_k') + h_k f_{k+\frac{1}{5}}''' \end{bmatrix} \qquad (2.15)$$

The coefficient matrix of the system of equations (2.14) and (2.15) in the unknowns $a_{k,2}$, k = 1,2, … , n-1 is seen to be nonsingular and hence the coefficients $a_{k,2}$, k = 1,2, … , n-1, are uniquely determined and so are, therefore, the coefficients $a_{k,3}$, $a_{k,4}$, $a_{k,5}$, k = 1,2, … ,n-1.

III. Theorem of Convergence

Let $f \in C^l(I), l = 5, 6$. Then for the unique spline $s_\Delta(x)$ of Theorem 1 associated with the function f , we have

$$\left\| s_\Delta^{(5)}(x) - f^{(5)}(x) \right\| \begin{cases} O(\omega_5(H)), & if\ f \in C^5(I), \\ K_3 H \left\| f^{(6)} \right\| + O(\omega_5(H)), & if\ f \in C^6(I), \end{cases} \qquad (3.1)$$

and if

$\frac{\max h_k}{\min h_k} \leq \lambda \leq \infty$and H = $\max_{0 \leq k \leq n-1} h_k$, then

$$\left\| s_\Delta^{(q)}(x) - f^{(q)}(x) \right\| \begin{cases} O(H^{4-q}\omega_5(H)), & if\ f \in C^5(I), \\ K_2 H^{6-q} \left\| f^{(6)} \right\| + O(H^{5-q}\omega_5(H)), & if\ f \in C^6(I), \end{cases} \qquad (3.2)$$

q = 0,1,2,3,4.

Where K_2 and $K_3$are some constants involving λ $(0 \leq \frac{1}{5} \leq 1)$.

Auxiliary Lemmas

Now we give three lemmas that are used to obtain the proof of the Theorem of convergence theorem.

Lemma 3.1.1. Let $A_{k,2} = a_{k,2} - f_k''$.

Then we have for k = 1,2, … ,n-1.

$$|A_{k,2}| = \begin{cases} O(\sum_{\nu=0}^{k-1} h_\nu^3 \omega_5(h_\nu)), & if\ f \in C^5(I) \\ K_1 h_k^4 f^{(6)} + O(\sum_{\nu=0}^{k-1} h_\nu^4 \omega_6(h_\nu)), & if\ f \in C^6(I) \end{cases}$$

Where $K_1 = -0.005$

ProofFrom (2.15) we have

$A_{k+1,2} + 0.5\, A_{k,2} = (a_{k+1,2} - f_{k+1}'') + 0.5(a_{k,2} - f_k'') = \alpha_k$(say), k = 1,2, …… , n-2. (3.1.1)

$$\alpha_k = -1.7 \left[-\frac{2.4}{h_k^2}(f_{k+1} - f_k - h_k f_k') - \frac{2.4}{h_k}(f_{k+1}' - f_k') + h_k f_{k+\frac{1}{5}}''' \right] - [f_{k+1}'' + 5\, f_k'']$$

If $f \in C^5(I)$then by Taylor's formula

$$\alpha_k = O\left(h_k^3 \omega_5(h_k)\right). \tag{3.1.2}$$

Similarly if $if\ f \in C^5(I),$ then

$$\alpha_k = K_1 h_k^4 f_k^{(6)} + O\left(h_k^4 \omega_6(h_k)\right). \tag{3.1.3}$$

Also from (2.14)

$$|A_{1,2}| = |a_{1,2} - f_1''| = \begin{cases} O\left(h_0^3 \omega_5(h_\nu)\right),\ if\ f \in C^5(I) \\ K_2 h_0^4 f^{(6)} + O\left(h_0^4 \omega_6(h_\nu)\right),\ \ if\ f \in C^6(I) \text{Where} K_2 = -0.005 \end{cases} \tag{3.1.4}$$

From (3.1.1) and (3.1.2) and the derivatives for α_k we have

$$|A_{k,2}| = \begin{cases} O\left(\sum_{\nu=0}^{k-1} h_\nu^3 \omega_5(h_\nu)\right),\ if\ f \in C^5(I) \\ K_1 h_k^4 f^{(6)} + O(h_k^4 \omega_6(h_k)),\ \ if\ f \in C^6(I) \end{cases}$$

This proves the assertion of lemma.

Lemma 3.1.2.

Let $A_{k,4} = a_{k,4} - f_k^{(4)}$and $\frac{\max h_k}{\min h_k} \leq 0.2 \leq \infty$, $H = \max_{0 \leq k \leq n-1} h_k$.

Then we have for

k = 0,1, …, n-1. $|A_{k,4}| = \begin{cases} O\left(\omega_5(H)\right),\ if\ f \in C^5(I) \\ K_2 H^2 \|f^{(6)}\| + O(H\omega_6(H)),\ \ if\ f \in C^6(I) \end{cases}$

Where $K_2 = \frac{1}{5}$

Proof From (2.9) and (2.12) we see $A_{0,2} = 0$, then

$$A_{k,4} = a_{k,4} - f_k^{(4)} = \frac{156}{h_k^2} A_{k,2} + \beta_k,\ k = 0,1, \dots, n-1 \tag{3.1.5}$$

Where$\beta_k = -1.7\left[\frac{273.6}{h_k^4}(f_{k+1} - f_k - h_k f_k') - \frac{43.2}{h_k^3}(f_{k+1}' - f_k') - \frac{93.6}{h_k^2} f_k'' - \frac{24}{h_k} f_{k+\frac{1}{5}}'''\right]$

$$\beta_k = O\left(h_k \omega_5(H)\right),\ if\ f \in C^5(I). \tag{3.1.6}$$

If $f \in C^6(I)$, then (3.1.7) $\beta_k = K_2 h_k^2 f_k^{(6)} + O(h_k^2 \omega_6(h_k))$, where K_2=0.2

Using Lemma 3.1, we have for k =0,1, … , n-1.

$$|A_{k,4}| = \begin{cases} O\left(\frac{1}{h_k^2}\sum_{\nu=0}^{k-1} h_\nu^3 \omega_5(h_\nu)\right) + O(h_k\omega_5(h_k)) , & if\ f \in C^5(I) \\ K_2 h_k^2 f_k^{(6)} + O\left(h_k^2\omega_6(h_k)\right), & if\ f \in C^6(I) \end{cases}$$

The result clearly holds for k=0. Hence if $\frac{\max h_k}{\min h_k} \leq 0.2 \leq \infty$, H = $\max_{0 \leq k \leq n-1} h_k$,

we have from (3.1.5) to (3.1.7)

$$|A_{k,4}| = \begin{cases} O\left(\omega_5(H)\right), & if\ f \in C^5(I) \\ K_2 H\|f^{(6)}\| + O(H\omega_6(H)) , & if\ f \in C^6(I) \end{cases} \qquad k = 0,1, \dots , n\text{-}1.$$

This proves Lemma 3.1.2.

Lemma 3.1.3 Let $A_{k,5} = a_{k,5} - f_k^{(5)}$

Then we have for k = 0,1, … , n-1

$$|A_{k,5}| = \begin{cases} O\left(\omega_5(H)\right), & if\ f \in C^5(I) \\ K_3 H\|f^{(6)}\| + O\left(H\omega_6(H)\right), & if\ f \in C^6(I) \end{cases}$$

Where $K_2 = 0.2$

Proof: Following a similar method we can get the results for $|A_{k,5}|$ hence we omitted the proof.

IV. Proof of Theorem 2

Let $x \in [x_k, x_{k+1}]$, k = 0,1, …… , n-1

Then from (2.3) we have

$$s_k^{(5)}(x) = a_{k,5} \tag{4.1}$$

and $s_k^{(5)}(x) = a_{k,4} + (x - x_k)\, a_{k,5}$ (4.2)

Therefore

$$\left|s_k^{(5)}(x) - f^{(5)}(x)\right| = \left|s_k^{(5)}(x) - f_k^{(5)} + f_k^{(5)} - f^{(5)}(x)\right|$$

$$\leq \left| a_{k,5} - f_k^{(5)}\right| + \left|f_k^{(5)} - f^{(5)}(x)\right|.$$

If $f \in C^5(I)$ then using Lemma 3.1.3, we have

$$\left|s_k^{(5)}(x) - f^{(5)}(x)\right| = O\left(\omega_5(H)\right). \tag{4.3}$$

Again from (4.2)

$$s_k^{(4)}(x) - f^{(4)}(x) = (a_{k,4} - f^{(4)}) + (x - x_k)(a_{k,5} - f_k^{(5)}) - [f^{(4)}(x) - f_k^{(4)} - (x - x_k)f_k^{(5)}] \quad (4.4)$$

$$= A_{k,4} + (x - x_k)A_{k,5} - (x - x_k)(f^{(4)}(\eta_k) - f_k^{(5)}), \ x_k \le \eta_k \le x$$

Thus,

$$\left|s_k^{(5)}(x) - f^{(5)}(x)\right| \le |A_{k,4}| + H|A_{k,5}| + H\omega_5(H).$$

Now applying Lemma 3.1.2 and 3.1.3 we get,

$$\left|s_k^{(4)}(x) - f^{(4)}(x)\right| = O(\omega_5(H)) + H\, O(\omega_5(H)) = O(\omega_5(H)). \quad (4.5)$$

Now,

$$|s_k'''(x) - f'''(x)| = \left|\int_{x_{k+0.2}}^{x}[s_k^{(4)}(t) - f^{(4)}(t)]dt\right| \le (x - x_{k+0.2})\left|s_k^{(4)}(x) - f^{(4)}(x)\right|$$

$$|s_k'''(x) - f'''(x)| = (H\omega_5(H)) \quad (4.6)$$

Set h(x_k) = h(x_{k+1}) = 0.

So by Rolle's theorem, there exists a

μ_k , $x_k < \mu_k < x_{k+1}$, such that$h'(\mu_k) = s_k''(\mu_k) - f''(\mu_k) = 0$.

This gives$|s_k''(x) - f''(x)| = \left|\int_{\mu_k}^{x}[s_k'''(t) - f'''(t)]dt\right| \le (x - \mu_k)|s_k'''(x) - f'''(x)|$

$= O(HH\omega_5(H))$

$$|s_k''(x) - f''(x)| = (H^2\omega_5(H)) \quad (4.7)$$

Again using interpolatory conditions (2.1) we can write

$$|s_k'(x) - f'(x)| = \left|\int_{x_k}^{x}[s_k''(t) - f''(t)]dt\right|$$

$$|s_k'(x) - f'(x)| = (H^3\omega_5(H)) \quad (4.8)$$

Similarly

$$|s_k(x) - f(x)| = \left|\int_{x_k}^{x}[s_k'(t) - f'(t)]dt\right| = (H^4\omega_5(H)) \quad (4.9)$$

This proves the theorem for $f \in C^5(I)$. Next we consider the case when $f \in C^6(I)$. Then from Lemma 3.1.3

$$\left|s_k^{(5)}(x) - f^{(5)}(x)\right| = \left|(a_{k,5} - f^{(5)}) + (x - x_k)f_k^{(5)}(\xi_k)\right|, \ x_k \le \xi_k \le x$$

$$\le K_3H\|f^{(6)}\| + O(H\omega_6(H)).$$

Again

$$s_k^{(4)}(x) - f^{(4)}(x) = A_{k,4} + (x - x_k)A_{k,5} + \frac{(x-x_k)^2}{2}f^{(5)}(\xi_k), \ x_k \le \xi_k \le x$$

Which on using Lemma 3.2 and Lemma 3.3, gives

$$\left|s_k^{(4)}(x) - f^{(4)}(x)\right| \leq K_3 H \left\|f^{(6)}\right\| + \left(H\omega_6(H)\right). \tag{4.10}$$

From (4.10) on using the method of successive integration we at once have

$$\left|s_k^{(q)}(x) - f^{(q)}(x)\right| \leq K_3 H^{6-q} \left\|f^{(6)}\right\| + \left(H^{6-q}\omega_6(H)\right), \text{ q = 1,2,3,4.} \tag{4.11}$$

This proves the theorem of convergence for $f \in C^6(I)$.

V. Results and Discussions

The present paper dealt with the (0,1;3) lacunary interpolation problem for which we have found a unique spline $S_{n,5}^{(2)}$ (1.1) which can interpolate the given function. Also, the Error bounds (4.10) have been found and shown that this spline function is convergent (4.11). Dealing with such types of problems we can conclude that a similar approach can be applied for other lacunary data provided the function exists uniquely also there would be a need for some interpolatory conditions.

VI. Future Prospects and Applications

As discussed in [XXI] spline functions can be used to find numerical solutions to differential equations and to obtain quadrature formula for a given curve. Splines are used to approximate solutions to partial differential equations in physics and engineering simulations. Because of flexibility and smoothness spline functions are useful in other fields, like Computer Science, Image processing, Mechanical Engineering, BioSciences, etc. Splines (especially B-splines and NURBS) are used in CAD software to create smooth, flexible curves and surfaces. In computer graphics, splines are used for keyframe interpolation, ensuring smooth animations in movies and games. Spline interpolation helps in reconstructing and filtering signals while maintaining smoothness.

Spline interpolation can be used in Terrain models to generate smooth elevation models from scattered data points. Also, they help in mapping and remote sensing applications to estimate missing or sparse data points. Presently Data Science, Machine Learning, and Artificial Intelligence are fast-growing fields, Spline regression methods (e.g., cubic splines, B-splines) provide Deep Learning Integration, Combining spline interpolation with neural networks (e.g. spline-based activation functions) can improve interpretability and efficiency. They can be used in the field of Biomedical and Genomic Data Analysis. They can help in reconstructing complex 3D molecular and cellular structures from experimental data. Spline-based interpolation can assist in modeling complex genetic interactions for personalized treatments. We can say that the Spline function interpolation technique can be used in so many other fields not only in Mathematics.

VII. Conclusion

In this research, we investigated the (0,1;3) lacunary interpolation problem and successfully derived a unique spline function $S_{n,5}^{(2)}$ which is capable of interpolating the given function. Additionally, we established the error bounds (4.10) and demonstrated the convergence of the spline function (4.11), ensuring the reliability and accuracy of our approach.

The findings indicate that the proposed method is effective for handling lacunary interpolation problems and can be extended to other cases with similar data structures. However, the applicability of this approach requires the existence of a unique interpolating function, along with appropriate interpolatory conditions. Future studies may explore generalized lacunary interpolation frameworks and their applications in various domains, further enhancing the scope and efficiency of spline-based interpolation techniques.

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Conflict of interest

The authors declare no conflict of interest available for this paper.

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