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DYNAMICAL MODE OF CASE STUDY ON MASS-SPRING SYSTEM ON A MASSLESS CART: COMPARED ANALYTICAL AND NUMERICAL SOLUTIONS

Rabab Jarrar[1], Rabia Safdar[2], Noorhan F. AlShaikh Mohammad[3]
Olivia Florea[4], Jihad Asad[5]

[1,3,5] Department of Physics, Faculty of Applied Science, Palestine Technical University- Kadoorie, P.O. Box 7, Tulkarm, Palestine.

[2]Department of Mathematics, Lahore College for Women University, Lahore 54000, Pakistan.

[4]Department of Mathematics and Computer Science, Transilvania University of Brasov, Romania.

Email: [1]r.jarrar@ptuk.edu.ps, [2]safdar1109@gmail.com, [3]n.ibraheem@ptuk.edu.ps , [4]olivia.florea@unitbv.ro, [5]j.asad@ptuk.edu.ps

Corresponding Author: **Jihad Asad**

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Abstract

In this research, we study the dynamical behaviors of a mass-spring system on a massless moving cart. The Lagrangian of the system was first constructed, which resulted in obtaining the Euler-Lagrange equation (ELE) of the system. As a next step, we used the Laplace transformation technique to attain an exact solution for ELE of the system. Furthermore, numerical and simulation techniques were applied with the help of MATLAB software, where we solved ELE numerically for some specified initial conditions. Simulation results indicate that they are in good agreement with the exact analytical solution. Finally, some simulation results were presented in this research.

Keywords: Mass-Spring System; Lagrangian; Euler-Lagrange equation; Laplace Transformation; Simulation; MATLAB Software**.**

I. Introduction

In classical mechanics, the Lagrangian technique has been used widely in solving many interesting and real-world systems. As known, this technique is based mainly on a scalar concept which is energy (i.e., kinetic and potential energies). The first step in this technique is constructing the Lagrangian of the system by deriving both kinetic and potential energies respectively. For more details about Lagrangian technique and some ingesting physical systems solved the reader can refer to the following books [XII, XV, X, IXVIII, III]. Applying the Lagrangian technique to any system will yield differential equations (Des) known as equations of motion or ELEs, which are in general second-order differential equations (ordinary in some cases and partial in other cases).

The derived ELEs, which describe the behavior of the system, may need to be solved analytically in certain cases. To do so, it is essential to understand how to solve ordinary or even partial differential equations, as well as the techniques that can be applied. For readers interested in exploring this further, we recommend consulting the references [III, I, XIX, XXV, VI]. In other complicated situations, the obtained ELEs cannot be handled analytically, so some numerical techniques can be of great importance.

Laplace transformation has its importance due to getting algebraic equations from the governing differential equations of the physical problem. The basic definition and properties can be seen in [XVIII] and [XXIV]. The first step is to solve the differential equation into an algebraic equation by applying Laplace transformation. Afterward, the implementation of the initial condition and inverse Laplace gives the solution of the differential equation. The applications of Laplace transformation on different physical problems reader can refer to the following articles [IV, XVII, II, XI, XVI].

Granular dampers operate using a system of Vacuum Packed Particles (VPPs) and can be modeled mathematically as a mass-spring-damper system. The functionality of the granular damper relies on a jamming mechanism, which considers the rearrangement of granules during cycles of loading and unloading [XXIII, XIV]. Recently, there has been increasing awareness of “smart devices” both in academic research and industrial applications. For example, "intelligent" magnetorheological (MR) dampers [XXII, XIII] are gradually replacing classic hydraulic dampers in many commercial trucks and vehicles. Furthermore, "smart dampers" are being investigated to stabilize structures during earthquakes, storms, and tsunamis.

This study describes a unique "intelligent" damper based on the jamming mechanism of VPPs, which contributes to the developing field of "smart devices". VPP systems consist of a soft, hermetically sealed envelope filled with granular material, which can vary in shape and typically contain millimeter-sized particles. In its inactive state, a VPP system behaves like a dense fluid. However, when a vacuum or "under pressure" is created by removing air from the system, the particles reorganize to form a stiff skeleton capable of transmitting external stresses [VII, XXI, XX]. The friction forces generated at the contact locations between individual granules affect the system's overall stiffness [V].

In the literature, numerous helpful numerical approaches can play an essential role in solving ordinary and partial differential equations. Numerical simulation is important to study the nature of the nonlinear system. Some different kinds of numerical methods and simulations with applications can be seen [XVI] and [XVII]. We use here ode45 build in code in MATLAB software to simulate our results numerically and analyze its behaviour concerning analytical solutions derived in sec. 4.

We arranged the rest of this paper as follows: In section 3, we give a brief description of the system and derive the ELE from the constructed Lagrangian, while in section 4 we applied Laplace transforms and solved the ELE of the system analytically, while in section 5 numerical solution and simulation results was presented. Finally, we end our work with a conclusion part in section 6.

II. Problem Statement

Consider a mass-spring system with mass and spring constant, included within a massless cart moving horizontally with constant velocity as in Fig. 1 below. The constant velocity is provided and maintained by an external agent. The Lagrangian $(L = T - V)$ of the system can easily be derived and given as:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}k(q - ut)^2. \tag{1}$$

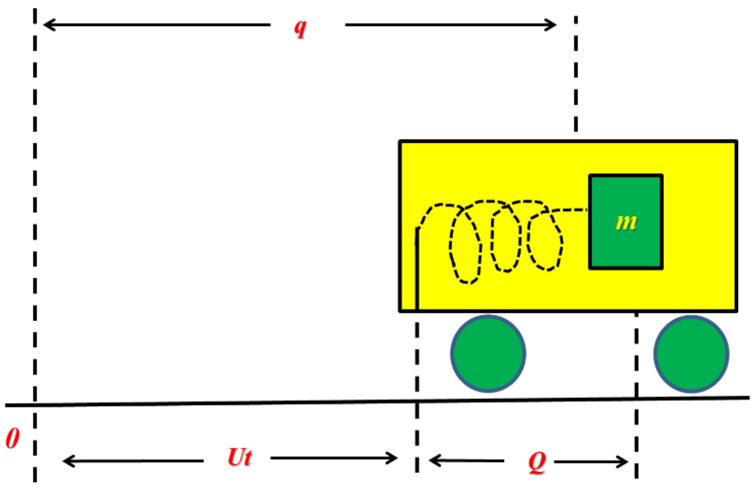

Fig. 1. Geometry of the problem [IV, sec 8.2]

Substituting equation (1) into the following relation, $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$, we got the equations of motion of the system (i.e., known also as Euler- Lagrange Equation; ELE):

$$m\ddot{q} + k(q - ut) = 0. \tag{2}$$

For the special case, where the cart is stationary $u = 0$, equation (2) is reduced to the well-known equation (i.e., the equation of a simple harmonic oscillator) given as:

$$\ddot{q} + \omega^2 q = 0. \tag{3}$$

with $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the system. As it is clear in many classical textbooks like [5] the solution of equation (3) takes the form:

$$q(t) = A\cos(\omega t) + \mathrm{B}\sin(\omega t). \tag{4}$$

where A, B and ω are constants to be determined from the initial condition of the system.

III. Approach and Results

Analytical solution of the system

In this section, we aimed to solve equation (2) analytically using Laplace transformation techniques. On applying Laplace transformation on equation (2) with the following initial condition $q(0) = \alpha$ and $\dot{q}(0) = \beta$ we got:

$$m(s^2Q(s) - sq(0) - \dot{q}(0)) = -k(Q(s) - \frac{u}{s^2}). \tag{5}$$

After simple steps, equation (5) can be rewritten as:

$$Q(s) = \frac{ms^3\alpha + ms^2\beta + uk}{s^2[ms^2 + k]}. \tag{6a}$$

After decomposition in simple fractions, the image through the Laplace transform will have the following form:

$$Q(s) = u\frac{1}{s^2} + \alpha\frac{s}{s^2 + \sqrt{\frac{k}{m}}^2} + \frac{\beta - u}{\sqrt{\frac{k}{m}}}\frac{\sqrt{\frac{k}{m}}}{s^2 + \sqrt{\frac{k}{m}}^2} \tag{6b}$$

To get the final form of analytical solution we apply the inverse Laplace transform on equation (6b) and we have

$$q(t) = ut + \alpha\cos(\omega t) + \frac{(\beta - u)}{\omega}\sin(\omega t). \tag{7}$$

In the above step, we apply the partial fraction technique and make use of inverse Laplace transform tables. For the case, $u = 0$ the solution will be as

$$q(t) = \alpha\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t). \tag{8}$$

Numerical solution and simulation results

In this section, the numerical solution of equation (2) with the simulation results will be discussed for the values of the parameters and the four initial conditions specified in section 3.1. Here we have used the ode45 build code in MATLAB for the following cases.

Case 1. $\boldsymbol{q(0) = 1, \dot{q}(0) = 0}$

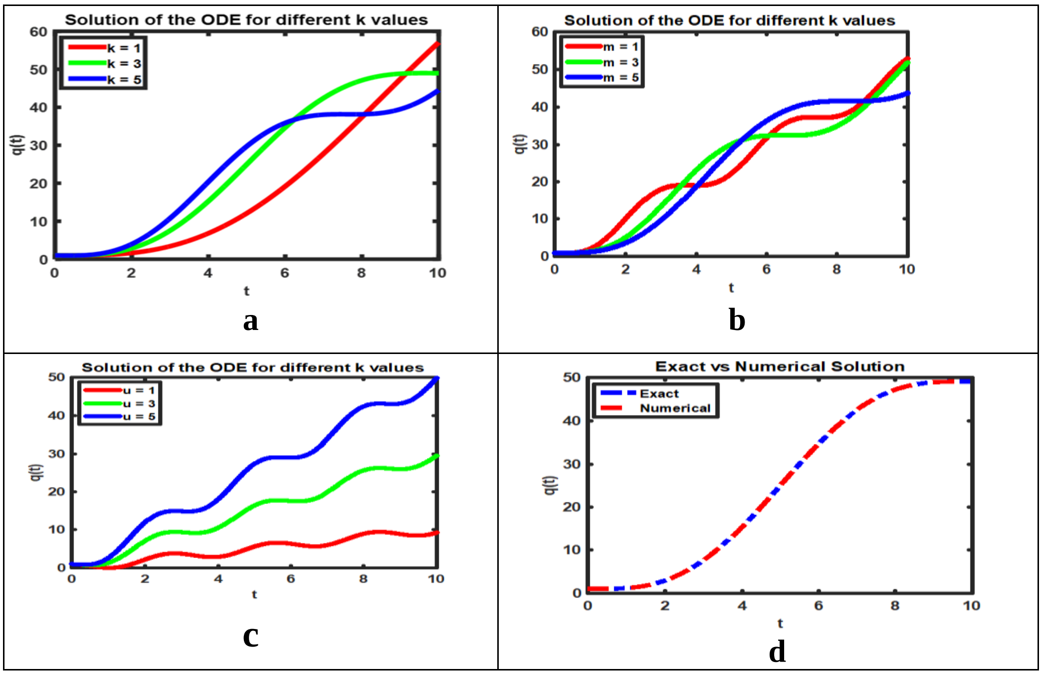

Fig. 2. Plot of q(t) vs time where (a) u=5, m=7 and k= 1, 3, 5 (b) u=5, k=3 and m=1, 3, 5 (c) m=1, k=3 and u= 1, 3, 5 (d) u=5, k=3 and m= 7.

Case 2. $q(0) = 0, \dot{q}(0) = 1$

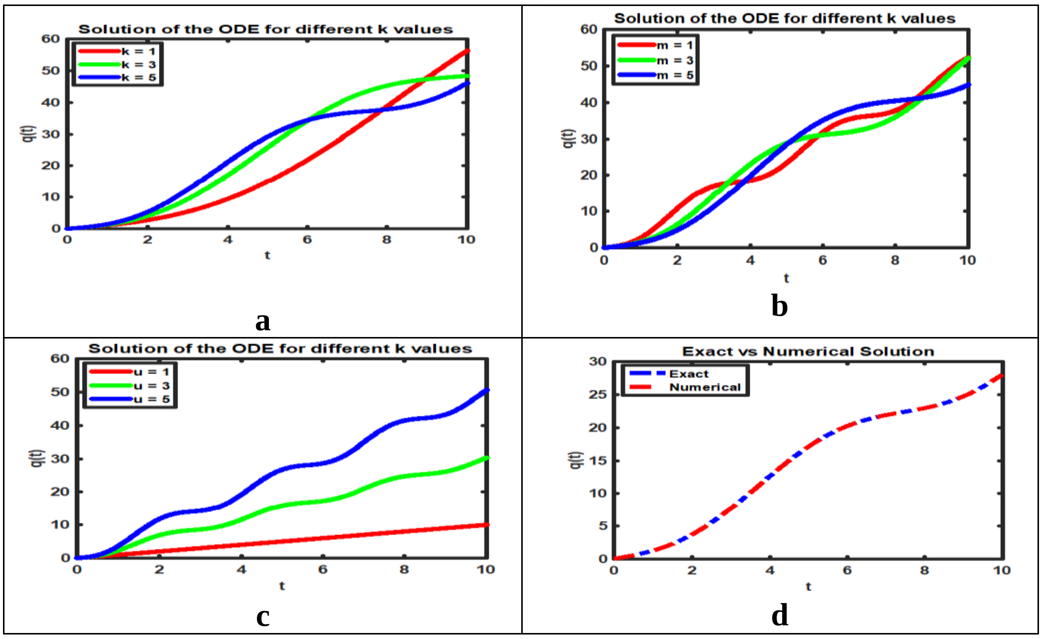

Fig. 3. Plot of q(t) vs time where (a) u=5, m=7 and k= 1, 3, 5 (b) u=5, k=3 and m=1, 3, 5 (c) m=1, k=3 and u= 1, 3, 5 (d) u=5, k=3 and m= 7.

To check the effect of u on the variation of q(t) against time we plot Fig. 5 below.

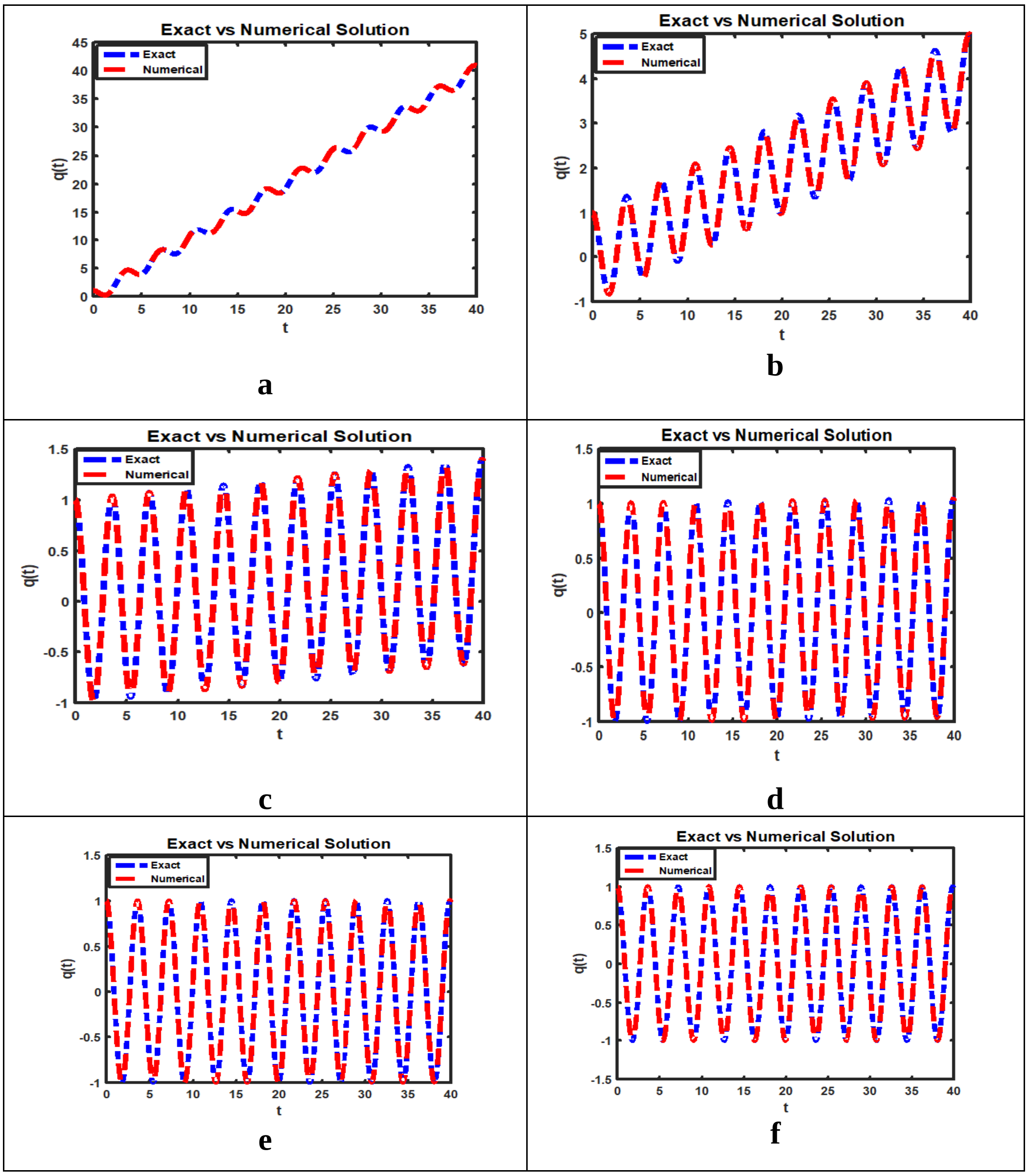

Fig. 4. Plot of q(t) vs time where m=3, k=9 with (a) u= 1 (b) u=0.1 (c) u=0.01 (d) u=0.001 (e) u=0.0001 and (f) u= 0

IV. Graphical Results and Discussion

The obtained results provide a detailed analysis of the system's dynamic response, highlighting the effects of varying key parameters such as *u*, *m*, and *k*.

Figures 2(a-d) and 3(a-d) present the time evolution of *q(t)* under two different sets of initial conditions: Case 1 (Condition 1) and Case 2 (Condition 2), respectively.

- **Effect of stiffness *k***:

  Figures 2a and 3a illustrate the behavior of *q(t)* over time for fixed values *u=5* and *m=7*, with *k* varying as *k=1,3,5*. The comparison between Fig. 2a (Case 1) and Fig. 3a (Case 2) reveals how different initial conditions influence the system's response, particularly in terms of amplitude and phase of oscillations.

- **Effect of mass *m***:

  Figures 2b and 3b depict the dynamic response for fixed values *u=5* and *k=3*, while *m* varies as *m=1,3,5*. These figures demonstrate how increasing the mass affects the frequency and amplitude of oscillations. Case 1 (Fig. 2b) and Case 2 (Fig. 3b) comparisons highlight how mass-dependent dynamics interact with initial conditions, leading to noticeable shifts in oscillatory patterns.

- **Effect of parameter *u***:

  Figures 2c and 3c show the response for *m=1*, *k=3*, and varying *u=1,3,5*. These plots emphasize the impact of *u* on the system's oscillatory behavior, revealing that as *u* increases, the oscillation amplitude increases, while phase shifts become more pronounced.

- **Consistent behavior under specific conditions**:

  Figures 2d and 3d show the time evolution of *q(t)* for fixed parameters *u=5*, *k=3*, and *m=7*. The agreement between the numerical and analytical solutions in both cases underscores the robustness of the numerical approach in accurately capturing system dynamics.

- **Special Case Analysis:**

  Figure 4 illustrates the system behavior for a special case where *u=0*, with fixed parameters *m=7* and *k=5*. As expected, the plot confirms the occurrence of pure harmonic motion, consistent with Eq. (4). When u decreases from *1* to progressively smaller values (down to zero), as shown in Figures 4a-f, the system transitions from shifted oscillations to simple harmonic motion.

- **Effect of increasing *u***:

  Figures 5(a-f) further explore the influence of varying uuu on the oscillation patterns. As uuu increases, the system exhibits noticeable oscillatory shifts due to the coupling effect introduced by the parameter uuu. This shift highlights the system's sensitivity to external forces, which is particularly important when applying this study to real-world nonlinear dynamic systems.

These results provide a deeper understanding of how key parameters influence system dynamics, offering a reliable basis for analyzing and designing complex spring-mass systems, oscillatory devices, and other engineering structures.

V. Conclusion

This study offers a comprehensive investigation of a harmonic oscillator mounted on a massless moving cart, providing insights into its dynamic behavior through both analytical and numerical approaches. By deriving the equation of motion as a second-order nonhomogeneous differential equation, we obtained an exact analytical solution using the Laplace transformation technique. In parallel, the system was numerically analyzed under various initial conditions, with results demonstrating high consistency and accuracy when compared to the analytical solution, as shown in Fig. 2(d) and Fig. 3(d). This close agreement confirms the validity and reliability of the proposed numerical approach, making it a valuable tool for analyzing similar systems.

Moreover, the comparative analysis highlights the robustness of both methods and provides a useful framework for addressing more complex nonlinear systems, such as spring-mass assemblies, rigid body systems, and general vibrational models involving natural frequencies and mode shapes. These findings can be directly applied to engineering applications involving dynamic systems, contributing to the design and optimization of structures subject to oscillatory motion. The methods and outcomes of this study lay the groundwork for future research on more advanced dynamical models and their practical implementations.

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Conflicts of Interest

The authors declare that they have no conflict of interest exists.

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