



AN ADAPTED ANALYTICAL SOLUTION TO THE MULHOLLAND EQUATION: MODIFIED DIRECT ITERATION PROCEDURE

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Abstract

The Mulholland equation is a third-order ordinary differential equation characterized by its nonlinearity. It particularly represents a central restoring force. Mulholland equations have many real applications in various fields, such as modern control theory, phase plane analysis, stability analysis, bifurcation analysis, etc. This paper employed the modified direct iteration method to solve the Mulholland equation, analytically incorporating all its terms (sometimes in approximated forms) during each iterative step. The analytical solutions are compared with existing results. The analytical solutions also showed remarkable precision when compared with numerical outcomes. It also becomes clear that compared to other approaches already in use, the modified direct iteration method is substantially easier to use, more accurate, efficient, and uncomplicated. Moreover, the fourth approximated frequency exhibited only a 0.022 percentage error. The suggested method can be widely applied to different engineering issues, while it is primarily demonstrated in nonlinear models with strong nonlinear factors.

Keywords: Mulholland Equation, Direct iteration procedure, Analytical solution, Modified Direct iteration procedure, Mathematica.

I. Introduction

Nonlinear systems play a significant role in mathematics, medical science, life science, and other related fields. As a result, research on nonlinear systems has greatly benefited these areas. However, studying nonlinear systems is challenging and delicate

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since even small changes to valid parameters can cause sudden changes in their characteristics. In many real-life situations, the governing equations are nonlinear, which requires special attention and different approaches to solving them. In recent decades, numerous researchers have taken an interest in studying nonlinear problems [I], with a particular focus on employing either numerical methods [XX] or analytical approximation methods [XX]. Various analytical approaches have been developed to construct approximate periodic solutions for nonlinear differential equations. Researchers have also formulated diverse procedures for solving such equations and determining their physical properties, such as the perturbation method [XIV-XX], Harmonic Balance method [XVII, XXII], Cubication method [II], Homotopy perturbation technique [VII], Multiple scales [XV], Mickens iteration method [XVIII], Hu's iteration method [XII], Lim and Wu's iteration method [XVI], Haque's iteration method [IV, V], Iterative homotopy harmonic balancing approach [III], and Modified extended iteration method [VI-XI]. Additionally, only a few researchers have investigated the Nonlinear Third Order Differential Equation, using various methods applied by Ismail and Abu-Zinadah [XIII], Mulholland [XIX], Rahman et al. [XXII], and Ramos [XXIII]. The proposed method stands out for its ease, simplicity, and applicability to higher-order approximation solutions. The perturbation method is the most widely used approach, which assumes that the nonlinear term is small. While perturbation methods provide accurate results for small nonlinearity, they become less effective when nonlinearity is strong. While the Harmonic Balance Method is powerful for analyzing periodic behaviors in nonlinear systems, it has significant limitations, especially when dealing with small nonlinearities are of concern. An alternative well-known technique is the homotopy perturbation method [VII]. This method was introduced by Ji-Huan He in 1999. The homotopy perturbation technique has been employed by researchers and engineers to address nonlinear issues since it reliably transforms a challenging problem into a more manageable one for analysis. A diverse array of nonlinear problems can be solved effectively using the homotopy perturbation method, providing approximations that rapidly converge to exact solutions. This method is particularly effective for addressing both weakly and strongly nonlinear equations. In such cases, Mickens [XVII] developed a technique called Harmonic Balance, which has been further developed by Lim and Wu [XVI], Hu [XII], and others [III, VI-XI] to handle strong nonlinear problems. However, a significant disadvantage of the Harmonic balance method is the difficulty in predicting whether a first-order harmonic balance calculation will accurately approximate a given nonlinear differential equation's periodic solution. On the other hand, the concept of the Iteration method was initially proposed by R E Mickens in 1987, and currently, the most commonly employed iteration methods are based on Mickens' approach [XVII]. This method provides a dependable and effective way to tackle a wide range of scientific and engineering problems, especially those involving highly nonlinear systems. Mickens, in addition, has introduced the direct iteration method, which can handle nonlinearities and other complexities in the problem, making it a reliable method for solving a wide range of differential equations. This method is computationally efficient and can be easily implemented using a computer program, making it a practical method for solving complex nonlinear

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differential equations. This research aims to explore the approximate analytical solutions and their associated amplitudes and frequencies for the Mulholland equation by utilizing the modified direct iteration procedure. After using the recommended approach, more accurate, realistic, and consistent findings are found. The obtained results exhibit a rapid convergence towards the exact values, with considerably fewer errors than those reported in the literature.

II. The Method

The Modified Direct Iteration Procedure is performed in four steps:

- (i) Examining an ordinary differential equation of second order;
- (ii) Putting it into standard form;
- (iii) Using the modified direct iteration technique; and
- (iv) Using the extended iteration technique

(i) Assume that the generic form of the Mulholland equation is

$$\ddot{u} + F(\ddot{u}, \dot{u}, u) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0, \quad (1)$$

and additionally presume that it is rewriteable to the form

$$\ddot{u} + f(\ddot{u}, \dot{u}, u) = 0 \quad (2)$$

where over dots signify differentiation concerning time, t , and A stand for the oscillator's amplitude.

(ii) For this system, we select its natural frequency Ω . Next, if we add $\Omega^2 u$ to both sides of equation (2), we get

$$\ddot{u} + \Omega^2 u = \Omega^2 u - f(\ddot{u}, \dot{u}, u) \equiv G(\ddot{u}, \dot{u}, u). \quad (3)$$

(iii) The Modified Direct Iteration Approach is now formulated as

$$\ddot{u}_{j+1} + \Omega^2 u_{j+1} = G(u_j, \dot{u}_j, \ddot{u}_j); \quad j = 0, 1, 2, 3, \dots \quad (4)$$

(iv) The extended iteration scheme

$$\begin{aligned} \ddot{u}_{j+1} + \Omega_j^2 u_{j+1} = & G(\ddot{u}_0, \dot{u}_0, u_0) + G_u(\ddot{u}_0, \dot{u}_0, u_0)(u_j - u_0) \\ & + G_{\dot{u}}(\ddot{u}_0, \dot{u}_0, u_0)(\dot{u}_j - \dot{u}_0) + G_{\ddot{u}}(\ddot{u}_0, \dot{u}_0, u_0)(\ddot{u}_j - \ddot{u}_0) \end{aligned} \quad (5)$$

in addition to the initial estimate

$$u_0(t) = A \cos(\Omega_0 t). \quad (6)$$

Thus, u_{j+1} fulfills the initial requirements.

$$u_{j+1}(0) = A, \quad \dot{u}_{j+1}(0) = 0. \quad (7)$$

The oscillator's frequencies at each stage of the iteration are determined by the requirement that secular terms do not occur during the whole solution $x_{j+1}(t)$

The steps listed above provide the order of the solutions: $u_0(t), u_1(t), u_2(t), \dots$.

The approach may commence with any sequence of approximations; however, owing to increasing algebraic complexity, the solution is typically restricted to a lower order, usually the second.

At this juncture, the subsequent observations deserve attention:

- (a) Having the solutions for j smaller than $j + 1$ is necessary to determine the solution $u_{j+1}(t)$.
- (b) Due to the need for secular terms to be missing, the linear differential equation $u_{j+1}(t)$ permits the determination of Ω_j .

III. Solutions Procedure

Take into account the Mulholland equation as follows,

$$\ddot{u} + \ddot{u} + \dot{u} + u = \epsilon (1 - u^2 - \dot{u}^2 - \ddot{u}^2) (\ddot{u} + \dot{u}) \quad (8)$$

After inserting $\Omega^2 u$ to both sides of Equation (8), we have

$$\begin{aligned} \ddot{u} + \ddot{u} + \Omega^2 \dot{u} + \Omega^2 u &= \Omega^2 \dot{u} + \Omega^2 u - (\dot{u} + u) \\ &+ \epsilon (1 - u^2 - \dot{u}^2 - \ddot{u}^2) (\ddot{u} + u) \end{aligned} \quad (9)$$

Now the iteration scheme of Equation (9), is

$$\begin{aligned} \ddot{u}_{j+1} + \ddot{u}_{j+1} + \Omega_j^2 \dot{u}_{j+1} + \Omega_j^2 u_{j+1} &= \Omega_j^2 \dot{u}_j + \Omega_j^2 u_j - (\dot{u}_j + u_j) \\ &+ \epsilon (1 - u_j^2 - \dot{u}_j^2 - \ddot{u}_j^2) (\ddot{u}_j + u_j) \end{aligned} \quad (10)$$

For $j = 0$, the Equation (10) can be written as

$$\begin{aligned} \ddot{u}_1 + \ddot{u}_1 + \Omega_0^2 \dot{u}_1 + \Omega_0^2 u_1 &= \Omega_0^2 \dot{u}_0 + \Omega_0^2 u_0 - (\dot{u}_0 + u_0) \\ &+ \epsilon (1 - u_0^2 - \dot{u}_0^2 - \ddot{u}_0^2) (\ddot{u}_0 + u_0) \end{aligned} \quad (11)$$

where $\epsilon = 1$.

Hence, the Equation (11) will be,

$$\begin{aligned} \ddot{u}_1 + \ddot{u}_1 + \Omega_0^2 \dot{u}_1 + \Omega_0^2 u_1 &= \Omega_0^2 \dot{u}_0 + \Omega_0^2 u_0 - (\dot{u}_0 + u_0) \\ &+ (1 - u_0^2 - \dot{u}_0^2 - \ddot{u}_0^2) (\ddot{u}_0 + u_0) \end{aligned} \quad (12)$$

Here the initial approximate root is,

$$u_0 = A_0 \cos \theta \quad (13)$$

$$\text{After differentiating, } \dot{u}_0 = -\Omega_0 A_0 \sin \theta \quad (14)$$

$$\text{Again, } \ddot{u}_0 = -\Omega_0^2 A_0 \cos \theta \quad (15)$$

By replacing the values of u , \dot{u}_0 , and \ddot{u}_0 from equations (13), (14), and (15) into the right-hand side of Equation (12), we obtain the frequency Ω_0 , the first approximation, u_1 , and A_0 is the amplitude.

$$\ddot{u}_1 + \dot{u}_1 + \Omega_0^2 \dot{u}_1 + \Omega_0^2 u_1 = m_{c11} \cos\theta + m_{c13} \cos 3\theta + m_{s11} \sin\theta + m_{s13} \sin 3\theta \quad (16)$$

where,

$$\begin{aligned} m_{c11} &= -A_0 + 0.75 A_0^3 \Omega_0^2 + 0.25 A_0^3 \Omega_0^4 + 0.75 A_0^3 \Omega_0^6 \\ m_{c13} &= 0.25 A_0^3 \Omega_0^2 - 0.25 A_0^3 \Omega_0^4 + 0.25 A_0^3 \Omega_0^6 \\ m_{s11} &= 0.25 A_0^3 \Omega_0 - A_0 \Omega_0^3 + 0.75 A_0^3 \Omega_0^3 + 0.25 A_0^3 \Omega_0^5 \\ m_{s13} &= 0.25 A_0^3 \Omega_0 - 0.25 A_0^3 \Omega_0^3 + 0.25 A_0^3 \Omega_0^5 \end{aligned} \quad (17)$$

Solving for A_0 and Ω_0 after setting the coefficients of $\cos\theta$ and $\sin\theta$ to zero, yields

$$A_0 = 0.8917992518437051 \quad (18)$$

$$\Omega_0 = 0.9177879325003003 \quad (19)$$

After simplification, Equation (16) reduces to

$$\ddot{u}_1 + \dot{u}_1 + \Omega_0^2 \dot{u}_1 + \Omega_0^2 u_1 = \frac{1}{8 \Omega_0^2 (9 \Omega_0^2 + 1)} (3(m_{s123} \cos 3\theta - m_{c123} \sin 3\theta) - (m_{s123} \sin 3\theta + m_{c123} \cos 3\theta)) \quad (20)$$

where,

$$\begin{aligned} m_{s123} &= 0.14112353483408657, \\ m_{c123} &= 0.1295214772625104 \end{aligned} \quad (21)$$

Solving Equation (21) and satisfying the initial conditions from Equation (6), we obtain

$$u_1 = A_1 \cos\theta + (0.0050817256223556215 \cos 3\theta - 0.009160241301783287 \sin 3\theta) \quad (22)$$

The initial approximation to Equation (8) is shown here

Inserting $j = 1$ in Equation (10), we obtain the second approximation u_2 , the amplitude A_1 , and the frequency Ω_1 . Finally, we use $\epsilon = 1$.

$$\begin{aligned} \ddot{u}_2 + \dot{u}_2 + \Omega_1^2 \dot{u}_2 + \Omega_1^2 u_2 &= \Omega_1^2 \dot{u}_1 + \Omega_1^2 u_1 \\ &\quad - (\dot{u}_1 + u_1) + (1 - u_1^2 - \dot{u}_1^2 - \ddot{u}_1^2) (\dot{u}_1 + \ddot{u}_1) \end{aligned} \quad (23)$$

$$\begin{aligned} \text{Here, } u_1 &= A_1 \cos\theta + (0.0050817256223556215 \cos 3\theta \\ &\quad - 0.009160241301783287 \sin 3\theta) \end{aligned} \quad (24)$$

After differentiating,

$$\begin{aligned} \dot{u}_1 &= \Omega_1 (-A_1 \sin\theta + 3(-0.0050817256223556215 \sin 3\theta - \\ &\quad 0.009160241301783287 \cos 3\theta)) \end{aligned} \quad (25)$$

Again,

$$\ddot{u}_1 = \Omega_1^2 (-A_1 \cos\theta + 3^2(-0.0050817256223556215 \cos 3\theta + 0.009160241301783287 \sin 3\theta)) \quad (26)$$

Now, the values of Equation (24), (25), and (26) are put on the right side of Equation (23). We get,

$$\begin{aligned} \ddot{u}_2 + \ddot{u}_2 + \Omega_1^2 \dot{u}_2 + \Omega_1^2 u_2 = & m_{c21} \cos\theta + m_{c23} \cos 3\theta + m_{c25} \cos 5\theta \\ & + m_{c27} \cos 7\theta + m_{c29} \cos 9\theta + m_{s21} \sin\theta + m_{s23} \sin 3\theta + m_{s25} \sin 5\theta \\ & + m_{s27} \sin 7\theta + m_{s29} \sin 9\theta \end{aligned} \quad (27)$$

where,

$$\begin{aligned} m_{c21} = & -A_1 + 0.002290060325445822 A_1^2 \Omega_1 + 0.001042472582074128 A_1 \Omega_1^2 + \\ & 0.01397474546147796 A_1^2 \Omega_1^2 + 0.75 A_1^3 \Omega_1^2 - 0.020610542929012397 A_1^2 \Omega_1^3 + \\ & 0.0004938028020351091 A_1 \Omega_1^4 - 0.0038112942167667155 A_1^2 \Omega_1^4 + 0.25 A_1^3 \Omega_1^4 - \\ & 2.03287907341032 \times 10^{-19} A_1 \Omega_1^5 - 0.034350904881687336 A_1^2 \Omega_1^5 + \\ & 0.013332675654947945 A_1 \Omega_1^6 + 0.034301647950900446 A_1^2 \Omega_1^6 + 0.75 A_1^3 \Omega_1^6 \end{aligned} \quad (28)$$

$$\begin{aligned} m_{c23} = & -0.005081725622355622 + 7.538921370231491 \times 10^{-7} \Omega_1 + \\ & 0.013740361952674932 A_1^2 \Omega_1 - 0.040650040923317726 \Omega_1^2 + \\ & 0.02794949092295592 A_1^2 \Omega_1^2 + 0.25 A_1^3 \Omega_1^2 - 0.027460368817650245 \Omega_1^3 + \\ & 0.041221085858024795 A_1^2 \Omega_1^3 + 0.000011292166581717668 \Omega_1^4 + \\ & 0.0228677653006003 A_1^2 \Omega_1^4 - 0.25 A_1^3 \Omega_1^4 + 0.00006106526309886733 \Omega_1^5 + \\ & 0.013740361952674932 A_1^2 \Omega_1^5 + 0.0003048884977063769 \Omega_1^6 \Omega_1^6 + \\ & 0.06860329590180089 \Omega_1^6 + 0.25 A_1^3 \Omega_1^6 \end{aligned} \quad (29)$$

$$\begin{aligned} m_{c25} = & 0.00011637458232558088 A_1 \Omega_1 + 0.01145030162722911 A_1^2 \Omega_1 - \\ & 0.00027590890567845606 A_1 \Omega_1^2 + 0.01397474546147796 A_1^2 \Omega_1^2 + \\ & 0.0006284227445581368 A_1 \Omega_1^3 - 0.020610542929012397 A_1^2 \Omega_1^3 - \\ & 0.0006534684608173961 A_1 \Omega_1^4 - 0.019056471083833584 A_1^2 \Omega_1^4 - \\ & 0.0006284227445581366 A_1 \Omega_1^5 + 0.04809126683436226 A_1^2 \Omega_1^5 - \\ & 0.0035287296884139382 A_1 \Omega_1^6 + 0.034301647950900446 A_1^2 \Omega_1^6 \end{aligned} \quad (30)$$

$$\begin{aligned} m_{c27} = & 0.00016292441525581322 A_1 \Omega_1 - 0.00027590890567845606 A_1 \Omega_1^2 - \\ & 0.0006284227445581368 A_1 \Omega_1^3 + 0.0009148558451443545 A_1 \Omega_1^4 + \\ & 0.003142113722790684 A_1 \Omega_1^5 - 0.0035287296884139382 A_1 \Omega_1^6 \end{aligned} \quad (31)$$

$$\begin{aligned} m_{c29} = & -4.423170086924818 \times 10^{-8} \Omega_1 - 0.000002582984144041228 \Omega_1^2 + \\ & 3.980853078232342 \times 10^{-7} \Omega_1^3 + 0.000023246857296371057 \Omega_1^4 - \\ & 0.00000358276777040911 \Omega_1^5 - 0.00020922171566733946 \Omega_1^6 \end{aligned} \quad (32)$$

$$\begin{aligned} m_{s21} = & 0.000054866978003942364 A_1 \Omega_1 + 0.0012704314055889056 A_1^2 \Omega_1 + 0.25 \\ & A_1^3 \Omega_1 - 0.025190663579904043 A_1^2 \Omega_1^2 - 0.9985185915938947 A_1 \Omega_1^3 - \\ & 0.01143388265030015 A_1^2 \Omega_1^3 + 0.75 A_1^3 \Omega_1^3 - 2.134523027080837 \times 10^{-19} A_1 \Omega_1^4 + \\ & 0.006870180976337466 A_1^2 \Omega_1^4 + 0.004444225218315981 A_1 \Omega_1^5 - \\ & 0.019056471083833584 A_1^2 \Omega_1^5 + 0.25 A_1^3 \Omega_1^5 - 0.0618316287870372 A_1^2 \Omega_1^6 \end{aligned} \quad (33)$$

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$$m_{s23} = 0.009160241301783287 + 4.182283919167834 \times 10^{-7} \Omega_1 + 0.007622588433533433 A_1^2 \Omega_1 + 0.25 A_1^3 \Omega_1 + 0.0732751453850331 \Omega_1^2 - 0.05038132715980809 A_1^2 \Omega_1^2 - 0.015233884700485147 \Omega_1^3 + 0.0228677653006003 A_1^2 \Omega_1^3 - 0.25 A_1^3 \Omega_1^3 - 0.000020355087699622457 \Omega_1^4 - 0.041221085858024795 A_1^2 \Omega_1^4 + 0.000033876499745153 \Omega_1^5 + 0.007622588433533433 A_1^2 \Omega_1^5 + 0.25 A_1^3 \Omega_1^5 - 0.0005495873678898061 \Omega_1^6 - 0.1236632575740744 A_1^2 \Omega_1^6 \quad (34)$$

$$m_{s25} = -0.00007260760675748841 A_1 \Omega_1 + 0.006352157027944528 A_1^2 \Omega_1 - 0.0004422234128372073 A_1 \Omega_1^2 - 0.025190663579904043 A_1^2 \Omega_1^2 - 0.0003920810764904377 A_1 \Omega_1^3 - 0.01143388265030015 A_1^2 \Omega_1^3 - 0.0010473712409302283 A_1 \Omega_1^4 + 0.034350904881687336 A_1^2 \Omega_1^4 + 0.00039208107649043753 A_1 \Omega_1^5 + 0.026679059517367015 A_1^2 \Omega_1^5 - 0.0056558047010232294 A_1 \Omega_1^6 - 0.0618316287870372 A_1^2 \Omega_1^6 \quad (35)$$

$$m_{s27} = -0.0001016506494604838 A_1 \Omega_1 - 0.0004422234128372073 A_1 \Omega_1^2 + 0.0003920810764904377 A_1 \Omega_1^3 + 0.0014663197373023192 A_1 \Omega_1^4 - 0.001960405382452188 A_1 \Omega_1^5 - 0.0056558047010232294 A_1 \Omega_1^6 \quad (36)$$

$$m_{s29} = -8.609947146804092 \times 10^{-7} \Omega_1 + 1.326951026077447 \times 10^{-7} \Omega_1^2 + 0.000007748952432123684 \Omega_1^3 - 0.000001194255923469702 \Omega_1^4 - 0.00006974057188911315 \Omega_1^5 + 0.000010748303311227316 \Omega_1^6 \quad (37)$$

If the coefficients of $\sin\theta$ and $\cos\theta$ are set to zero, then secular terms can be removed. We get

$$A_1 = 0.9253113651974292 \quad (38)$$

$$\Omega_1 = 0.8989289973953188 \quad (39)$$

After simplification Equation (27) reduces to

$$\ddot{u}_2 + \ddot{u}_2 + \Omega_1^2 \dot{u}_2 + \Omega_1^2 u_2 = \frac{1}{8\Omega_1^2(9\Omega_1^2+1)} (3 (m_{s223} \cos 3\theta - m_{c223} \sin 3\theta) - (m_{s223} \sin 3\theta + m_{c223} \cos 3\theta)) + \frac{1}{24\Omega_1^2(25\Omega_1^2+1)} (5 (m_{s225} \cos 5\theta - m_{c225} \sin 5\theta) - (m_{s225} \sin 5\theta + m_{c225} \cos 5\theta)) + \frac{1}{48\Omega_1^2(49\Omega_1^2+1)} (7 (m_{s227} \cos 7\theta - m_{c227} \sin 7\theta) - (m_{s227} \sin 7\theta + m_{c227} \cos 7\theta)) + \frac{1}{80\Omega_1^2(81\Omega_1^2+1)} (9 (m_{s229} \cos 9\theta - m_{c229} \sin 9\theta) - (m_{s229} \sin 9\theta + m_{c229} \cos 9\theta)) \quad (40)$$

where,

$m_{s223} = 0.1176025916001788$	$m_{s225} = -0.018807894037033677$
$m_{s227} = -0.0030919226632581923$	$m_{s229} = -0.0000310829693900055$
$m_{c223} = 0.1837894422541152$	$m_{c225} = 0.03252908774451423$
$m_{c227} = 0.00004331645916890218$	$m_{c229} = -0.00009915853597625951$

After simplifying Equation (27) and satisfying the initial conditions (16), we obtain
$$u_2 = A_2 \cos \theta + (0.0031604423016027275 \cos 3\theta - 0.00030781543233243214 \cos 5\theta - 0.00001377285769682708 \cos 7\theta - 4.203662445482452 \times 10^{-8} \cos 9\theta - 0.012508962531353019 \sin 3\theta - 0.00034981370355512966 \sin 5\theta + 0.000001771054971225466 \sin 7\theta + 2.149710602409783 \times 10^{-7} \sin 9\theta) \quad (41)$$

The second approximation to Equation (8) is shown here.

Using successive iteration procedures, we have the third and fourth approximate roots. Which are

$$u_3 = A_3 \cos \theta + (0.0021595980810987987 \cos 3\theta - 0.0005932249405990492 \cos 5\theta - 0.000012762307706707014 \cos 7\theta + 0.000001464940256110536 \cos 9\theta + 1.549057602317828 \times 10^{-7} \cos 11\theta + 3.064820058486627 \times 10^{-9} \cos 13\theta - 3.338642489560138 \times 10^{-10} \cos 15\theta - 2.436744631797067 \times 10^{-11} \cos 17\theta - 4.195485804173445 \times 10^{-13} \cos 19\theta + 1.365205491594932 \times 10^{-14} \cos 21\theta + 5.954503570530431 \times 10^{-16} \cos 23\theta + 2.905484038601132 \times 10^{-18} \cos 25\theta - 4.593930538124977 \times 10^{-20} \cos 27\theta - 0.009852151255022173 \sin 3\theta + 0.00002361856379276861 \sin 5\theta + 0.000029384614896270703 \sin 7\theta + 0.000001924067339787109 \sin 9\theta - 2.86205774937422 \times 10^{-8} \sin 11\theta - 9.055612981026059 \times 10^{-9} \sin 13\theta - 4.111422299317064 \times 10^{-10} \sin 15\theta + 2.690250115653538 \times 10^{-12} \sin 17\theta + 8.104421062780697 \times 10^{-13} \sin 19\theta + 2.331039009476541 \times 10^{-14} \sin 21\theta - 2.547101748623923 \times 10^{-17} \sin 23\theta - 7.926750625791116 \times 10^{-18} \sin 25\theta - 3.593546088191745 \times 10^{-20} \sin 27\theta) \quad (42)$$

The third approximation to Equation (8) is shown here.

$$u_4 = A_4 \cos \theta + (0.0029403545700440142 \cos 3\theta - 0.0003643473496711236 \cos 5\theta - 0.00001224685185335027 \cos 7\theta + 0.000002861491911699528 \cos 9\theta + 2.226064290049228 \times 10^{-7} \cos 11\theta - 1.647203075393092 \times 10^{-8} \cos 13\theta - 2.792233668398611 \times 10^{-9} \cos 15\theta + 1.185331743068799 \times 10^{-11} \cos 17\theta + 2.592617795825459 \times 10^{-11} \cos 19\theta + 1.364984383692718 \times 10^{-12} \cos 21\theta - 1.349041634527807 \times 10^{-13} \cos 23\theta - 1.89501611279779 \times 10^{-14} \cos 25\theta - 1.663564389312344 \times 10^{-16} \cos 27\theta + 1.228939553045029 \times 10^{-16} \cos 29\theta + 8.961933430219188 \times 10^{-18} \cos 31\theta - 1.761535530851664 \times 10^{-19} \cos 33\theta - 6.217410966855476 \times 10^{-20} \cos 35\theta - 3.075138800689011 \times 10^{-21} \cos 37\theta + 9.89971581023628 \times 10^{-23} \cos 39\theta + 1.916971248329097 \times 10^{-23} \cos 41\theta + 7.624953047755094 \times 10^{-25} \cos 43\theta - 2.170437900640091 \times 10^{-26} \cos 45\theta - 3.482064356370395 \times 10^{-27} \cos 47\theta - 1.218679381714174 \times 10^{-28} \cos 49\theta + 2.172732675353362 \times 10^{-30} \cos 51\theta + 3.432031566373689 \times 10^{-31} \cos 53\theta + 1.060810405206066 \times 10^{-32} \cos 55\theta - 9.650022895495361 \times 10^{-35} \cos 57\theta - 1.644172935311821 \times 10^{-35} \cos 59\theta - 4.151968178253881 \times 10^{-37} \cos 61\theta + 2.274390032391235 \times 10^{-39} \cos 63\theta + 3.348673033214568 \times 10^{-40} \cos 65\theta + 5.735919329373594 \times 10^{-42} \cos 67\theta)$$

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$$\begin{aligned}
 & - 3.637444233662013 \times 10^{-44} \cos 690 - 2.231848077122583 \\
 & \times 10^{-45} \cos 710 - 1.706013820153083 \times 10^{-47} \cos 730 \\
 & + 1.648949858100978 \times 10^{-49} \cos 750 + 2.306885701878429 \\
 & \times 10^{-51} \cos 770 + 1.626624949038507 \times 10^{-54} \cos 790 \\
 & - 2.663570121195699 \times 10^{-56} \cos 810 - 0.011129361032745507 \sin 30 \\
 & - 0.00013263999226264725 \sin 50 + 0.00003430105709045807 \sin 70 \\
 & + 0.000001810092831778395 \sin 90 - 2.319327327582501 \\
 & \times 10^{-7} \sin 110 - 2.553265343482475 \times 10^{-8} \sin 130 + 9.21983201819607 \times \\
 & 10^{-10} \\
 & \sin 150 + 2.828948998511205 \times 10^{-10} \sin 170 + 6.82378333973132 \times 10^{-12} \\
 & \sin 190 \\
 & - 2.075871336682801 \times 10^{-12} \sin 210 - 1.794078535819143 \\
 & \times 10^{-13} \sin 230 + 5.496574424458987 \times 10^{-15} \sin 250 \\
 & + 1.678767059256976 \times 10^{-15} \sin 270 + 6.71452969171976 \\
 & \times 10^{-17} \sin 290 - 6.786806880662981 \times 10^{-18} \sin 310 \\
 & - 8.506475529832716 \times 10^{-19} \sin 33 - 1.653813648432383 \\
 & \times 10^{-20} \sin 350 + 3.35099210951309 \times 10^{-21} \sin 370 \\
 & + 2.893364120330763 \times 10^{-22} \sin 390 + 3.541243460230331 \\
 & \times 10^{-24} \sin 410 - 8.936009107192946 \times 10^{-25} \sin 430 \\
 & - 6.308621378984203 \times 10^{-26} \sin 450 - 7.390129417882552 \\
 & \times 10^{-28} \sin 470 + 1.294404963628203 \times 10^{-28} \sin 490 \\
 & + 7.987723825122605 \times 10^{-30} \sin 510 + 1.009635829850507 \\
 & \times 10^{-31} \sin 530 - 9.560533148871466 \times 10^{-33} \sin 550 \\
 & - 5.148232737458204 \times 10^{-34} \sin 570 - 6.138362917758361 \\
 & \times 10^{-36} \sin 590 + 3.352409706003413 \times 10^{-37} \sin 610 \\
 & + 1.443262613290526 \times 10^{-38} \sin 630 + 1.231350083483733 \\
 & \times 10^{-40} \sin 650 - 5.104130109082876 \times 10^{-42} \sin 670 \\
 & - 1.386507542025245 \times 10^{-43} \sin 690 - 4.301334471949013 \\
 & \times 10^{-46} \sin 710 + 2.434366010975398 \times 10^{-47} \sin 730 \\
 & + 2.59443433832655 \times 10^{-49} \sin 750 - 4.691952572641194 \\
 & \times 10^{-52} \sin 770 - 1.175356075039077 \times 10^{-53} \sin 790 \\
 & - 1.28410523620339 \times 10^{-56} \sin 810)
 \end{aligned} \tag{43}$$

The fourth approximation to Equation (8) is shown here.

And corresponding amplitudes- frequencies are

$$A_2 = 0.923680268952660 \tag{44}$$

$$\Omega_2 = 0.9099148145391889 \tag{45}$$

and

$$A_3 = 0.9229214386590673 \tag{46}$$

$$\Omega_3 = 0.9046512499391295 \tag{47}$$

IV. Results and Discussions

A modified direct iteration method has been used in this paper to solve the nonlinear Mulholland equation. It has been claimed that our approach produces superior results to those of other existing methods. The recommended method provides ideal values for all initial amplitude values, no matter how small or large. In contrast, the majority of approaches are less accurate for smaller initial amplitude values and more inaccurate for larger ones. In addition, the recommended method produces excellent results for higher-order solutions, while the majority of methods produce decent results for first-order solutions but poor results for higher-order solutions. The approximate frequencies and the corresponding amplitudes for the first, second, third, and fourth order have been determined here, and are denoted by $\Omega_0, A_0; \Omega_1, A_1; \Omega_2, A_2$ and Ω_3, A_3 respectively for $\varepsilon = 1$. All the results are given in Table 1. We have also included the current findings to compare the estimated frequencies in Table 2 for $\varepsilon = 1$.

Table 1 : Approximate amplitudes and corresponding frequencies of $\ddot{u} + \ddot{u} + \dot{u} + u = (1 - u^2 - \dot{u}^2 - \ddot{u}^2)(\ddot{u} + \dot{u})$ for $\varepsilon = 1$.

ε	A_0 Ω_0	A_1 Ω_1	A_2 Ω_2	A_3 Ω_3
$\varepsilon = 1$	0.891799 0.917788	0.925311 0.898929	0.923680 0.909915	0.922921 0.904651

Table 2 : Comparison of the approximate frequencies produced by the suggested approach with the perturbation method [XIX], the method of averaging [XIX], and the exact frequency, Ω_e of $\ddot{u} + \ddot{u} + \dot{u} + u = (1 - u^2 - \dot{u}^2 - \ddot{u}^2)(\ddot{u} + \dot{u})$ for $\varepsilon = 1$.

Exact Frequency Ω_e	Perturbation method [XIX] Er(%)	Method of Averaging [XIX] Er(%)	Proposed Method Er(%)
0.904446	0.9184 1.543	0.916667 1.351	0.904651 0.022

Er(%) denotes percentage error.

V. Conclusion

This article has effectively refined the modified direct iteration method and employed it in nonlinear Mulholland equations. The obtained solutions exhibit higher accuracy compared to existing and exact outcomes. The approximated frequencies yielded by this approach demonstrate a noteworthy concurrence with both the precise frequency values and the numerical solutions.

In conclusion, we summed up:

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- The modified direct iteration method offers a powerful way to analyze random oscillations. It is also a quick and effective way to determine the estimated frequencies and periodic solutions to the nonlinear Mulholland equations.
- For every value of the initial amplitude, both small and big, the recommended method produces optimum values. However, the majority of approaches produce less accuracy for smaller initial amplitude values and more for bigger ones.
- The recommended approach is incredibly simple, straightforward, and free of complication.
- The recommended approach outperforms other current approaches.
- The nonlinear Mulholland equation has a 0.022% error for the fourth approximated frequency when the real system constant parameter is set to 1.

Thus, it can be concluded that the modified direct iteration method produces remarkably accurate results, making it suitable for a wide range of nonlinear systems.

Conflicts of Interest

The authors declare themselves to be impartial.

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