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A NOVEL QUADRATURE RULE FOR INTEGRATION OF ANALYTIC FUNCTIONS

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Abstract

This paper introduces a novel quadrature rule of precision 5 designed for the numerical integration of definite integrals involving analytic functions. The proposed method synergistically combines Simpson's $\frac{1}{3}rd$ *rule with a quadrature rule derived from Hermite interpolation of degree 3. By harnessing the strengths of both techniques, we establish a new quadrature rule that delivers superior precision, thereby ensuring enhanced accuracy in integration tasks. The theoretical framework underpinning the new rule is developed, and we provide a comprehensive analysis of its convergence properties. Numerical experiments demonstrate the superior performance of the proposed method in comparison to traditional quadrature techniques. The results reveal significant improvements in accuracy and efficiency when applied to various classes of analytic functions. This work aims to advance numerical integration strategies and demonstrates the valuable potential of hybrid methods in enhancing computational performance for the integration of analytic functions.*

Keywords: Simpson's $\frac{1}{3}rd$ rule, Hermite interpolation, Degree of Precision, Analytic functions

I. Introduction

Numerical quadrature is a fundamental tool in numerical analysis, enabling the approximation of definite integrals when analytical solutions are difficult or impossible to obtain. Numerous real-world problems involve integrals that may be challenging to solve analytically due to their complexity or the characteristics of the functions involved. Numerical quadrature provides efficient algorithms to compute these integrals to a high degree of accuracy, allowing researchers and practitioners to obtain results quickly. Numerical quadrature is applicable in various domains, including physics, engineering, economics, and data science. This approach is commonly employed in simulations, optimization challenges, and cases involving continuous functions where integration is necessary. It serves important functions, such as determining areas, volumes, and the overall accumulated value of a quantity. Numerical quadrature methods can handle complex functions, including those that are highly oscillatory, discontinuous, or have singularities. This versatility makes it possible to approximate integrals that would otherwise be intractable. Quadrature forms the backbone of many numerical methods, including finite element analysis, numerical solutions to differential equations, and Monte Carlo simulations. Accurate integration is crucial in these methods to ensure reliable and valid results. Numerical quadrature is an important and evolving field of research. Efforts are continually being made to enhance accuracy, optimize computational efficiency, and expand the applicability of these techniques to a diverse array of problems. New quadrature rules, such as those incorporating adaptive strategies or hybrid methods, contribute to this evolving field. Grasping the concepts of numerical quadrature is of significant importance for students and professionals in mathematics, physics, and engineering fields. This knowledge serves as a valuable foundation for further exploration in numerical analysis and computational methods, empowering individuals with the skills necessary to address complex challenges effectively.

In our study, we will focus on evaluating the integral [I]

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz \tag{1}$$

Where z_0 , h are complex numbers; $f(z)$ is an analytic function defined in the domain $\mathcal{H} \subset \mathbb{C}$.

Simpson's Rule and Trapezoidal Rule, while effective for low-order polynomials, struggle to provide accurate results for functions that exhibit higher-order variations or steep gradients. These methods typically have precision up to third or second order, respectively, which is insufficient for functions with more complex behaviour. Gauss-Legendre Quadrature, while more accurate, still becomes less efficient when the degree of the function increases, requiring more nodes to maintain accuracy.

The method combines Simpson's $\frac{1}{3}rd$ rule, which is typically of third-order accuracy, with a quadrature rule based on Hermite interpolation of degree 3, enhancing the precision to fifth-order. This improvement is especially beneficial for functions that are smooth and analytic. Also this new rule provides more accurate result as compared to other quadrature rules of degree of precision less than 5.

The research aims to improve the accuracy of existing numerical integration methods by creating a quadrature rule with higher precision. The new rule seeks to achieve fifth-order precision by merging Simpson's $\frac{1}{3}rd$ rule, traditionally third-order accurate, with quadrature rule based on Hermite interpolation of degree 3, which enhances the method's ability to capture more intricate details of the function being integrated. The method is particularly tailored for the integration of analytic functions, which are smooth and differentiable. The research seeks to leverage the smoothness of analytic functions to provide a more accurate approximation over a fixed number of function evaluations, surpassing traditional methods in terms of accuracy for these types of functions.

B.P Acharya [I] worked on the integration of analytic functions. E. Denich et al [III] investigated numerical quadrature for oscillating functions. Jun Zhoua Pieter et al [IV] has worked on numerical quadrature for Gregory quads. K Malik et al [V] contributed some derivative-based schemes for numerical cubature. Memon K et al [VII] worked on a derivative-based scheme for Riemann-Stieltjes Integral. P.A.A. Magalhaes et al [VIII] discovered quadrature rules using spline interpolation. Authors [IX,XI] studied on derivative-based quadrature rules.

II. Mathematical Model

The core idea is to combine the strengths of Simpson's $\frac{1}{3}rd$ rule with quadrature rule based on Hermite interpolation of degree 3 to achieve higher precision and better accuracy for integrating analytic functions.

II.i. Simpson's $\frac{1}{3}rd$ rule

Simpson's $\frac{1}{3}rd$ rule [X] is a widely used method for numerical integration and is based on quadratic interpolation. It approximates the integral of a function $f(x)$over the interval $[a, b]$.

The rule can be written as:

$$\int_a^b f(x)\,dx \cong \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{2}$$

where $h = \frac{b-a}{2}$

This rule achieves third-order accuracy because it exactly integrates polynomials of degrees up to 3. In light of equation (1) we can write

$$\int_0^1 f(x)\,dx \cong \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] \tag{3}$$

II.ii. Hermite interpolation-based quadrature rule

The two-point quadrature rule based on Hermite interpolation is expressed as follows [II]

$$\int_0^1 f(x)\,dx \cong \frac{1}{2} f\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right) \tag{4}$$

II.iii. Formulating $\int_{z_0-h}^{z_0+h} f(z)\,dz$ using Simpson's $\frac{1}{3}rd$ rule

From equation (1)

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\,dz$$

Using transformation $t = \frac{1}{2h}(z - z_0 + h)$ in the above integral, we have

$$I(f) = 2h \int_0^1 f(z_0 + 2th - h)\,dt$$

$$= \frac{h}{3}[f(z_0 - h) + 4f(z_0) + f(z_0 + h)] = S_L(f) \tag{5}$$

(From equation (3))

Using Taylor's series to functions present on the right-hand side of equation (5) about z_0

$$S_L(f) = 2hf(z_0) + \frac{h^3}{3} f''(z_0) + \frac{2h^5}{3 \times 4!} f^{iv}(z_0) + \frac{2h^7}{3 \times 6!} f^{vi}(z_0) + \ldots\ldots$$

Now applying Taylor's series to the function $f(z_0 + 2th - h)$ about z_0 we have

$$2h \int_0^1 f(z_0 + 2th - h)\,dt = 2hf(z_0) + \frac{h^3}{3} f''(z_0) + \frac{2h^5 f^{iv}(z_0)}{5 \times 4!} + \frac{2h^7}{7 \times 6!} f^{vi}(z_0) + \ldots\ldots \tag{6}$$

Therefore

$$I(f) - S_L(f) = \frac{-h^5}{90} f^{iv}(z_0) - \frac{8h^7}{21 \times 6!} f^{vi}(z_0) + \ldots\ldots \tag{7}$$

The error associated with the quadrature rule defined in equation (5) is:

$$E_{S_L}(f) = \frac{-h^5}{90} f^{iv}(z_0) \tag{8}$$

II.iv. Formulating $\int_{z_0-h}^{z_0+h} f(z)\,dz$ using Quadrature rule based on Hermite interpolation

Proceeding in the same way as in sub-section 2.3 together with the help of the quadrature rule defined in equation (4) we have

$$I(f) \cong h\left[f\left(\frac{-2h}{\sqrt{12}} + z_0\right) + f\left(\frac{2h}{\sqrt{12}} + z_0\right)\right] = H_L(f) \tag{9}$$

Now applying Taylor's series[VI] to functions present on the right-hand side of equation (9) we have

$$H_L(f) = h\left[f\left(\frac{-2h}{\sqrt{12}} + z_0\right) + f\left(\frac{2h}{\sqrt{12}} + z_0\right)\right]$$

$$= 2hf(\, z_0) + \frac{h^3}{3} f''(z_0) + \frac{h^5}{12 \times 9} f^{iv}(z_0) + \frac{2^7 h^7}{\left(\sqrt{12}\right)^6} \frac{f^{vi}(z_0)}{6!} + \ldots\ldots$$

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz$$

$$= 2h \int_0^1 f(z_0 + 2th - h)\, dt$$

$$= 2hf(\, z_0) + \frac{h^3}{3} f''(z_0) + \frac{2h^5 f^{iv}(z_0)}{5 \times 4!} + \frac{2h^7}{7 \times 6!} f^{vi}(z_0) + \ldots\ldots.$$

(from (6))

Now

$$I(f) - H_L(f) = \frac{h^5}{135} f^{iv}(z_0) + \frac{h^7}{6!} \cdot \frac{40}{189} f^{vi}(z_0) + \ldots\ldots \quad (10)$$

The error term of the formula represented in equation (9) is: $E_{H_L}(f) = \frac{h^5}{135} f^{iv}(z_0)$

II.v. Formulation of Novel Quadrature Rule

27 times of equation (10) is:

$$27[I(f) - H_L(f)] = \frac{h^5}{5} f^{iv}(z_0) + \frac{h^7}{6!} \cdot \frac{27 \times 40}{189} f^{vi}(z_0) + \ldots\ldots\ldots \quad (11)$$

18 times of equation (7) is:

$$18[\, I(f)\ -\ S_L(f)] = \frac{-h^5}{5} f^{iv}(z_0)\ - \frac{h^7}{6!} \cdot \frac{8 \times 18}{21} \cdot f^{vi}(z_0) - \ldots\ldots\ldots \quad (12)$$

Now adding equations (10) and (11) we get

$$I(f) = \frac{3}{5} H_L(f) + \frac{2}{5}\, S_L(f) + \frac{h^7 f^{vi}(z_0)}{45} \left\{\frac{27 \times 40}{189 \times 6!} - \frac{8 \times 18}{21 \times 6!}\right\} + \ldots\ldots\ldots$$

Therefore $$I(f) = \frac{3}{5} H_L(f) + \frac{2}{5}\, S_L(f) = C_L(f) \quad (13)$$

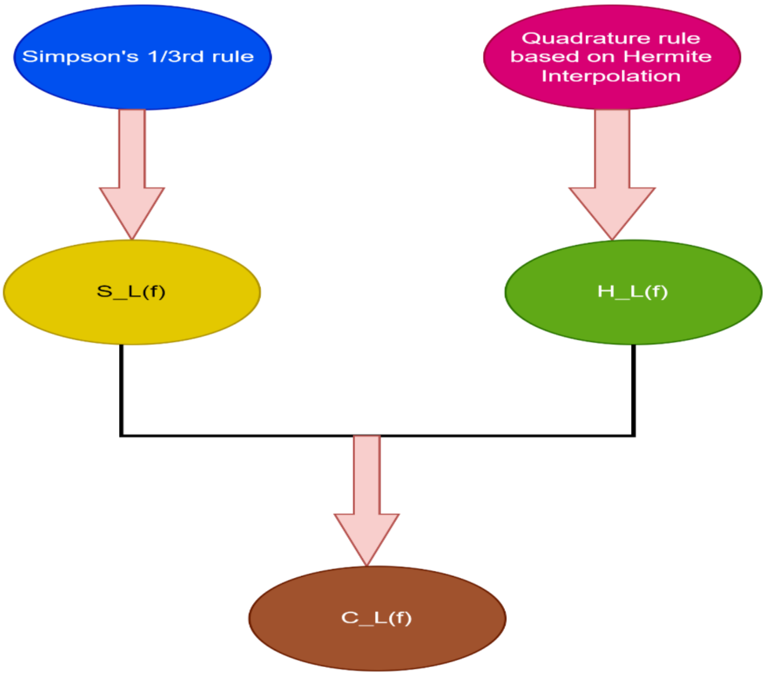
Where $H_L(f) = h\left[f\left(\frac{-2h}{\sqrt{12}} + z_0\right) + f\left(\frac{2h}{\sqrt{12}} + z_0\right)\right]$

$$S_L(f) = \frac{h}{3}[f(z_0 - h) + 4f(z_0) + f(z_0 + h)]$$

The new novel quadrature formula defined in equation (12) is our desired formula.

$$E_{CL}(f) = I(f) - C_L(f) = \frac{h^7 f^{vi}(z_0)}{45}\left\{\frac{27\times40}{189\times6!} - \frac{8\times18}{21\times6!}\right\} + \ldots\ldots.. \qquad (14)$$

The diagrammatic representation of the construction of $C_L(f)$ is as follows



III. Error Analysis and Precision Evaluation

Error analysis of the proposed quadrature rule (13) is discussed.

Theorem-1

If $\boldsymbol{f(z)}$ is analytic in a domain $\mathcal{H} \supset [z_0 - h, z_0 + h\,]$ then the truncation error for the quadrature rule

$C_L(f)$ denoted by $E_{C_L}(f)$ and is given by $E_{C_L}(f) = \frac{h^7 f^{vi}(z_0)}{45}\left\{\frac{27\times40}{189\times6!} - \frac{8\times18}{21\times6!}\right\}$.

Proof:

The proof follows directly from equation (13).

Theorem-2

Error bound of the constructed quadrature rule $C_L(f)$ is given by $\left|E_{C_L}(f)\right| \leq \frac{h^7 M}{45}\left\{\frac{27\times40}{189\times6!} - \frac{8\times18}{21\times6!}\right\}$ where $= max\left|f^{vi}(z_0)\right|$.

Proof:

Proof follows directly from theorem 1.

The local truncation error for the quadrature rule based on Simpson's $\frac{1}{3}rd$ rule over a subinterval $[z_0 - h , z_0 + h]$ is given by the following expression:

$$E_{S_L}(f) = \frac{-h^5}{90} f^{iv}(z_0)$$

This error term is proportional to the fourth derivative of the integrand f indicating that the rule is accurate to fourth order. More specifically, the global error for the quadrature rule based on Simpson's $\frac{1}{3}rd$ rule is (h^5) , where the coefficient depends on f^{iv}, which controls the magnitude of the truncation error for smooth analytic functions.

Similarly, the local truncation error for the Hermite-based quadrature rule is: $E_{H_L}(f) = \frac{h^5}{135} f^{iv}(z_0)$

As in previous analyses, this error is proportional to the fourth derivative and demonstrates fourth-order accuracy. The distinction between the two rules arises from the different constant multipliers, which reflect the distinct weights used in the Hermite interpolation-based method. Despite this difference, the asymptotic accuracy of both rules is identical in terms of order.

When the quadrature rule based on Simpson's $\frac{1}{3}rd$-rule and Hermite-based rule are combined, and the resulting error term is influenced by higher-order derivatives of the integrand. The leading-order truncation error of the combined quadrature rule is:

$$E(f) = \frac{h^7 f^{vi}(z_0)}{45} \left\{ \frac{27 \times 40}{189 \times 6!} - \frac{8 \times 18}{21 \times 6!} \right\}$$

The error term now incorporates the sixth derivative, highlighting an opportunity for enhanced accuracy in combined rule, which outperforms each of the individual component rules. This result demonstrates that the combined quadrature rule achieves sixth-order accuracy with $O(h^7)$ behaviour.

The quadrature rules based on Simpson's $\frac{1}{3}rd$ rule and the Hermite-based rule—each exhibit a third degree of precision. However, the combination of these two rules yields a fifth degree of precision. This demonstrates that the proposed method provides improved precision for the integration of analytic functions, making it highly efficient for applications where higher accuracy is required.

IV. Results and Discussion

The performance of the newly constructed quadrature rule is examined by applying it to test functions. The goal is to evaluate the accuracy of the proposed method compared to well-known quadrature methods. $H_L(f)$ and $S_L(f)$. We have taken the following test functions to ensure the computational efficiency of the new quadrature rule. $C_L(f)$.

1. $I_1 = \int_{-\frac{i}{2}}^{\frac{i}{2}} \cos z \, dz$, $Exact\ value = 1.0421906109874948i$
2. $I_2 = \int_{-i}^{i} e^z \, dz$, $Exact\ Value = 1.682941969615793i$

The numerical results for each test function are summarized in the tables below. The tables provide the error for each quadrature method. The proposed method is compared with $H_L(f)$ and $S_L(f)$.

Table 1. Computation of error and approximate values for the integral

$$I_1 = \int_{-\frac{i}{2}}^{\frac{i}{2}} \cos z \, dz$$

Quadratur Rules	Approximate Values	Error=|approx. value − exact value|	Degree of Precision
$S_L(f)$	1.0425419884021268i	0.00035137741463198147	3
$H_L(f)$	1.041956823470835i	0.00023378751665981135	3
$C_L(f)$	1.0421908894433516i	$2.784558568169615\times10^{-7}$	5

Table 2. Computation of error and approximate values for the integral $I_1 = \int_{-i}^{i} e^z \, dz$

Quadrature Rules	Approximate Values	Error=|approx. value − exact value|	Degree of Precision
$S_L(f)$	1.69353487057876i	0.010592900962967056	3
$H_L(f)$	1.6758236553899861i	0.007118314225806888	3
$C_L(f)$	1.6829081414654956i	$3.382815029739916\times10^{-5}$	5

The following graph shows a comparison of exact and approximate values for quadrature rules $S_L(f)$,$H_L(f)$ and $C_L(f)$ for I_1.

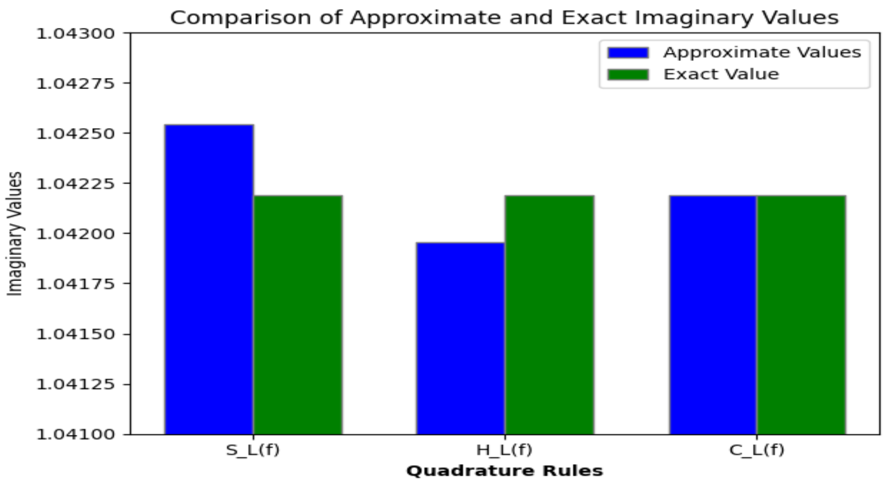


Fig.1

Fig.2 provides the comparison of degree of precision with error of quadrature rules $S_L(f)$,$H_L(f)$ and $C_L(f)$ for I_1.

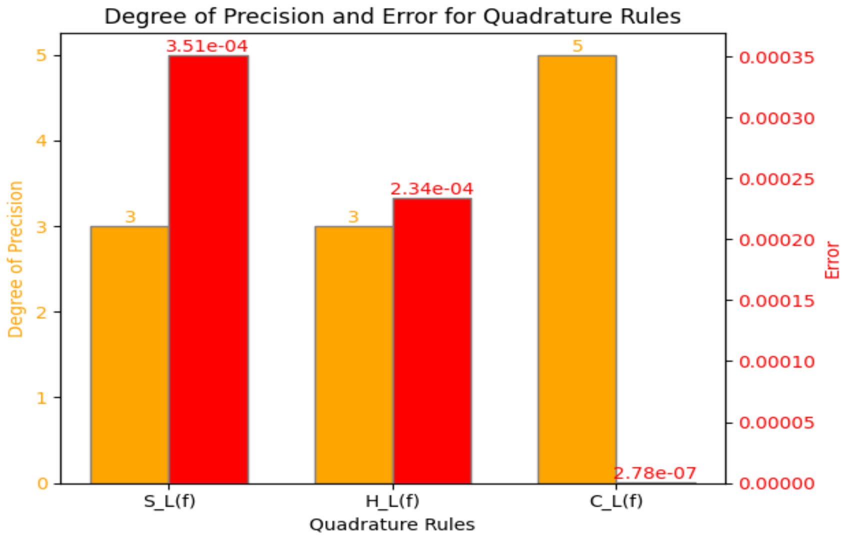


Fig. 2.

The following graph is about the comparison of exact values and approximate values of quadrature rules $S_L(f)$,$H_L(f)$ and $C_L(f)$ for I_2.

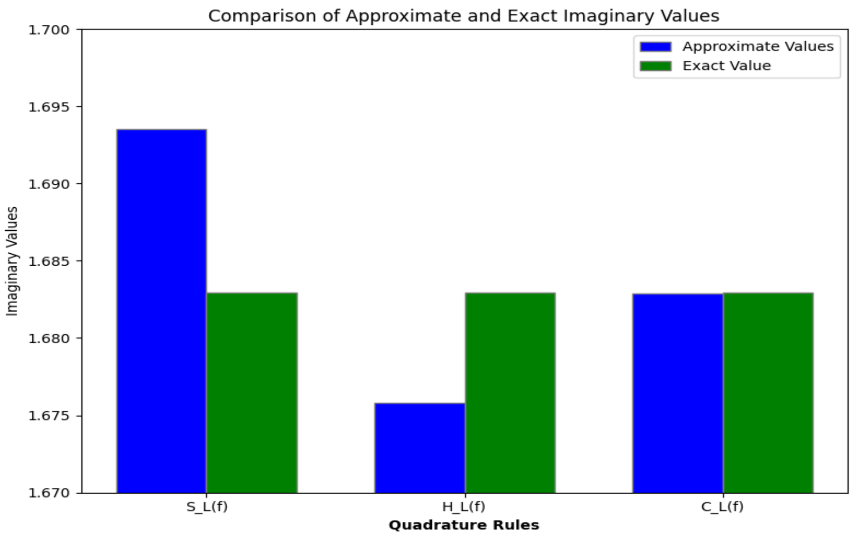


Fig. 3.

Following figure is about the comparison of degree of precision with error of quadrature rules $S_L(f)$,$H_L(f)$ and $C_L(f)$ for I_2.

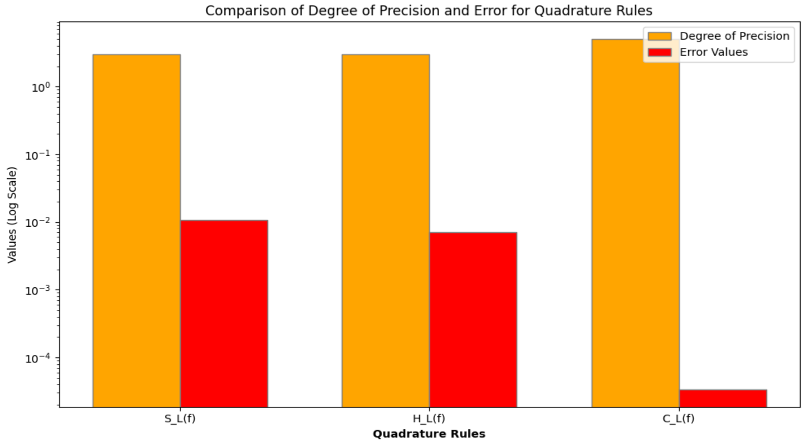


Fig. 4.

V. Conclusion

In this work, we introduced a novel quadrature rule that combines Simpson's $\frac{1}{3}rd$-rule with a quadrature method derived from Hermite interpolation of degree 3. Through thorough theoretical analysis and numerical experiments, we have shown that the proposed rule reaches higher precision, inspiring confidence in our ability to integrate analytic functions more effectively. The results confirmed that this hybrid approach significantly improves accuracy and efficiency compared to traditional quadrature techniques. This work underscores the potential of combining classical methods in innovative ways to advance numerical integration strategies, providing a valuable tool for applications requiring precise and efficient integration of analytic functions. Future research may further extend this methodology to more complex function classes and explore its application in higher-dimensional integrals.

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Conflict of Interest:

There is no conflict of interest regarding this paper.

References

I. B.P. Acharya,: 'Compound Birkhoff-Young rule for numerical integration of analytic functions', *Int. J. Math. Educ. Sci. Technol.*. Vol. 14(1), pp.101-103, 1983. 10.1080/0020739830140116

II. E. Suli and David F. Mayer, : An Introduction to Numerical Analysis, *Cambridge University press,2003.* http://www.statslab.cam.ac.uk/~lab85/resources/schedules2223.pdf

III. E. Denich, P. Novati,: 'Numerical quadrature for integrals involving oscillating Functions'. Journal of Approximation Software, Vol. 1, pp:1-14, 2024. https://ojs.unito.it/index.php/JAS/article/view/10097/8890

IV. Jun Zhoua, Pieter J. Barendrecht, Michael Barton, Jirí Kosinka ,: 'Numerical quadrature for Gregory quads'. *Applied Mathematics and Computation*, vol. 453, pp: 1-14, 2023. https://www.cs.rug.nl/~jiri/papers/23ZhBaBaKo.pdf

V. K Malik, M. M. Shaikh, M.S. Chandio, A.W. Shaikh,: 'Some new and efficient derivative based schemes for numerical cubature'. J. Mech. Cont. & Math. Sci.,Vol.1. ,pp:67-78. 2020. 10.26782/jmcms.2020.10.00005.

VI. Lin Ma,: 'The Taylor's theorem and its application, highlights in science', engineering and Technology. Vol.72, pp:1368-1372. 10.54097/98knf072

VII. Memon K., M. M. Shaikh, M. S. Chandio, A. W. Shaikh,: 'A Modified Derivative-Based Scheme for the Riemann-Stieltjes Integral', Sindh University Research Journal,Vol. 52(01), pp:37-40, 2020 10.26692/sujo/2020.03.06

VIII. P. A. A. Magalhaes, P.A.A. Magalhaes Junior, C. A Magalhaes, A.L.M.A Magalhaes,: 'New Formulas of Numerical Quadrature Using Spline Interpolation'. Archieves of Computational Methods in Engineering, Vol. 28, pages 553–576, 2021. 10.1007/s11831-019-09391-3

IX. Rana, K.: 'Harmonic Mean and Contra-Harmonic Mean Derivative-Based Closed Newton-Cotes Quadrature'. Integr. J. Res. Arts Humanit. Vol. 2, pp: 55– 61, 2022 . 10.55544/ijrah.2.3.36

X. R. L. Burden, J.D Faires, : Numerical Analysis, *Cengage Learning*, Ninth Edition, 2011. https://www.pdfdrive.com/numerical-analysis-9th-edition-e157182502.html

XI. S. Mahesar, M.M. Shaikh, M.S. Chandio, A. W. Shaikh ,: 'Some New Time and Cost Efficient Quadrature Formulas to Compute Integrals Using Derivatives with Error Analysis'. *Symmetry*, Vol.14(12), pp:2611, 2022. 10.3390/sym14122611