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ANALYSIS OF SERIAL QUEUES LINKED WITH NON-SERIAL SERVICE CHANNELS CHARACTERIZED BY FEEDBACK AND CUSTOMERS' BEHAVIOUR

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Abstract

This research primarily presents a model involving R-serial service channels connected to S non-serial service channels. Feedback mechanisms are applied to the serial queues, while balking and reneging behaviors, triggered by urgent calls/messages or customer impatience, are analyzed in both serial and non-serial queues. After developing the queuing model, the system's differential-difference equations are formulated in a compact form, and their solutions are derived by reducing them to the steady-state form for unlimited waiting capacity. Marginal probabilities and mean queue lengths are calculated to evaluate the system's performance in this scenario.

Keywords: Differential-difference equations, Exponential, Impatient behaviour, Poisson, Probabilities, Queue discipline, Service channels, Steady-state, Urgent message, Waiting space, .

I. Introduction

Queuing theory is an extensive field, with numerous researchers contributing significantly, as highlighted in the references. Building on these various studies, a more realistic and generalized queuing model has been developed.

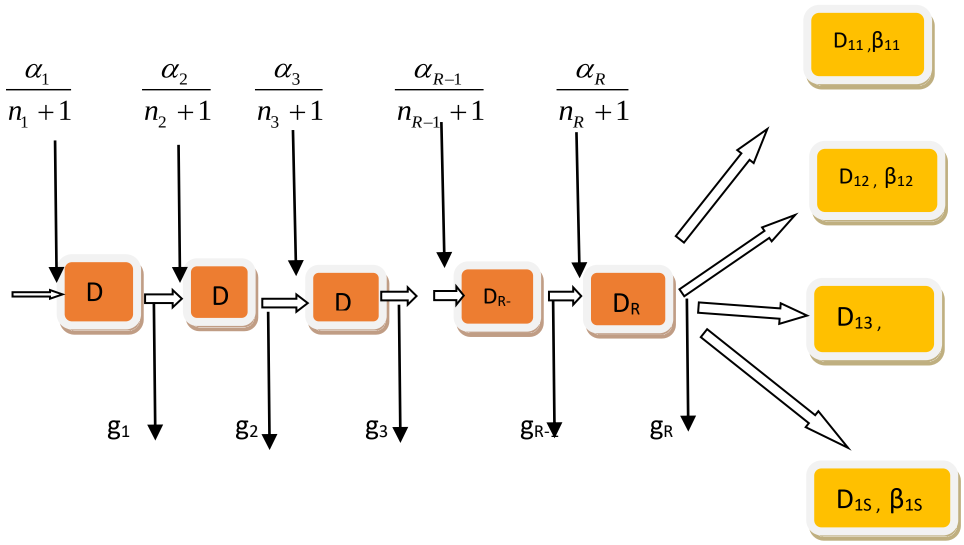

Fig. 1.

II. Model description

A queuing model (Fig. 1) consists of R serial service channels $M_1, M_2, M_3, ..., M_{R-1}, M_R$, each with its respective servers $D_1, D_2, D_3, ..., D_{R-1}, D_R$. The last channel M_R in the serial queue is connected to S non-serial service channels $M_{11}, M_{12}, M_{13}, ..., M_{1S-1}, M_{1S}$ which also have their respective servers $D_{11}, D_{12}, D_{13}, ..., D_{1S-1}, D_{1S}$. Customers arrive at these service channels in a Poisson stream, with the arrival rates at serial and non-serial channels denoted as α_a and α_{1b} respectively. If new customers choose not to enter any queue, the Poisson input rate becomes $\frac{\alpha_a}{n_a+1}$, where n_a is the number of customers in the queue at the a^th channel. Similarly, if customers hesitate to join non-serial queues, demonstrating balking behavior, the arrival rate at the non-serial queue is $\alpha_{1b} / m_b + 1$, where m_b is the number of customers in queue M_{1b}.

The service times for the servers Da (a = 1, 2, 3,…, R) and D_{1b} (b = 1, 2, 3,…, S) follow mutually independent negative exponential distributions, with rates β_a and β_{1b}, respectively. After completing service at Da, a customer may leave the a^th service channel with probability g_a, proceed to the next channel with probability $\frac{p_a}{n_{a+1}+1}(a=1,2,3,...,R-1)$, or return to a previous channel for re-service with probabilities $\frac{r_{al}}{n_a+1}(l=1,2,3,...,a-1)$, where $g_a + \frac{p_a}{n_{a+1}+1} + \sum_{l=1}^{a-1}\frac{d_{al}}{n_l+1} = 1 \ (a=1,2,,..,R-1)$

. Once service at D_R is complete, a customer either exits the system with probability g_R, returns for re-service with probabilities $\frac{d_{Rl}}{n_l+1}\,(l=1,2,...,R-1)$, or joins a non-serial queue M1b(b = 1, 2, 3,…, S) with probability $\frac{q_{Rb}}{m_b+1}$, where $g_R+\sum_{l=1}^{R-1}\frac{d_{Rl}}{n_l+1}+\sum_{b=1}^{S}\frac{q_{Rb}}{m_b+1}=1$. Occasionally, customers receive an important call or message while waiting in line, causing them to renege immediately, regardless of the queue length. In such cases, the average reneging rate in the a^{th} serial service channel due to urgent calls/messages is denoted as ω_a , and the rate due to customer impatience after a waiting period T_{0a} is defined as $A_{n_a}=\frac{\beta_a e^{\frac{-\beta_a T_{0a}}{n_a}}}{\left(1-e^{\frac{-\beta_a T_{0a}}{n_a}}\right)}$. Likewise, the reneging rates in non-serial queues are expressed as ω_{1b} and due $A_{1bm_b}=\frac{\beta_{1b} e^{\frac{-\beta_{1b} T_{0b}}{m_b}}}{\left(1-e^{\frac{-\beta_{1b} T_{0b}}{m_b}}\right)}$ to urgent calls and impatience, respectively.

It is assumed that service begins as soon as a customer arrives at an available service channel. The present queuing model is analyzed under the assumption of unlimited capacity. The steady-state marginal probabilities and the mean queue length of the system are computed for unlimited capacity under a first-come, first-served discipline.

For instance, consider the hierarchical structure of a district administration, with various officers such as Patwaris, Kanoongos, Sadar Kanoongos, Sub-Tehsildars, Tehsildars, SDMs, ADMs, and DMs representing the servers in the model. People may approach these officers directly or move up the hierarchy to address their issues. Balking and reneging are common in such administrative setups. Senior officers may send individuals back to lower-level officers for re-service if some information is missing. Typically, the officers call people in a random order for hearings. The final authority, such as the DM, may direct individuals to other departments (like Education, Health, or Electricity) if their issues pertain to those areas, corresponding to the non-serial servers in this model.

III. Methodology

The difference-differential equation approach is preferred for this study, and iterative methods, generating functions, and linear operators will be used to solve the equations.

"Defining the probability $P(n_1, n_2, n_3, ...n_R; m_1, m_2, m_3, ...m_S; t)$ at a time 't' such that n_a customers (either leave the queue or after service by a[th] service channel either get out of the system or enter the next phase or join back its any previous service channel) waiting in the a[th] service channel before the server D_a(a=1,2,3,…,R-1) , n_R customers (either renege or after getting service from the R[th] service channel either leave the system or join back all its previous channels or join any non-serial service channels) waiting in R[th] service channel in front of server D_R ; m_b customers (which leave the system after service) waiting in M_{1b} before the server D_{1b}(b=1,2,3,…,S)"

"The Equations of the system are considered in the compact form by using the operators $X_{a}.,\ X._{a},\ and\ \ X._{a,a+1}.$ on the vector $\tilde{n} = (n_1, n_2, n_3, ...n_R)$ defined by

$$X_a.\left(\tilde{n}\right) = (n_1, n_2, ..., n_i - 1, ...n_R);\ X._a\left(\tilde{n}\right) = (n_1, n_2, ..., n_a + 1, ..., n_R);$$

$$X._{a,a+1}.(\tilde{n}) = (n_1, n_2, ..., n_a + 1, n_{a+1} - 1, ..., n_R)."$$

III.i. Differential-difference Equations

These types of equations of the system are as under

$$\frac{d}{dt}P(\tilde{n}, \tilde{m}; t) = -\left[\sum_{a=1}^{R}\frac{\alpha_a}{n_a+1} + \sum_{b=1}^{S}\frac{\alpha_{1b}}{m_b+1} + \sum_{a=1}^{R}\delta(n_a)(\beta_a + \omega_a + A_{an_a}) + \sum_{b=1}^{S}\delta(m_b)(\beta_{1b} + \omega_{1b} + A_{1bm_b})\right]P(\tilde{n}, \tilde{m}; t)g$$

$$\sum_{a=1}^{R}\frac{\alpha_a}{n_a}P\left(X_a.(\tilde{n}), \tilde{m}; t\right) + \sum_{b=1}^{S}\frac{\alpha_{1b}}{m_b}P\left(\tilde{n}, X_{b.}(\tilde{m}); t\right) + \sum_{a=1}^{R-1}\frac{\beta_a p_a}{n_{a+1}}P\left(X_{.a,a+1.}(\tilde{n}), \tilde{m}; t\right)$$

$$+\sum_{a=1}^{R}\left(\beta_a g_a + A_{an_a+1} + \omega_a\right)P\left(X._a\left(\tilde{n}\right), \tilde{m}; t\right)$$

$$+\sum_{a=2}^{R}\beta_a\sum_{l=1}^{a-1}\frac{d_{al}}{n_l}P\left((n_1, n_2, ...n_{l-1}, .., n_l - 1, n_{l+1}, .., n_a + 1, .., n_R), \tilde{m}; t\right) \tag{1}$$

$$+\sum_{b=1}^{S}\frac{\beta_b}{m_b}q_{Rb}P(n_1, n_2, ..., n_R + 1, X_{b.}(\tilde{m}); t) + \sum_{b=1}^{S}\left(\beta_{1b} + A_{1bm_b+1} + \omega_{1b}\right)P(\tilde{n}; X_{.b}(\tilde{m}); t)$$

$$"n_a \geq 0, m_b \geq 0;\ (a = 1, 2, ..., R; b = 1, 2, ..., S)"$$

where

$$\delta(n_a) = \begin{cases} 1 & if \quad n_a \neq 0 \\ 0 & if \quad n_a = 0 \end{cases}$$

And $P\left(\tilde{n},\tilde{m};t\right)=\tilde{0}$ "if any of the arguments is negative".

III.ii. Steady–State Equations

The equations obtained from (1) by equating the time derivative to zero in the differential-difference Equation are as under:

$$\left[\sum_{a=1}^{R}\frac{\alpha_a}{n_a+1}+\sum_{b=1}^{S}\frac{\alpha_{1b}}{m_b+1}+\right]P\left(\tilde{n},\tilde{m}\right)+$$

$$\left[\sum_{a=1}^{R}\delta\left(n_a\right)\left(\beta_a+\omega_a+A_{an_a}\right)+\sum_{b=1}^{S}\delta(m_b)\left(\beta_{1b}+\omega_b+A_{1bm_b}\right)\right]P\left(\tilde{n},\tilde{m}\right)$$

$$=\sum_{a=1}^{R}\frac{\alpha_a}{n_a}P\left(X_{a}.\left(\tilde{n}\right),\tilde{m}\right)+\sum_{b=1}^{S}\frac{\alpha_{1b}}{m_b}P\left(\tilde{n},X_{b}\left(\tilde{m}\right)\right)+\sum_{a=1}^{R-1}\frac{\beta_a p_a}{n_{a+1}}P\left(X_{a,a+1}\left(\tilde{n}\right),\tilde{m}\right)$$

$$+\sum_{a=1}^{R}\left(\beta_a g_a+A_{an_a+1}+\omega_a\right)P\left(X._a\left(n\right),\tilde{m}\right)$$

$$+\sum_{a=2}^{R}\beta_a\sum_{l=1}^{a-1}\frac{d_{al}}{n_l}P\left((n_1,n_2,...n_{l-1},..,n_l-1,n_{l+1},..,n_a+1,..,n_R),\tilde{m}\right) \tag{2}$$

$$+\sum_{b=1}^{S}\beta_R\frac{q_{Rb}}{m_b}P(n_1,n_2,...,n_R+1,X_{b.}(\tilde{n}))+\sum_{b=1}^{S}\left(\beta_{1b}+A_{1bm_b+1}+\omega_{1b}\right)P(\tilde{n};X_{.b}(\tilde{m}))$$

for $"n_a\geq 0,m_b\geq 0;(a=1,2,...,R;b=1,2,....,S)"$

III.iii. Steady-State Solutions

These equations (2) are satisfied by the following Steady-State solutions

$$P\left(\tilde{n},\tilde{m}\right)=P\left(\tilde{0},\tilde{0}\right)\left(\frac{1}{n_1!}\frac{\left(\alpha_1+\sum_{a=2}^{R}\frac{\beta_a d_{a1}\rho_a}{(n_a+1)(\beta_a+\omega_a+A_{an_a+1})}\right)^{n_1}}{\prod_{a=1}^{n_1}(\beta_1+A_{1a}+\omega_a)}\right).$$

$$\left(\frac{1}{n_2!}\frac{\left(\alpha_2+\frac{\beta_1 p_1\rho_1}{(n_1+1)(\beta_1+\omega_1+A_{1n_1+1})}+\sum_{a=3}^{R}\frac{\beta_a d_{a2}\rho_a}{(n_a+1)(\beta_a+\omega_a+A_{an_a+1})}\right)^{n_2}}{\prod_{a=1}^{n_2}(\beta_2+\omega_2+A_{2a})}\right).$$

....

$$\cdot\left(\frac{1}{n_{R-1}!}\frac{\left(\alpha_{R-1}+\frac{\beta_{R-1}p_{R-1}\rho_{R-1}}{(n_{R-2}+1)(\beta_{R-2}+\omega_{R-2}+A_{R-2n_{R-2}+1})}+\frac{\beta_R d_{R,R-1}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{R,n_R+1})}\right)^{n_{R-1}}}{\prod_{a=1}^{n_{R-1}}\left(\beta_{R-1}+A_{R-1,a}+\omega_{R-1}\right)}\right)$$

$$\cdot\left(\frac{1}{n_R!}\frac{\left(\alpha_R+\frac{\beta_{R-1}p_{R-1}\rho_{R-1}}{(n_{R-1}+1)(\beta_{R-1}+\omega_{R-1}+A_{R-1,n_{R-1}+1})}\right)^{n_R}}{\prod_{a=1}^{n_R}\left(\beta_R+A_{R,a}+\omega_R\right)}\right)$$

$$\left(\frac{1}{m_1!}\frac{\left(\alpha_{11}+\frac{\beta_R d_{R1}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_1}}{\prod_{b=1}^{m_1}\left(\beta_{11}+A_{11b}+\omega_{11}\right)}\right)\cdot\left(\frac{1}{m_2!}\frac{\left(\alpha_{12}+\frac{\beta_R d_{R2}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_2}}{\prod_{b=1}^{m_2}\left(\beta_{12}+A_{12b}+\omega_{12}\right)}\right)\cdot$$

$$\left(\frac{1}{m_3!}\frac{\left(\alpha_{13}+\frac{\beta_{TR} d_{R3}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_3}}{\prod_{b=1}^{m_3}\left(\beta_{13}+A_{13b}+\omega_{13}\right)}\right)\cdots\cdots\left(\frac{1}{m_S!}\frac{\left(\alpha_{1R}+\frac{\beta_R d_{RS}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_S}}{\prod_{b=1}^{m_S}\left(\beta_{1S}+A_{1Sb}+\omega_{1S}\right)}\right)$$

for . $$"n_a \geq 0, m_b \geq 0; (a=1,2,3,...,R; b=1,2,3,....,S)"$$

Here,

$$"\rho_1=\alpha_1+\sum_{a=2}^{R}\frac{\beta_a d_{a1}\rho_a}{(n_a+1)(\beta_a+\omega_a+A_{an_a+1})}$$

$$\rho_2=\alpha_2+\frac{\beta_1 p_1\rho_1}{(n_1+1)(\beta_1+\omega_1+A_{1n_1+1})}+\sum_{a=3}^{R}\frac{\beta_a d_{a2}\rho_a}{(n_a+1)(\beta_a+\omega_a+A_{an_a+1})}\ \cdots\cdots\cdots$$

$$\rho_{R-1}=\alpha_{R-1}+\frac{\beta_{R-2}p_{R-2}\rho_{R-2}}{(n_{R-2}+1)(\beta_{R-2}+\omega_{R-2}+A_{R-2,n_{R-2}+1})}+\frac{\beta_R d_{R,R-1}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}$$

$$\rho_R=\alpha_R+\frac{\beta_{R-1}p_{R-1}\rho_{R-1}}{(n_{R-1}+1)(\beta_{R-1}+\omega_{R-1}+A_{R-1,n_{R-1}+1})}$$

(4) R-equations

For the sake of simplicity, we take

$$k_{ab} = \frac{\beta_a d_{ab}}{(n_a+1)\left(\beta_a + A_{a\,n_a+1} + \omega_a\right)} \, (a = 1,2,3,\ldots,R-1)$$

and

$$h_a = (n_a+1)(\beta_a + \omega_a + A_{a\,n_a+1}) \, (a = 1,2,3,\ldots,R)$$

in result (4)

Then,

$$\rho_1 = \alpha_1 + k_{21}\rho_2 + K_{31}\rho_3 + K_{41}\rho_4 + \ldots + K_{R-2,1}\rho_{R-2} + K_{R-1,1}\rho_{R-1} + K_{R,1}\rho_R$$

$$\rho_2 = \alpha_2 + \frac{\beta_1 p_1 \rho_1}{h_1} + k_{32}\rho_3 + k_{42}\rho_4 + \ldots + k_{R-2,2}\rho_{R-2} + k_{R-1,2}\rho_{R-1} + k_{R,2}\rho_R$$

$$\rho_3 = \alpha_3 + \frac{\beta_2 p_2 \rho_2}{h_2} + k_{43}\rho_4 + k_{53}\rho_5 + \ldots + k_{R-2,3}\rho_{R-2} + k_{R-1,3}\rho_{R-1} + k_{R,3}\rho_R$$

....

$$\rho_{R-1} = \alpha_{R-1} + \frac{\beta_{R-2} p_{R-2} \rho_{R-2}}{h_{R-2}} + k_{R,R-1}\rho_R$$

$$\rho_R = \alpha_R + \frac{\beta_{R-1} p_{R-1} \rho_{R-1}}{h_{R-1}}$$

(5) R-equations $\rho_{1,}\rho_{2,}\rho_{3.........}\rho_{R-1,}\rho_R$ are solved with the help of (5) R-equations

R-equations are elucidated for ρ_R (using determinants), we get

$$\rho_R = \left(\frac{\begin{array}{l} \alpha_R \Delta_{R-1} + \frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \alpha_{R-1}\Delta_{R-2} + \frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \cdot \frac{p_{R-2}\beta_{R-2}}{h_{R-2}} \alpha_{R-2}\Delta_{R-3} + \ldots \\ + \left(\frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \cdot \frac{p_{R-2}\beta_{R-2}}{h_{R-2}} \cdot \frac{p_{R-3}\beta_{R-3}}{h_{R-3}} \ldots \frac{p_3\beta_3}{h_3} \cdot \frac{p_2\beta_2}{h_2} .\alpha_2 \Delta_1 \right) \\ + \left(\frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \cdot \frac{p_{R-2}\beta_{R-2}}{h_{R-2}} \cdot \frac{p_{R-3}\beta_{R-3}}{h_{R-3}} \ldots \frac{p_3\beta_3}{h_3} \cdot \frac{p_2\beta_2}{h_2} \cdot \frac{p_1\beta_1}{h_1} \alpha_1 \right) \end{array}}{\begin{array}{l} \Delta_{R-1} - \frac{p_{R-1}\beta_{R-1}}{h_{R-1}} .k_{R,R-1}\Delta_{R-2} - \frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \cdot \frac{p_{R-2}\beta_{R-2}}{h_{R-2}} .k_{R,R-2}\Delta_{R-3} - \\ \ldots - \left(\frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \cdot \frac{p_{R-2}\beta_{R-2}}{h_{R-2}} \ldots \frac{p_2\beta_2}{h_2} k_{R,2}\Delta_1 \right) \\ - \left(\frac{p_{R-1}\beta_{R-1}}{h_{R-1}} \cdot \frac{p_{R-2}\beta_{R-2}}{h_{R-2}} \ldots \frac{p_2\beta_2}{h_2} \cdot \frac{p_1\beta_1}{h_1} .k_{R,1} \right) \end{array}} \right)$$

where

$$\Delta_R = \begin{vmatrix} 1 & -k_{21} & -k_{31} & \ldots & -k_{R-2,1} & -K_{R-1,1} & -k_{R,1} \\ \frac{-p_1\beta_1}{h_1} & 1 & -k_{32} & \ldots & -k_{R-2,2} & -k_{R-1,2} & -k_{R,2} \\ 0 & \frac{-p_2\beta_2}{h_2} & 1 & \ldots & -k_{R-2,3} & -k_{R-1,3} & -k_{R,3} \\ 0 & 0 & \frac{-p_3\beta_3}{h_3} & \ldots & -k_{R-2,4} & -k_{R-1,4} & -k_{R,4} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & -k_{R-1,R-2} & -k_{R,R-2} \\ 0 & 0 & 0 & \ldots & \frac{-p_{R-2}\beta_{R-2}}{h_{R-2}} & 1 & -k_{R,R-1} \\ 0 & 0 & 0 & \ldots & 0 & \frac{-p_{R-1}\beta_{R-1}}{h_{R-1}} & 1 \end{vmatrix}$$

Then

$$\Delta_{R-1} = \begin{vmatrix} 1 & -k_{2,1} & -k_{3,1} & \cdots & -k_{R-2,1} & -k_{R-1,1} \\ \frac{-p_1\beta_1}{h_1} & 1 & -k_{3,2} & \cdots & -k_{R-2,2} & -k_{R-1,2} \\ 0 & \frac{-p_2\beta_2}{h_2} & 1 & \cdots & -k_{R-2,3} & -k_{R-1,3} \\ 0 & 0 & \frac{-p_3\beta_3}{h_3} & \cdots & -k_{R-2,4} & -k_{R-1,4} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \cdots & -k_{R-2,R-3} & -k_{R-1,R-3} \\ 0 & 0 & 0 & \cdots & 1 & -k_{R-1,R-2} \\ 0 & 0 & 0 & \cdots & \frac{-p_{R-2,R-2}}{h_{R-2}} & 1 \end{vmatrix}$$

Continuing in this way

$$\Delta_3 = \begin{vmatrix} 1 & -k_{2,1} & -k_{3,1} \\ \frac{-p_1\beta_1}{h_1} & 1 & -k_{3,2} \\ 0 & \frac{-p_2\beta_2}{h_2} & 1 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 1 & -k_{2,1} \\ \frac{-p_1\beta_1}{h_1} & 1 \end{vmatrix}, \ \Delta_1 = |1| = 1$$

“Since the value of ρ_R has already been calculated, we can determine ρ_{R-1} by substituting ρ_R into the final equation of (2.8). Similarly, we can compute ρ_{R-2} by inserting the values of ρ_{R-1} and ρ_R into the second-to-last equation $\rho_{R-3}, \rho_{R-4}, ---, \rho_3, \rho_2,$ and ρ_1.

All the parameters $\rho_1, \rho_2, \rho_3, ---, \rho_{R-1}, \rho_R$ have been calculated except $P(\tilde{0},\tilde{0})$ which can be determined by normalizing conditions.

$$\sum_{\substack{\tilde{m}=\tilde{0} \\ \tilde{n}=\tilde{0}}}^{\infty} P(\tilde{n},\tilde{m}) = 1$$

and with the restriction that "the traffic intensity of each service channel of the system is less than unity".

Thus,

$$P\left(\tilde{n},\tilde{m}\right)=P\left(\tilde{0},\tilde{0}\right)\left(\frac{1}{n_1\,!}\left(\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}\ (\beta_1+A_{1i}+\omega_1)}\right)\right).$$

$$\left(\frac{1}{n_2\,!}\left(\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}(\beta_2+A_{2i}+\omega_2)}\right)^{n_2}\right)$$

$$\cdot\left(\frac{1}{n_3\,!}\left(\frac{(\rho_3)^{n_3}}{\prod_{a=1}^{n_3}\ (\beta_3+A_{3a}+\omega_3)}\right)\right)\cdots\left(\frac{1}{n_{R-1}\,!}\left(\frac{(\rho_{R-1})^{n_{R-1}}}{\prod_{a=1}^{n_{R-1}}\ (\beta_{R-1}+A_{R-1,a}+\omega_{R-1})}\right)\right)$$

$$\cdot\left(\frac{1}{n_R!}\left(\frac{(\rho_R)^{n_R}}{\prod_{a=1}^{n_R}\quad(\beta_R+A_{R,a}+\omega_R)}\right)\right)\cdot\left(\frac{1}{m_1!}\frac{\left(\alpha_{11}+\frac{\beta_R d_{R1}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_1}}{\prod_{b=1}^{m_1}(\beta_{11}+A_{11b}+\omega_{11})}\right)\quad .$$

$$\left(\frac{1}{m_2\,!}\frac{\left(\alpha_{12}+\frac{\beta_R d_{R2}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_2}}{\prod_{y=1}^{m_2}\left(\beta_{12}+A_{12y}+\omega_{12}\right)}\right)\cdot\left(\frac{1}{m_3\,!}\frac{\left(\alpha_{13}+\frac{\beta_R d_{R3}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_3}}{\prod_{b=1}^{m_b}\left(\beta_{13}+A_{13b}+\omega_{13}\right)}\right)\cdots\cdots\cdots$$

$$\left(\frac{1}{m_S\,!}\frac{\left(\alpha_{1S}+\frac{\beta_R d_{RS}\rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}\right)^{m_S}}{\prod_{y=1}^{m_S}\left(\beta_{1S}+A_{1Sb}+\omega_{1S}\right)}\right)\qquad(6)$$

$"n_a \geq 0, m_b \geq 0; (a = 1, 2, 3, ..., R; b = 1, 2, 3, ..., S)"$

has been determined completely.

Additionally, since the current queuing model is analyzed under steady-state conditions and the derivative $P(\tilde{n}, \tilde{m})$ does not rely on any particular queue discipline, in the long run, customer reneging due to impatience becomes constant. As a result, the reneging rates are represented as A_a for the serial queues, and A_{1bmb} simplifies to A_{1b} for the non-serial queues. Putting the difference-differential equations (1), in equations (2), and the solutions (3) and (6), we get solutions in steady-state as under

$$P(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0}) \left(\frac{1}{n_1 !} \left(\frac{\rho_1}{(\beta_1 + A_1 + \omega_1)} \right)^{n_1} \right) \cdot \left(\frac{1}{n_2 !} \left(\frac{\rho_2}{(\beta_2 + A_2 + \omega_2)} \right)^{n_2} \right)$$

$$\left(\frac{1}{n_3 !} \left(\frac{\rho_3}{(\beta_3 + A_3 + \omega_3)} \right)^{n_3} \right) \cdots \left(\frac{1}{n_{R-1} !} \left(\frac{\rho_{R-1}}{\beta_{R-1} + A_{R-1} + \omega_{R-1})} \right)^{n_{R-1}} \right).$$

$$\left(\frac{1}{n_R !} \left(\frac{\rho_R}{(\beta_R + A_{Ra} + \omega_R)} \right)^{n_R} \right) \cdot \left(\frac{1}{m_1 !} \frac{\left(\alpha_{11} + \frac{\beta_R d_{R1} \rho_R}{(n_R + 1)(\beta_R + \omega_R + A_{Rn_R+1})} \right)^{m_1}}{\prod_{b=1}^{m_1} (\beta_{11} + A_{11b} + \omega_{11})} \right)$$

$$\left(\frac{1}{m_2 !} \frac{\left(\alpha_{12} + \frac{\beta_R d_{R2} \rho_R}{(n_R + 1)(\beta_R + \omega_R + A_{Rn_R+1})} \right)^{m_2}}{\prod_{y=1}^{m_2} (\beta_{12} + A_{12y} + \omega_{12})} \right) \cdot \left(\frac{1}{m_3 !} \frac{\left(\alpha_{13} + \frac{\beta_R d_{R3} \rho_R}{(n_R + 1)(\beta_R + \omega_R + A_{Rn_R+1})} \right)^{m_3}}{\prod_{b=1}^{m_b} (\beta_{13} + A_{13b} + \omega_{13})} \right) \cdots\cdots\cdot$$

$$\left(\frac{1}{m_S !} \frac{\left(\alpha_{1S} + \frac{\beta_R d_{RS} \rho_R}{(n_R + 1)(\beta_R + \omega_R + A_{Rn_R+1})} \right)^{m_S}}{\prod_{y=1}^{m_S} (\beta_{1S} + A_{1Sb} + \omega_{1S})} \right)$$

$$n_a \geq 0, m_b \geq 0; \ (a = 1, 2, 3, ..., R; b = 1, 2, 3, ..., S) \tag{7}$$

Where,

$$\rho_1 = \alpha_1 + \sum_{a=2}^{R} \frac{\beta_a d_{a1} \rho_a}{(n_a + 1)(\beta_a + \omega_a + A_{an_a+1})}$$

$$\rho_2 = \alpha_2 + \frac{\beta_1 p_1 \rho_1}{(n_1+1)(\beta_1+\omega_1+A_{1n_1+1})} + \sum_{a=3}^{R} \frac{\beta_a d_{a2} \rho_a}{(n_a+1)(\beta_a+\omega_a+A_{an_a+1})}$$

$$\rho_3 = \alpha_3 + \frac{\beta_2 p_2 \rho_2}{(n_2+1)(\beta_2+\omega_2+A_{2n_2+1})} + \sum_{a=4}^{R} \frac{\beta_a d_{a3} \rho_a}{(n_a+1)(\beta_a+\omega_a+A_{an_a+1})}$$

……

$$\rho_{R-1} = \alpha_{R-1} + \frac{\beta_{R-2} p_{R-2} \rho_{R-2}}{(n_{R-2}+1)(\beta_{R-2}+\omega_{R-2}+A_{R-2,n_{R-2}+1})} + \frac{\beta_R d_{R,R-1} \rho_R}{(n_R+1)(\beta_R+\omega_R+A_{Rn_R+1})}$$

$$\rho_R = \alpha_R + \frac{\beta_{R-1} p_{R-1} \rho_{R-1}}{(n_{R-1}+1)(\beta_{R-1}+\omega_{R-1}+A_{R-1,n_{R-1}+1})}$$

As discussed earlier, we can find ρ_R from above R-equations with the help of determinants or ρ_R can be derived easily directly by taking

$$k_{ab} = \frac{d_{ab}\beta_a}{(n_a+1)\left(\beta_a + A_a + \omega_a\right)} (b = 1, 2, 3, ..., R-1)$$

and

$$h_a = (n_a+1)\left(\left(\beta_a + A_a + \omega_a\right)\right)(a = 1, 2, 3, .., S)$$

in results (6) and (7) and in the values of $\Delta_{R-1}, ..., \Delta_3, \Delta_2$.

Using the normalizing condition, $P\left(\tilde{0},\tilde{0}\right)$ can be calculated as under

$$1 = \sum_{\substack{\tilde{m}=\tilde{0} \\ \tilde{n}=0}}^{\infty} P\left(\tilde{n},\tilde{m}\right) = P\left(\tilde{0},\tilde{0}\right)\left(\sum_{n_1=0}^{\infty} \frac{1}{n_1!}\left(\frac{\rho_1}{\mu_1 + C_1 + \alpha_1}\right)^{n_1}\right).$$

$$\left(\sum_{n_2=0}^{\infty} \frac{1}{n_2!}\left(\frac{\rho_2}{\mu_2 + C_2 + \alpha_2}\right)^{n_2}\right).\left(\sum_{n_3=0}^{\infty} \frac{1}{n_3!}\left(\frac{\rho_3}{\mu_3 + C_3 + \alpha_3}\right)^{n_3}\right)....$$

$$\left(\sum_{n_{R-1}=0}^{\infty}\frac{1}{n_{R-1}!}\left(\frac{\rho_{R-1}}{\mu_{R-1}+C_{R-1}}\right)^{n_{R-1}}\right).$$

$$\left(\sum_{n_a=0}^{\infty}\frac{1}{n_a!}\left(\frac{\rho_a}{\mu_a+C_a}\right)^{n_R}\right).\sum_{m_1=0}^{\infty}\left(\frac{\rho_{11}}{\mu_{11}}\right)^{m_1}.\sum_{m_2=0}^{\infty}\left(\frac{\rho_{12}}{\mu_{12}}\right)^{m_2}.\sum_{m_3=0}^{\infty}\left(\frac{\rho_{13}}{\mu_{13}}\right)^{m_3}\dots\sum_{m_S=0}^{\infty}\left(\frac{\rho_{1b}}{\mu_{1b}}\right)^{m_S}$$

Where,

$$\rho_{1b}=\alpha_{1b}+\frac{\beta_R\rho_R d_{Rb}}{(n_R+1)(\beta_R+\omega_R+A_R)}, b=1,2,3.\dots,S$$

Therefore,

$$1=P\left(\tilde{0},\tilde{0}\right)\left(\prod_{a=1}^{R}e^{\left(\frac{\rho_a}{(\beta_a+A_a+\omega_a)}\right)}\right)\left(\prod_{b=1}^{S}e^{\left(\frac{\rho_{1b}}{(\beta_{1b}+A_{1b}+\omega_{1b})}\right)}\right)$$

Thus,

$$P\left(\tilde{0},\tilde{0}\right)^{-1}=\left(\prod_{a=1}^{R}e^{\left(\frac{\rho_a}{(\beta_a+A_a+\omega_a)}\right)}\right)\left(\prod_{b=1}^{S}e^{\left(\frac{\rho_{1b}}{(\beta_{1b}+A_{1b}+\omega_{1b})}\right)}\right) \quad (13)$$

Thus $P\left(\tilde{n},\tilde{m}\right)$ has been calculated completely.

III.iv. Steady-State Marginal Probabilities

"The steady-state marginal probability of the service channel P_1 having m_1 customers waiting for service before the server D_1 denoted by $P\left(n_1\right)$, is determined as under "

$$P\left(n_1\right)=\sum_{\substack{n_2=0=n_3,\dots,=n_R\\ \tilde{m}=0}}^{\infty}P\left(\tilde{n},\tilde{m}\right)=\frac{\frac{1}{n_1!}\left(\frac{\rho_1}{\beta_1+A_1+\omega_1}\right)^{n_1}}{e^{\left(\frac{\rho_1}{\beta_1+A_1+\omega_1}\right)}}$$

Similarly

$$P(n_a)=\frac{\frac{1}{n_a!}\left(\frac{\rho_a}{\beta_a+A_a+\omega_a}\right)^{n_a}}{e^{\left(\frac{\rho_a}{\beta_a+A_a+\omega_a}\right)}} for a=2,3,4,\dots,R-1,R$$

and

$$P\left(m_b\right)=\left(\rho_{1b}\right)^{m_b}\left(1-\rho_{1b}\right), b=1,2,3,4,\dots,S$$

IV. Mean Queue Length

Marginal mean queue length before the server P_1 is derived by

$$L_1 = \sum_{n_1=0}^{\infty} n_1 P(n_1) = \sum_{n_1=0}^{\infty} \frac{\frac{n_1}{n_1!}\left(\frac{\rho_1}{\beta_1 + A_1 + \omega_1}\right)^{n_1}}{e^{\left(\frac{\rho_1}{\beta_1 + A_1 + \omega_1}\right)}} = \frac{\rho_1}{\beta_1 + A_1 + \omega_1}$$

Similarly

$$L_a = \frac{\rho_a}{\beta_a + A_a + \omega_a}; a = 1, 2, 3, ..., R$$

$$L_{1b} = \frac{\rho_{1b}}{\beta_{1b} + \omega_{1b} + A_{1b}}; b = 1, 2, 3, ..., S$$

Therefore, the Mean queue length of the system =

$$L = \sum_{a=1}^{R} L_a + \sum_{b=1}^{S} L_{1b} = \sum_{a=1}^{R} \frac{\rho_a}{\beta_a + A_a + \omega_a} + \sum_{b=1}^{S} \frac{\rho_{1b}}{\beta_{1b} + \omega_{1b} + A_{1b}}$$

V. Concluding Remarks

(i) The important concept of balking and reneging has been introduced in the present study because balking occurs either due to heavy rush at the service station or due to the limited capacity of the system and reneging either due to urgent call or due to impatient behaviour of the customer have a bearing effect on the direct as well as indirect cost of the business.

(ii) If feedback is limited from each service channel to its previous service channel in the present model, then its results would resemble the results of the queuing model discussed by

(iii) This model is efficient in various service stations, toll plazas, and places where services are connected with different fields such as health care, and business setups.

(iv) More the service rate, the better the efficiency of the system.

(v) By analyzing various customers' behaviour at servers, desired results are found such as mean queue length is satisfactorily within the limit.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding this article.

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