JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-19, No.-11, November (2024) pp 44 - 55 ISSN (Print) 0973-8975

DOUBLE ELZAKI DECOMPOSITION METHOD FOR SOLVING PDES ARISING DURING LIQUID DROP FORMATIONS

Inderdeep Singh[1], Parvinder Kaur[2]

[1,2] Department of Physical Sciences, SBBSU, Jalandhar-144030, Punjab India.

Emails: [1]inderdeeps.ma.12@gmail.com , [2]parvinder29m@gmail.com

Corresponding Author: **Parvinder Kaur**

https://doi.org/10.26782/jmcms.2024.11.00004

(Received: August 11, 2024; Revised: October 21, 2024; Accepted: November 06, 2024)

Abstract

Partial differential equations are essential to every branch of science and engineering. They are regarded as the fundamental components of the majority of mathematical and physical simulations with practical uses. Numerous partial differential equations may be useful in the description of a physical phenomenon that could help in a deeper comprehension of its behaviour. The importance of PDEs has drawn more attention in recent years, which motivates researchers to solve these equations analytically and numerically. In this study, we propose a new hybrid technique for solving partial differential equations arising during liquid drop formations. The proposed hybrid technique is the combustion of double Elzaki transform and the classical Adomian decomposition method. To illustrate the simplicity and accuracy of the proposed scheme, some experimental work has been carried out.

Keywords: Double Elzaki transform; Adomian decomposition method; Rosenau Hyman equations; Test examples.

I. Introduction

Nonlinear PDEs have numerous applications in biology, physics, chemistry, and engineering. Several analytical, semi-analytical, and numerical schemes have been established to find the solutions of linear and nonlinear mathematical models. In this study, a novel technique based on the mixture of double Elzaki transform and the classical Adomian decomposition method has been established for finding the soliton solutions of the nonlinear Rosenau-Hyman equation of the form:

$$u_t + \rho(u^n)_x + \sigma(u^n)_{xxx} = 0 \quad (1)$$

where ρ and σ are arbitrary constants. This equation was discovered by Philip Rosenau and J.M. Human during the research of compacton solutions in 1993. Such an equation has been produced during the formation of a liquid drop pattern.

Sumudu transform-based Adomian decomposition method has been implemented for solving some of Burger's equations in [I]. Double integral transform has been used to find the solutions of a singular system of hyperbolic equations in [II]. In [III], double Laplace transform has been used to find the solutions of Telegraph equations. Elzaki transform has been introduced in [IV] for solving some applications of sciences and engineering. Double Elzaki transform has been introduced and implementation of such transform to find the solutions of some applications of differential equations in [V]. A combination of the Elzaki transform and the projected differential transform method has been established in [VI] for solving linear and nonlinear partial differential equations. A comparison between the Laplace transform and with the Elzaki transform has been presented in [VII] for solving some differential equations. Some ordinary differential equations with variable coefficients have been solved with the help of Elzaki transform in [VIII]. Two techniques based on the Elzaki transform and the Sumudu transform have been implemented for solving some differential equations in [IX]. In [X], wave equations have been solved with the help of the double Elzaki transform and the results are compared with the exact solutions and the solutions obtained by the double Laplace transform method. The double Elzaki transform-based Adomian decomposition method has been utilized for solving nonlinear partial differential equations in [XI]. In [XII], a double Elzaki transform-based decomposition method has been established for solving third-order Korteweg-De Vries equations. Convergence of the double Elzaki transform has been established to illustrate the accuracy and efficiency of the double Elzaki transform-based technique in [XIII]. In [XIV], Adomian polynomials have been used with the Elzaki transform method for solving third-order Korteweg-De Vries equations. Elzaki transform-based efficient techniques have been established for solving various applications of sciences and engineering in [XV-XVIII].

This research paper is constituted as follows: Section II contains the full information regarding the double Elzaki transform and its properties. The proposed technique has been presented in Section III for solving nonlinear Rosenau-Hyman equations. In Section IV, some computational work has been carried out by using the proposed technique for solving such equations. In Section V, results and discussion have been presented. The future scope has been presented in Section VI. Section VII contains the conclusion part of the research paper.

II. Double Elzaki Transform and its Properties

In this Section, we discuss about the double Elzaki transform, inverse double Elzaki transform and various its properties.

II.i. Introduction to Double Elzaki Transform

Let $f(x,t)$ with $x\,,t\ >0$ be a function. This function can be expressed in the form of an infinite series. Therefore, the double Elzaki transform is written as:

$$E_2\,\{f(x,t);u,v\} = T(u,v) = uv\int_0^\infty\int_0^\infty f(x,t)\,e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}\,dxdt,$$

whenever integral exists.

II.ii. Inverse Double Elzaki Transform

The inverse of the double Elzaki transform can be written as:

$$E_2^{-1}\{T(u,v)\} = f(x,t), \ \ x,t > 0$$

The order of the function $f(x,t)$ is said to be of exponential, if for $a > 0, b > 0$ in the region belong to the interval $0 \le x < \infty, 0 \le t < \infty$, ∃ a positive constant k such that

$$|f(x,t)| \le k\, e^{\left(\frac{x}{a}+\frac{t}{b}\right)}$$

II.iii. Standard Properties of Double Elzaki Transform

In this Section, we will discuss some properties of the double Elzaki transform:

Linearity Property: If $f(x,y)$ and $g(x,y)$ be two function of $x,y > 0$ such that $[f(x,y)] = T_1(u,v)$ and $E_2[g(x,y)] = T_2(u,v)$, then

$$E_2\{\, af(x,y) + b\, g(x,y)\} = a\, E_2\{f(x,y)\} + b\, E_2\,\{g(x,y)\} = a\, T_1(u,v) + \, bT_2(u,v)$$

Change Shifting Property: If $E_2\{f(x,y)\} = T(u,v)$,

then

$$E_2\{f(ax,by)\} = \frac{1}{ab}\, T(au,bv)$$

First Shifting Property :

(a) If $E_2\{f(x,y)\} = T(u,v)$,

then

$$E_2\{\, e^{ax+by}\, f(x,y) = T\, \left[\frac{u}{1-au}, \frac{v}{1-bv}\right]$$

(b) If $E_2\{f(x,y)\} = T(u,v)$,

then

$$E_2\{\, e^{-ax-by}\, f(x,y) = \ T\, \left[\frac{u}{1-au}, \frac{v}{1-bv}\right]$$

II.iv. Double Elzaki Transform of Partial Derivatives

In this Section, we provide a double Elzaki transform of some partial derivatives:

a) $E_2\,\{\frac{\partial}{\partial x}\, f(x,y)\} = \frac{1}{u}\, T(u,v) - uT(0,v)$

b) $E_2\left\{\frac{\partial}{\partial y}\, f(x,y)\right\} = \frac{1}{v}\, T(u,v) - v\, T(u,0)$

c) $E_2\,\{\frac{\partial^2}{\partial x^2} f(x,y)\,\} = \frac{1}{u^2} T(u,v) - T(0,v) - u\frac{\partial}{\partial x}\, T(0,v)$

d) $E_2\,\{\frac{\partial^2}{\partial y^2} f(x,y)\,\} = \frac{1}{v^2} T(u,v) - T(u,0) - v\frac{\partial}{\partial y}\, T(u,0)$f

e) $E_2\,\{\frac{\partial^2}{\partial x \partial y}\, f(x,y)\,\} = \frac{1}{uv} T(u,v) - \frac{v}{u}\, T(u,0) - \frac{u}{v}\, T(0,0) + uv\, T(0,0)$

III. Proposed Technique for Solving Nonlinear for Rosenau-Hyman Equation

Consider the general PDE with the initial condition of the form:

$$L\,u(x,t) + N\,u(\,x,t) = g(x,t) \tag{2}$$

with initial condition

$$u(x,0) = h(x), \tag{3}$$

here L represents a linear differential operator $L = \frac{\partial}{\partial t}$, N represents the nonlinear differential operator, and $g(x,t)$ is the source term.

Implementation of double Elzaki transform in Equation (2) and single Elzaki transform in Equation (3), we obtain

$$E_2\big(L\,u(x,t)\big) + E_2\,\big(N\,u(x,t)\big) = E_2\,(g(x,t)) \tag{4}$$

and

$$E(u(x,0)) = \;E\big(h(x)\big) = T(u,0) \tag{5}$$

From Equation (4), we obtain

$$\frac{1}{v}T(u,v) - v\,T(u,0) = E_2\,\big(g(x,t)\big) - E_2\,\big(N\,u(x,t)\big)$$

This implies

$$T(u,v) = v^2 T(u,0) + vE_2\,\big(g(x,t)\big) - vE_2\,\big(N\,u(x,t)\big)$$

Or

$$E_2\big(u(x,t)\big) = v^2\,E\big(h(x)\big) + v\,E_2\,\big(g(x,t)\big) - \big\{vE_2\big(N\,u(x,t)\big)\big\}$$

Implementation of inverse double Elzaki transform on Equation (6), we obtain

$$u(x,t) = G(x,t) - E_2^{-1}\big\{vE_2\big(N\,u(x,t)\big)\big\} \tag{7}$$

where

$$G(x,t) = E_2^{-1}\big\{v^2\,E\big(h(x)\big) + v\,E_2\,\big(g(x,t)\big)\big\}$$

Assume that the solution is of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{8}$$

Write the nonlinear term as:

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u), \tag{9}$$

here $A_n(u)$ represents the Adomian polynomials and can be calculated as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\epsilon^n}\Big\{N\Big(\sum_{j=0}^{\infty} \epsilon^j u_j\Big)\Big\}_{\epsilon=0}, n = 0, 1, 2, 3, \ldots$$

Substituting the values from (8) and (9) in (7), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = S(x,t) - E_2^{-1}\{vE_2(\sum_{n=0}^{\infty} A_n(u))\} \tag{10}$$

From (10), we obtain

$$u_0(x,t) = S(x,t),$$

$$u_1(x,t) = -E_2^{-1}\{v\, E_2(A_0)\},$$

$$u_2(x,t) = -E_2^{-1}\{v\, E_2(A_1)\},$$

$$\vdots$$

The approximate solution of the problem is:

$$u(x,t) = \lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(x,t).$$

IV. Computational Work

In this Section, we perform some test examples to obtain the solutions of nonlinear PDEs, arising during liquid drop formation.

Example 1: Consider the Rosenau-Hyman equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0 \tag{11}$$

with initial condition $u(x,0) = 1 + x$. The exact solution is:

$$u(x,t) = \frac{1+x}{1+2t}$$

Implementation of double Elzaki transform on Equation (11), we obtain

$$E_2(u_t) = -E_2((u^2)_x + (u^2)_{xxx})$$

This implies

$$\frac{1}{v}T(u,v) - v.T(u,0) = -E_2((u^2)_x + (u^2)_{xxx}) \tag{12}$$

Applying a single Elzaki transform to the initial conditions, we obtain

$$E\big(u(x,0)\big) = T(u,0) = E(1+x) = u^2 + u^3$$

From (12), we obtain

$$\frac{1}{v}T(u,v) = v(u^2 + u^3) - E_2((u^2)_x + (u^2)_{xxx})$$

This implies

$$T(u,v) = v^2(u^2 + u^3) - v.E_2((u^2)_x + (u^2)_{xxx})$$

Implementation of inverse double Elzaki transforms, we obtain

$$E_2^{-1}(T(u,v)) = E_2^{-1}(v^2(u^2 + u^3)) - E_2^{-1}\big(v.E_2((u^2)_x + (u^2)_{xxx})\big)$$

This implies

$$u(x,t) = (1+x) - E_2^{-1}\big(v.E_2((u^2)_x + (u^2)_{xxx})\big)$$

Applying the Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = (1+x) - E_2^{-1}(v.E_2\{\sum_{n=0}^{\infty} A_n(u)\})$$

From the above Equation, we obtain

$$u_0(x,t) = 1 + x,$$
$$u_1(x,t) = -E_2^{-1}(v.E_2\{A_0\}),$$
$$u_2(x,t) = -E_2^{-1}(v.E_2\{A_1\}),$$
$$\vdots$$

Some of the Adomian polynomials are:

$$A_0 = 2(1 + x),$$
$$A_1 = -8(1 + x)t,$$
$$A_2 = 48(1 + x)t^2,$$
$$\vdots$$

The values of $u_0, u_1, u_2, \ldots$ are given by

$$u_0(x,t) = 1 + x,$$
$$u_1(x,t) = -2(1 + x)t,$$
$$u_2(x,t) = 4(1 + x)t^2,$$
$$\vdots$$

The solution is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Or

$$u(x,t) = (1 + x)(1 - (2t) + (4t^2) - \cdots)$$

The closed-form solution is:

$$u(x,t) = \frac{1+x}{1+2t}.$$

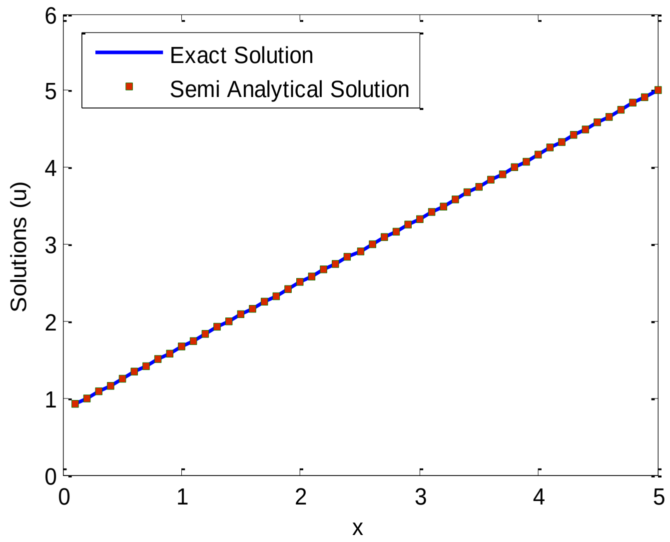

Fig. 1. Comparison of exact solutions and solutions obtained by double Elzaki ADM at $t = 0.1$

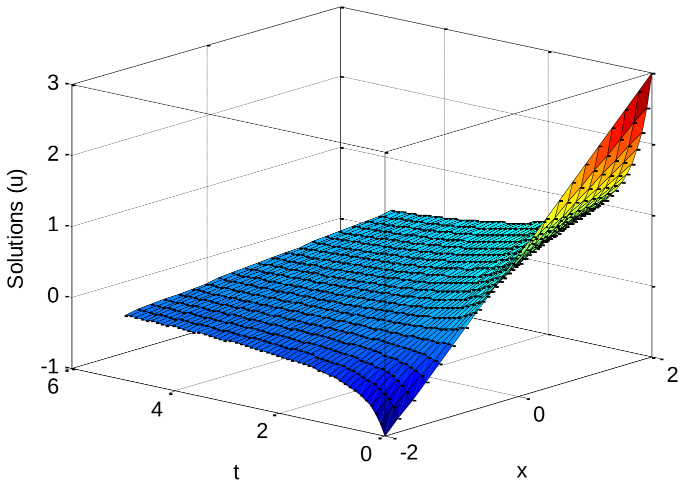

Fig. 2. Physical behavior of solutions of Example 1

Figure 1 shows the comparison of the solutions found by the double Elzaki transform-based Adomian decomposition method (taking the first ten terms of an infinite series) and exact solutions at $t = 0.1$ and different values of x. Figure 2 shows the dynamical and physical behavior of analytical solutions obtained by the double Elzaki transform-based Adomian decomposition method at different ranges of x and t.

Example 2: Consider the nonlinear Rosenau Hyman equation

$$u_t = uu_x + uu_{xxx} + 3u_x u_{xx} \tag{13}$$

with initial condition

$$u(x,0) = -\frac{8}{3}\cos^2\left(\frac{x}{4}\right)$$

The exact solution is:

$$u(x,t) = -\frac{8}{3}\cos^2\left(\frac{x-t}{4}\right)$$

Implementation of double Elzaki transform on Equation (13), we obtain

$$E_2(u_t) = -E_2(uu_x + uu_{xxx} + 3u_x u_{xx})$$

This implies

$$\frac{1}{v}T(u,v) - v.T(u,0) = -E_2(uu_x + uu_{xxx} + 3u_x u_{xx}) \tag{14}$$

Taking a single Elzaki transform to the initial conditions, we obtain

$$E\big(u(x,0)\big) = T(u,0) = -E\left(\frac{8}{3}\cos^2\left(\frac{x}{4}\right)\right) = -\frac{4}{3}\left\{u^2\left(\frac{u^2+8}{u^2+4}\right)\right\}$$

From (14), we obtain

$$\frac{1}{v}T(u,v) = -\frac{4}{3}v\left\{u^2\left(\frac{u^2+8}{u^2+4}\right)\right\} - E_2(uu_x + uu_{xxx} + 3u_xu_{xx})$$

This implies

$$T(u,v) = -\frac{4}{3}v^2\left\{u^2\left(\frac{u^2+8}{u^2+4}\right)\right\} - v.E_2(uu_x + uu_{xxx} + 3u_xu_{xx})$$

Implementation of inverse double Elzaki transforms, we obtain

$$E_2^{-1}(T(u,v)) = E_2^{-1}\left(-\frac{4}{3}v^2\left\{u^2\left(\frac{u^2+8}{u^2+4}\right)\right\}\right) - E_2^{-1}\left(v.E_2(uu_x + uu_{xxx} + 3u_xu_{xx})\right)$$

This implies

$$u(x,t) = -\frac{8}{3}\cos^2\left(\frac{x}{4}\right) - E_2^{-1}\left(v.E_2(uu_x + uu_{xxx} + 3u_xu_{xx})\right)$$

Using the Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = -\frac{8}{3}\cos^2\left(\frac{x}{4}\right) - E_2^{-1}\left(v.E_2\left\{\sum_{n=0}^{\infty} A_n(u)\right\}\right)$$

From the above Equation, we obtain

$$\begin{aligned} u_0(x,t) &= -\frac{8}{3}\cos^2\left(\frac{x}{4}\right), \\ u_1(x,t) &= -E_2^{-1}(v.E_2\{A_0\}), \\ u_2(x,t) &= -E_2^{-1}(v.E_2\{A_1\}), \\ &\vdots \end{aligned}$$

Some of the Adomian polynomials are:

$$\begin{aligned} A_0 &= -\frac{2}{3}\sin\left(\frac{x}{2}\right), \\ A_1 &= \frac{1}{3}\cos\left(\frac{x}{2}\right)t, \\ &\vdots \end{aligned}$$

The values of $u_0, u_1, u_2, \ldots$ are given by

$$\begin{aligned} u_0(x,t) &= -\frac{8}{3}\cos^2\left(\frac{x}{4}\right), \\ u_1(x,t) &= -\frac{2}{3}\sin\left(\frac{x}{2}\right)t, \\ u_2(x,t) &= \frac{1}{6}\cos\left(\frac{x}{2}\right)t^2, \\ &\vdots \end{aligned}$$

The solution is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Or

$$u(x,t) = -\frac{8}{3}\cos^2\left(\frac{x}{4}\right) - \frac{2}{3}\sin\left(\frac{x}{2}\right)t + \frac{1}{6}\cos\left(\frac{x}{2}\right)t^2 + \cdots$$

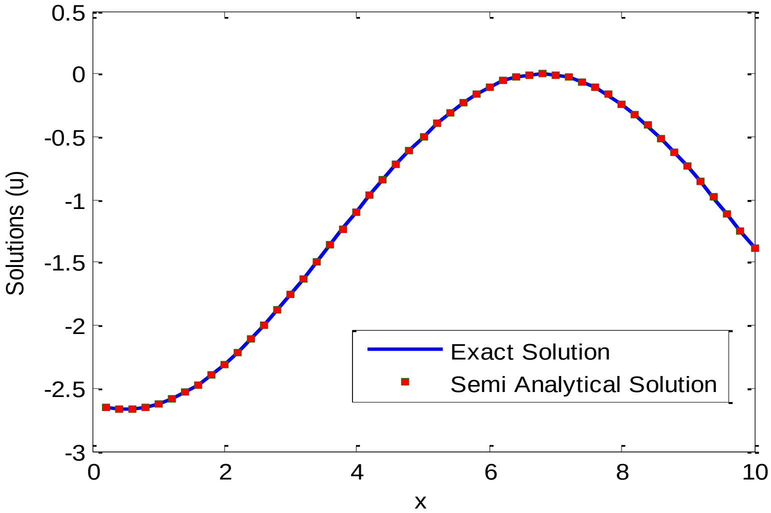

Fig. 3. Comparison of exact solution and solution obtained double Elzaki ADM at $t = 0.5$

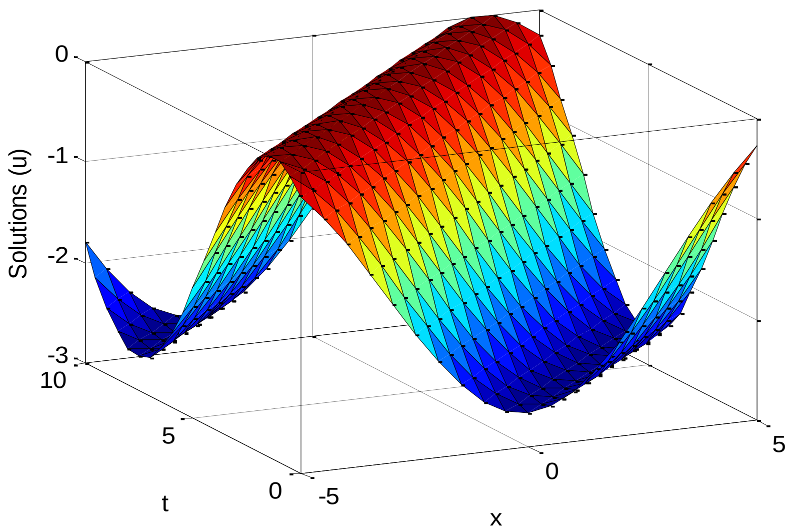

Fig. 4. Physical behavior of solutions of Example 2

Figure 3 shows the comparison of the solutions found by the double Elzaki transform-based decomposition method (taking the first three terms of an infinite series) and exact solutions at $t = 0.5$ and different values of x. Figure 4 shows the dynamical and physical behavior of solutions obtained with the help of the double Elzaki transform-based Adomian decomposition method at different ranges of x and t.

V. Results and Discussion

Figure 1 and Figure 2 show the comparison study of the solutions obtained with the help of proposed technique (taking first ten terms) and the exact solutions for different values of x and t. Figure 3 and Figure 4 show the comparison study of the solutions obtained with the help of proposed technique (taking first three terms) and the exact solutions for different values of x and t. The above numerical and graphical representations show that the semi-analytical solutions obtained by the double Elzaki transform-based Adomian decomposition method are very accurate and closer to the exact solutions. The numerical procedure based on the double Elzaki transform and Adomain decomposition method is very simple and fast for solving these partial differential equations.

VI. Future Scope

This technique is very easy to apply and can be used to solve a wide range of fractional and nonlinear differential equations and systems of differential equations including algebraic equations, ordinary differential equations, partial differential equations, and integral equations. This technique helps to break down complex problems, fosters collaboration, improves understanding, and lends robustness to the completed work. So this technique will be applicable for solving linear, nonlinear as well as fractional applications in the form of ordinary differential equations, partial differential equations, and integral equations.

VII. Conclusion

From the above computational data, it is concluded that the double Elzaki transform is a powerful mathematical tool, when combined with the Adomian decomposition method to find the solutions of Rosenau Hyman-type equations arising during the liquid drop pattern formation. The solutions are closer to the exact solution, when the terms of an infinite series may vary. For the future scope, this technique will be used to find the semi-analytical solutions of fractional nonlinear partial differential equations, which are arising during various applications of sciences and engineering.

Conflict of Interest

The authors declare that there is no conflict of interest regarding this article.

References

I. Ahmed, S., 'Application of Sumudu Decomposition Method for Solving Burger's Equation,' *Advances in Theoretical and Applied Mathematics*, Vol. 9(1), pp. 23-26, (2014).

II. Alderremy, A. A. and Elzaki, T.M., 'On the New Double Integral Transform for Solving Singular System of Hyperbolic Equations,' *Journal of Nonlinear Sciences and Applications*, Vol. 11, pp. 1207-1214, (2018). 10.22436/jnsa.011.10.08

III. Eltayeb, H. and Kilicman, A., 'A Note on Double Laplace Transform and Telegraphic Equations,' *Abstract and Applied Analysis*, 2013, pp. 1-6, (2013). 10.1155/2013/932578

IV. Elzaki, T., 'The New Integral Transform Elzaki Transform,' *Global Journal of Pure and Applied Mathematics*, Vol. 7(1), pp. 57-64, (2011). http://www.ripublication.com/gjpam.htm

V. Elzaki, T. M. and Hilal, E.M., 'Solution of Telegraph Equation by Modified Double Sumudu Transform 'Elzaki Transform,' *Mathematical Theory and Modeling*, Vol. 2, pp. 95-103, (2012). 10.4236/am.2015.63056

VI. Elzaki, T. and Hilal, M.A.,'Solution of Linear and Non-Linear Partial Differential Equations Using Mixture of Elzaki Transform and the Projected Differential Transform Method,' *Mathematical Theory and Modeling*, Vol. 2(4), pp. 50-59, (2012).

VII. Elzaki, T. and Elzaki, S.M., 'On the Connections Between Laplace and Elzaki Transforms,' *Advances in Theoretical and Applied Mathematics*, Vol. 6(1), pp. 1-10, (2011). http://www.ripublication.com/atam.htm

VIII. Elzaki, T., and Elzaki, S. M., 'On the Elzaki Transform and Ordinary Differential Equation with Variable Coefficients,' *Advances in Theoretical and Applied Mathematics*, Vol. 6(1), pp. 41-46, (2011). http://www.ripublication.com/atam.htm

IX. Elzaki, T., Elzaki, J.M. and Hilal, M.A., 'Elzaki and Sumudu Transforms for Solving Some Differential Equations,' *Global Journal of Pure and Applied Mathematics*, Vol. 8(2), pp. 167-173 (2012). http://www.ripublication.com/gjpam.htm

X. Hassaballa, I. A., and Salih, Y.A., 'On Double Elzaki Transform and Double Laplace Transform,' *IOSR Journal of Mathematics*, Vol. 11(1), pp. 35-41, (2015). 10.9790/5728-11163541

XI. Hassan, M. A., and Elzaki, T.M., 'Double Elzaki Transform Decomposition Method for Solving Non-Linear Partial Differential Equations,' *Journal of Applied Mathematics and Physics*, Vol. 8, pp. 1463-1471 (2020). 10.4236/jamp.2020.88112.

XII. Hassan, M. A., and Elzaki, T.M., 'Double Elzaki Transform Decomposition Method for Solving Third Order Korteweg-De Vries Equations,' *Journal of Applied Mathematics and Physics*, Vol. 9, pp. 21-30 (2021). 10.4236/jamp.2021.91003.

XIII. Idrees, M. I., Ahmed, Z., Awais, M. and Perveen, Z., 'On the Convergence of Double Elzaki Transform,' *International Journal of Advanced and Applied Sciences*, Vol. 5, pp. 19-24,(2018). 10.21833/ijaas.2018.06.003

XIV. Ige, O. E., Heilio, M., Oderinu, R.A. and Elzaki, T.M., 'Adomian Polynomial and Elzaki Transform Method of Solving Third Order Korteweg-De Vries Equations,' *Pure and Applied Mathematics*, Vol. 15, pp. 261-277, (2019). 10.12732/ijam.v32i3.7

XV. Sharma, S and Singh, I., 'Elzaki transform homotopy analysis techniques for solving fractional (2+1)-D and (3+1)-D nonlinear Schrodinger equations,' *Communications on Applied Nonlinear Analysis*, Vol. 31 (6s), pp. 305-317, (2024). 10.52783/cana.v31.1224

XVI. Singh, I. and Kumari, U., 'Elzaki transform homotopy perturbation method for solving two- dimensional time fractionl Rosenau-Hyman equation,' *MATHEMATIKA, MJIM*, Vol. 39 (2), pp. 159-171, (2023).

XVII. Singh, I. and Kumari, U., 'Solving 2D and 3D Telegraph equations with Elzaki transform and Homotopy perturbation method,' *Journal of Mechanics of Continua and Mathematical Sciences*, Specal Issue , Vol. 11, pp. 18-29, (2024). 10.26782/jmcms.spl.11/2024.05.00002

XVIII. ZeinEldin, R.A., Singh, I., Singh, G., Elgarhy, M. and El-Wahed Khalifa, H.A., 'Efficient technique for solving (3+1)-D fourth order parabolic PDEs with time fractional derivatives,' *Axioms*, Vol. 12(4), pp. 347, (2023). 10.3390/axioms12040347