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# SOLVING 2D MATHEMATICAL MODELS ARISING IN APPLIED SCIENCES WITH CAPUTO DERIVATIVES VIA HYBRID HPM

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**Abstract**

*This paper presents a novel approach for solving 2D mathematical models arising in applied sciences, specifically focusing on 2-dimensional time-fractional order Klein-Gordon (TFKGE) and sine-Gordon equations (TFSGE) using the Sumudu transform-homotopy perturbation method (STHPM). The amalgamation of the Sumudu transform with the homotopy perturbation method provides an effective analytical technique for tackling these time-fractional order partial differential equations. The solutions obtained illustrate the precision and efficiency of the method, offering valuable insights for modelling complex physical systems. In this study, we also solve the same numerical problems using the variational iteration method and perform a comparative analysis of the results. This study advances the application of fractional calculus methods to challenging problems in theoretical and applied physics.*

**Keywords:** Homotopy Perturbation Method, Klein-Gordon Equation, Sine-Gordon Equation, Sumudu Transform, Test Examples, Variational Iteration Method.

## I. Introduction

Fractional calculus enhances traditional calculus by dealing with derivatives and integrals of fractional (non-integral) orders. This mathematical framework is effective for modeling systems with memory effects and hereditary properties. This has led to increased interest in fractional versions of classical differential equations, as they offer a more comprehensive understanding of complex dynamical systems. The time-fractional Klein-Gordon (TFKGE) and time-fractional sine-Gordon equations (TFSGE) are important partial differential equations with widespread applications in various domains like physics, applied mathematics, and engineering. The time-fractional Klein-Gordon equation is crucial in quantum mechanics and field theory for describing scalar fields. In contrast, the time-fractional sine-Gordon equation is notable for its role in studying solitons, nonlinear optics, and condensed matter physics. Due to their numerous applications, researchers are highly motivated to find solutions to these equations. Various analytical and semi-analytical techniques are devised to tackle time-fractional Klein-Gordon equations, including the 3D Laplace method [II], a lie group approach [VI], wavelet method [XVI], Elzaki transform homotopy perturbation method [XXI], reduce differential equation [XXII], quadruple Laplace transform [XXIII], meshless numerical analysis [XXVI], homotopy perturbation Shehu transform method [XX]. Whereas the time-fractional sine-Gordon equation is employed with the reduce differential transform method [III], Mesoscopic lattice Boltzmann BGK model [VII], space-time spectral method [XXIV], finite difference scheme [XXV].

2D time-fractional order Klein-Gordon equation:

$$D_{t}^{\alpha}u(x,y,t)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y,t)+g\big(u(x,y,t)\big)=f(x,y,t),$$

$$(x,y)\in\Omega=[0,1]\times[0,1],\qquad t>0,\;\;1<\alpha\leq 2 \tag{1}$$

Subject to initial conditions

$$u(x,y,0)=g_1(x,y),\qquad u_t(x,y,0)=g_2(x,y),\qquad (x,y,t)\in\Omega,$$

2D time-fractional order sine-Gordon equation:

$$D_{t}^{2\alpha}u(x,y,t)+D_{t}^{\alpha}u(x,y,t)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y,t)-$$

$$p(x,y)\sin\big(u(x,y,t)\big)+\gamma\frac{dB(t)}{dt}=f(x,y,t),$$

$$(x,y)\in\Omega=[0,1]\times[0,1],\qquad t>0,\;\;0<\alpha\leq 1 \tag{2}$$

With initial conditions

$$u(x,y,0)=g_3(x,y),\qquad u_t(x,y,0)=g_4(x,y),\qquad (x,y)\in\Omega,$$

where $D_{t}^{\alpha}u=\frac{\partial^{\alpha}u}{\partial t^{\alpha}}$. In particular, if $\alpha=1$, equation (1) becomes a 2D Klein-Gordon equation, whereas equation (2) becomes a 2D sine-Gordon equation.

The Sumudu transform invented by Watugala (1990), is applied to handle differential equations like nonlinear heat-like equations, fractional PDEs, and control engineering problems in [I], and [X]. The properties of the sumudu transform are discussed in [IX]. Among the wide range of numerical schemes available, the homotopy perturbation method stands out as a versatile approach suitable for tackling fractional-order PDEs, this method was initially developed by J.H. He in 1999, [XI], [XII] and presented its efficacy by applying it on various PDEs in [XIII] and fractional PDEs in [V], [XIV], [XV], [XVII], and [XVIII]. In this study, we employ the sumudu transform-homotopy perturbation method (STHPM) to solve the 2-dimensional time fractional Klein-Gordon equation (TFKGE) and sine-Gordon equations (TFSGE). This hybrid scheme is also applied to solve various time-fractional PDEs in [IV], and

[VIII]. Our approach highlights the efficiency and accuracy of this hybrid scheme by applying TFKGE and TFSGE, providing new insights into the dynamics of these equations. The method's convergence is discussed in [XIX]. In XXVII variational iteration method is discussed.

This paper is organized as Section (sec.) II outlines the overview of fractional calculus and the Sumudu Transform. The homotopy perturbation method is discussed in Sec. III. The details of the hybrid Sumudu transform-based HPM, along with its convergence and error analysis are explored in sec. IV. Sec V contains the basic idea of the variational iteration method. The comparative analysis of numerical experiments is presented in sec. VI . And sec. VII contains the results and discussion. Sec.VIII offers a conclusion and suggestions for future research.

## II. Some Basic Definitions from Fractional calculus and Sumudu Transform

Fractional differential equations generalize classical differential equations by incorporating non-integer order derivatives. This approach offers enhanced modeling capabilities for complex systems in diverse scientific and engineering disciplines.

**Definition:** (see [IV]) A real function $g(t) \in C_\mu$ for $t > 0$ and $\mu \in R$ if there exist a real number $q \in R$ and $q > \mu$, such that $g(t) = t^q m(t)$, where $m(t) \in C[0,\infty)$ and $g(t) \in C_\mu^n$ if $g^{(n)} \in C_\mu$, $n \in N$.

**Definition:** The caputo fractional order derivative of the function $h(\tau)$ is (see [IV]):

$$\frac{\partial^\alpha}{\partial\tau^\alpha} h(\tau) = J^{(n-\alpha)} \frac{\partial^n}{\partial\tau^n} h(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau-\Omega)^{n-\alpha-1} h^n(\Omega) d\Omega,$$

where $h \in C_{-1}^n, n-1 < \alpha \le n, n \in N, \tau > 0$.

**Definition:** As defined in [IV], The Mittag-Leffler functions with two parameters $a$ and $b$ is: $E_{a,b}(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(an+b)}$, $a, b > 0$

### II.i. The Sumudu transform

Watugala pioneered the Sumudu transform which is applicable to a specific set of functions (see [X]):

$$X = \{g(t) | \exists\, N, \Delta_1, \Delta_2 > 0, |g(t)| < N e^{\frac{|t|}{\Delta_j}}, \text{ if } t \in (-1)^j \times [0,\infty)\},$$

the formula of the Sumudu transform is:

$$\mathbb{S}[g(t)] = \int_0^\infty g(ut) e^{-t} dt, \qquad u \in (\Delta_1, \Delta_2)$$

(a) The Sumudu transform in terms of Caputo fractional derivative is:

$$\mathbb{S}\left\{\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}h(\tau)\right\} = \frac{\mathbb{S}\{h(\tau)\}}{u^{\alpha}} - \sum_{k=0}^{\text{ṇ}-1}\frac{h^{k}(0)}{u^{\alpha-k}}, \quad \text{ṇ} - 1 < k \leq \text{ṇ}$$

(b) The Sumudu transform of several functions are:

1. $\mathbb{S}[1] = 1,$
2. $\mathbb{S}\left[\frac{t^{\text{ṇ}}}{\Gamma(\text{ṇ}+1)}\right] = u^{\text{ṇ}},$
3. $\mathbb{S}[e^{a\text{ṭ}}] = \frac{1}{1-au},$
4. $\mathbb{S}[af(\text{ṭ}) + bg(\text{ṭ})] = a\mathbb{S}[f(\text{ṭ})] + b\mathbb{S}[g(\text{ṭ})].$

## III. The Homotopy Perturbation Method

J.H.He is acknowledged for pioneering the development of the Homotopy perturbation technique [XI] in the course of his research. To understand the basic concept of the method, suppose a nonlinear partial differential equation:

$$\psi(u) = g(r), \qquad r \in \mathrm{X} \tag{3}$$

with boundary conditions as :

$$\mathrm{B}\left(u, \frac{\partial u}{\partial x}\right) = 0, \tag{4}$$

Break $\psi(u)$ as $\rho(u)$ and $\sigma(u)$, where

$\rho(u) =$ Linear operator,

$\sigma(u) =$ Non-linear operator,

Then, (3) can be written as:

$$\rho(u) + \sigma(u) - g(r) = 0,$$

In the topology, two continuous functions from one topological space to another are said to be homotopic if one can be continuously deformed into the other. The homotopy technique and its application are discussed in [XI], [XII], [XIII], and [XIV], the homotopy is defined as:

$$w(r, b): \dot{\mathrm{X}} \times [0, 1] \to \text{Ṛ},$$

which satisfies

$$\mathcal{H}(w, b) = (1 - b)[\rho(w) - \rho(u_0)] + b[\psi(w) - g(r)] = 0, \quad b \in [0,1], \quad r \in \mathcal{X}$$

Or

$$\rho(w) - \rho(u_0) - b\rho(w) + b\rho(u_0) + b[\rho(w) + \sigma(w) - g(r)] = 0,$$

Or

$$\mathcal{H}(w, b) = \rho(w) - \rho(u_0) + b\rho(u_0) + b[\sigma(w) - g(r)] = 0, \quad (5)$$

Here, $b$ is the embedding parameter, $b \in [0, 1]$

Consider the initial guess as $u_0$, and it adheres to the boundary conditions, from (4), we have

$$\mathcal{H}(w, 0) = \rho(w) - \rho(u_0) = 0$$

and

$$\mathcal{H}(w, 1) = \psi(w) - g(r) = 0$$

as $b \in [0, 1]$, the function $w(r, b)$ changes from $u_0(r)$to $u(r)$, and this is known as a deformation in topology. So the quantities $\rho(w) - \rho(u_0)$ and $\psi(w) - g(r)$ are termed as homotopy. Equation (3) can be solved and expressed as power series:

$$w = w_0 + bw_1 + b^2w_2 + \cdots$$

If $b = 1$, then the approximate solution of (1) is:

$$u = \lim_{b\to 1} w = w_0 + w_1 + w_2 + \cdots$$

## IV. Proposed Scheme: Hybrid Sumudu Transform-based Homotopy Perturbation Method [IV]

To address the procedure of the hybrid Sumudu transform-homotopy perturbation method, let's examine a general time-fractional non-linear PDE of the following form:

$$D_{t}^{\alpha} u(x, y, t) = Lu(x, y, t) + Nu(x, y, t) + q(x, y, t), \quad (6)$$

with $n - 1 < \alpha < n$, the subject to initial conditions

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y),$$

where

$D_{t}^{\alpha} u(x, y, t)$ = caputo fractional derivative,

$Lu(x, y, t)$ = linear term,

$Nu(x, y, t)$ = Non-linear term,

$q(x, y, t)$ = Source term.

Applying Sumudu transform (denoted as $\mathbb{S}$) on each side of (6)

$$\mathbb{S}\left[D_{t}^{\alpha} u(x, y, t)\right] = \mathbb{S}[Lu(x, y, t) + Nu(x, y, t) + q(x, y, t)],$$

Using differential property of sumudu transform and initial condition,

$$u^{-\alpha}\mathbb{S}[u(x,y,t)] - \sum_{k=0}^{n-1} u^{-(\alpha-k)}u^{(k)}(x,y,t) = \mathbb{S}[Lu(x,y,t) + Nu(x,y,t) + q(x,y,t)],$$

This implies

$$\mathbb{S}[u(x,y,t)] = \sum_{k=0}^{n-1} u^{(k)}f_k(x,y) + u^{\alpha}\mathbb{S}[Lu(x,y,t) + Nu(x,y,t) + q(x,y,t)],$$

Applying the inverse Sumudu on each side

$$u(x,y,t) = \mathbb{S}^{-1}\left[\sum_{k=0}^{n-1} u^{(k)}f_k(x,y)\right] + \mathbb{S}^{-1}\{ u^{\alpha}\mathbb{S}[Lu(x,y,t) + Nu(x,y,t) + q(x,y,t)]\},$$

Now, apply the homotopy perturbation technique, on each side

$$\sum_{n=0}^{\infty} p^{n}u_{n}(x,y,t) = \mathbb{S}^{-1}\left[\sum_{k=0}^{n-1} u^{(k)}f_k(x,y)\right] + p\mathbb{S}^{-1}\{ u^{\alpha}\mathbb{S}\left[L\left(\sum_{n=0}^{\infty} p^{n}u_{n}(x,y,t)\right) + N\left(\sum_{n=0}^{\infty} p^{n}H_{n}(u)\right) + q(x,y,t)\right]\} \quad (7)$$

Matching the similar powers of $p$ on both sides,

$$p^{o} = u_0 = \mathbb{S}^{-1}\left[\sum_{k=0}^{n-1} u^{(k)}f_k(x,y)\right], \quad \ldots \quad \ldots$$

$$p^{n} = u_{n} = \mathbb{S}^{-1}\left\{ u^{\alpha}\mathbb{S}\left[L\left(\sum_{n=0}^{\infty} p^{n}u_{n}(x,y,t)\right) + N\left(\sum_{n=0}^{\infty} p^{n}H_{n}(u)\right) + q(x,y,t)\right]\right\}.$$

where $H_{n}(u)$ is He's polynomial

$$H_{n}(u_0,u_1,u_2,\ldots) = \frac{1}{n!}\left[\frac{\partial^{n}}{\partial p^{n}}\left(N\left(\sum_{i=0}^{\infty} p^{i}u_i\right)\right)\right]_{p=0},$$

By using the values of $u_0, u_1, \ldots.$ in equation, the approximation for the solution of eq. (4) is

$$u(x,y,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^{n}u_{n}(x,y,t) = u_0 + u_1 + u_2 + \cdots$$

This solution generally converges rapidly.

### IV.i. Convergence Analysis

This Section contains some theorems to showcase the error analysis and convergence analysis of the suggested method:

**Theorem:** Let $w(r,t)$ and $w_{n}(r,t)$ be defined within the Banach space,

$$w(r,t) = \sum_{n=0}^{\infty} b^n w_n(r,t), \tag{8}$$

converges to the solution of the (6), if $\exists\ \chi \in (0,1)$ such that

$$\|w_{n+1}\| \le \chi \|w_n\|$$

See the proof of the theorem in [XIX].

**Note:** The greatest absolute truncation error of the series solution, given as: $\left|w(r,t) - \sum_{k=0}^{n} w_k(r,t)\right| \le \frac{\chi^{n+1}}{1-\chi}\|w_0\|$.

## V. Variational Iteration Method [XX VII]

Consider an initial valued differential equation

$$Lu(t) + Nu(t) = g(t), \quad t > 0, \tag{9}$$

$u^{(k)} = c_k,\ k = 0,1,\dots m-1,$

Where $L$differential operator of the highest order is, $N$ is the non-linear operator, $g(t)$ is a known function.

The correction functional for (9) is

$$u_{k+1}(\tau) = u_k(\tau) + \int_0^t \lambda(\tau)\left[\left(Lu_k(\tau) - Nu_k(\tau) - g(\tau)\right)\right]d\tau, \tag{10}$$

Here $\lambda(\tau)$ is the Lagrange multiplier corresponding to Caputo derivatives. For fractional derivatives of order $\alpha$, the Lagrange multiplier is taken as

$$\lambda(\tau) = \frac{(-1)^m}{(\alpha-1)!}(\tau - t)^{\alpha-1} = \frac{(\tau-t)^{\alpha-1}}{\Gamma[\alpha]}, \tag{11}$$

Then the series solution is $u(t) = \lim_{k\to\infty} u_k(t)$.

## VI. Numerical Experiments

In this section, a few numerical experiments are discussed to show the efficiency and efficacy of the proposed method by comparing the results with the variational iteration method.

**Example 1**. Consider the 2-D time-fractional Klein-Gordon equation

$$D_t^{\alpha} u(x,y,t) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y,t) + u^2 = x^2 y^2 t^2, \tag{12}$$

subject to initial conditions

$$u(x,y,0) = 0, \quad u_t(x,y,0) = xy, \qquad (x,y) \in \Omega, \tag{13}$$

The exact solution is:

$$u(x, y, t) = xyt, \qquad \text{for } \alpha = 2$$

**(a). Solution by STHPM:** Applying Sumudu transform on both sides of (12)

$$\mathbb{S}\left[D_t^{\alpha} u(x, y, t)\right] = \mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y, t) - u^2 + x^2 y^2 t^2\right],$$

By the differential property of the Sumudu transform

$$u^{-\alpha}\mathbb{S}[u(x, y, t)] - u^{-\alpha} u(x, y, 0) - u^{1-\alpha} u_t(x, y, 0) = \mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y, t) - u^2 + x^2 y^2 t^2\right],$$

Using the initial condition we obtain

$$\mathbb{S}[u(x, y, t)] = uxy + u^{\alpha}\mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y, t) - u^2 + x^2 y^2 t^2\right],$$

Applying the inverse Sumudu transform on each side of the above equation

$$u(x, y, t) = txy + \mathbb{S}^{-1} u^{\alpha}\mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y, t) - u^2 + x^2 y^2 t^2\right],$$

Applying the homotopy perturbation method on each side of the above equation

$$\sum_{n=0}^{\infty} p^n u_n = xyt + \mathbb{S}^{-1} u^{\alpha}\mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \sum_{n=0}^{\infty} p^n u_n - \sum_{n=0}^{\infty} p^n H_n(u) + x^2 y^2 t^2\right],$$

where$H_n(u)$ is He's polynomial

$$H_n(u_0, u_1, u_2, \dots) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right)\right]_{p=0}$$

Here some of the He's polynomials are :

$$H_0 = {u_0}^2, \quad H_1 = 2u_0 u_1, \quad H_2 = {u_1}^2 + 2u_0 u_1, \quad H_3 = 2u_1 u_2 + 2u_0 u_3, \dots,$$

Compare the like powers of p on both sides,

$$p^0 = u_0 = xyt,$$

$$p^1 = u_1 = \mathbb{S}^{-1} u^{\alpha}\mathbb{S}\left[(u_0)_{xx} + (u_0)_{yy} - H_0 + x^2 y^2 t^2\right] = 0,$$

$$p^2 = u_2 = \mathbb{S}^{-1} u^{\alpha}\mathbb{S}\left[(u_1)_{xx} + (u_1)_{yy} - H_1\right] = 0,$$

$$p^3 = u_3 = \mathbb{S}^{-1} u^{\alpha}\mathbb{S}\left[(u_2)_{xx} + (u_2)_{yy} - H_2\right] = 0,$$

and so on. The series solution is

$$u(x, y, t) = \sum_{n=0}^{\infty} u_n = xyt.$$

which is required exact solution.

**(b) Solution by VIM:** The solution of 2D TFKGE by variational iteration method.

The initial guess $u_0 = xyt$,

From equation (10), we have

$$u_1(\tau) = u_0(\tau) + \int_0^t \frac{(\tau - t)^{\alpha-1}}{\Gamma[\alpha]}\left[D_t^{\alpha} u_0 - \left(u_{0,xx} + u_{0,yy}\right) + u_n^2 - x^2y^2\tau^2\right]d\tau,$$

By using the properties of Caputo derivatives we get

$$u_1(t) = xyt\left(1 + \frac{1}{\Gamma[\alpha+1]}\right),$$

$$u_2(t) = xyt\left(1 + \frac{1}{\Gamma[\alpha+1]}\right)^2 - x^2y^2t^2\frac{\Gamma[3]}{\Gamma[\alpha+3]},$$

$$u_3(t) = xyt\left(1 + \frac{1}{\Gamma[\alpha+1]}\right)^3 - x^2y^2t^3\frac{\Gamma[3]}{\Gamma[\alpha+3]}\left(1 + \frac{1}{\Gamma[\alpha+1]}\right),$$

And so on. The series solution for the variational iteration method is

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t),$$

For $\alpha = 2$, the exact solution is $u(x,y,t) = xyt$.

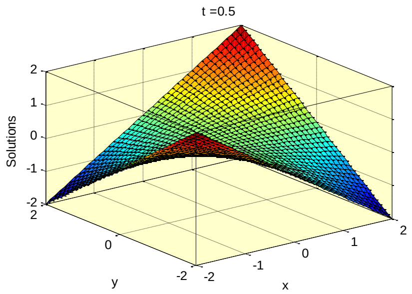


**Fig.1.**

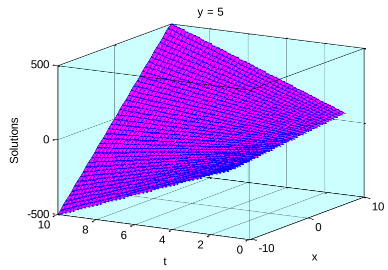


**Fig. 2.**

**Example 2:** Consider the 2-D time-fractional order Klein-Gordon equation

$$D_t^{\alpha}u(x,y,t) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x,y,t) + u^2 = 2x^2y^2 - 2x^2t^2 - 2y^2t^2 + x^4y^4t^4, \qquad (14)$$

subject to initial conditions

$$u(x,y,0) = 0, \quad u_t(x,y,0) = 0, \quad (x,y) \in \Omega, \qquad (15)$$

The exact solution is:

$$u(x,y,t) = x^2y^2t^2, \quad \text{for } \alpha = 2$$

**(b). Solution by STHPM:** First of all apply the Sumudu transform on each side of (14), then apply the differential property of the Sumudu transform and initial condition (15), after applying the inverse Sumudu transform

$$u(x,y,t) = \mathbb{S}^{-1}u^{\alpha}\mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y,t) - u^2 + 2x^2y^2 + 2x^2t^2 - 2y^2t^2 + x^4y^4t^4\right],$$

Applying the homotopy perturbation method on each side of the above equation

$$\sum_{n=0}^{\infty} p^n u_n = \mathbb{S}^{-1}u^{\alpha}\mathbb{S}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sum_{n=0}^{\infty} p^n u_n - \sum_{n=0}^{\infty} p^n H_n(u) + 2x^2y^2 + 2x^2t^2 - 2y^2t^2 + x^4y^4t^4\right],$$

where$H_n(u)$ is He's polynomial. Here some He's polynomials are

$$H_0 = u_0{}^2,\ \ H_1 = 2u_0u_1,\ \ \ H_2 = u_1{}^2 + 2u_0u_1,\ \ H_3 = 2u_1u_2 + 2u_0u_3, \ldots,$$

Compare the same powers of p on both sides

$$p^0 = u_0 = 0,$$

$$p^1 = u_1 = \mathbb{S}^{-1}u^{\alpha}\mathbb{S}\left[(u_0)_{xx} + (u_0)_{yy} - H_0 + 2x^2y^2 + 2x^2t^2 - 2y^2t^2 + x^4y^4t^4\right]$$

$$= 2x^2y^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2\Gamma(3)x^2\frac{t^{2+\alpha}}{\Gamma(\alpha+3)} - 2\Gamma(3)y^2\frac{t^{2+\alpha}}{\Gamma(\alpha+3)} + \Gamma(5)x^4y^4\frac{t^{4+\alpha}}{\Gamma(\alpha+5)},$$

$$p^2 = u_2 = \mathbb{S}^{-1}u^{\alpha}\mathbb{S}\left[(u_1)_{xx} + (u_1)_{yy} - H_1\right]$$

$$= 4x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 4y^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2\Gamma(3)\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + 12\Gamma(5)x^2y^4\frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)} + 12\Gamma(5)x^4y^2\frac{t^{2\alpha+4}}{\Gamma(2\alpha+5)},$$

and so on. The series solution is given by

$$u(x,y,t) = \textstyle\sum_{n\to 0}^{\infty} u_n = x^2y^2t^2,\ \ \ \text{for}\alpha = 2$$

which is required exact solution.

**(b). Solution by VIM:** The solution of 2D TFKGE by variational iteration method.

The initial guess $u_0 = 0$,

The values of various iterations from equation (10)

$$u_1(t) = \frac{2x^2y^2t^{\alpha}}{\Gamma[\alpha+1]} - \frac{2x^2t^{\alpha+2}}{\Gamma[\alpha+3]} - \frac{2y^2t^{\alpha+2}}{\Gamma[\alpha+3]} + \frac{2x^4y^4t^{\alpha+4}}{\Gamma[\alpha+5]},$$

$$u_2(t) = \frac{2x^2y^2t^{\alpha}}{\Gamma[\alpha+1]} - \frac{2x^2t^{\alpha+2}}{\Gamma[\alpha+3]} - \frac{2y^2t^{\alpha+2}}{\Gamma[\alpha+3]} + \frac{2x^4y^4t^{\alpha+4}}{\Gamma[\alpha+5]},$$

$$u_3(t) = \frac{2x^2y^2t^{\alpha}}{\Gamma[\alpha+1]} - \frac{2x^2t^{\alpha+2}}{\Gamma[\alpha+3]} - \frac{2y^2t^{\alpha+2}}{\Gamma[\alpha+3]} + \frac{2x^4y^4t^{\alpha+4}}{\Gamma[\alpha+5]} \ldots,$$

And so on. The series solution for the variational iteration method is

$$u(x, y, t) = \lim_{n\to\infty} u_n(x, y, t),$$

For $\alpha = 2$, the exact solution is $u(x, y, t) = x^2y^2t^2$.

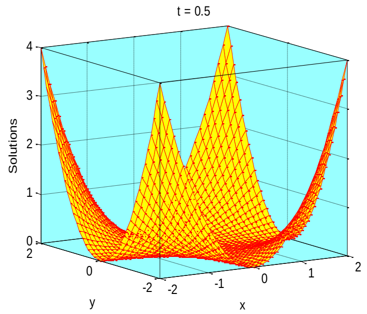


**Fig. 3.**

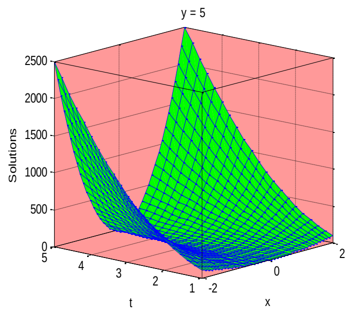


**Fig. 4.**

**Example 3:** Consider the 2-D time fractional sine-Gordon equation

$$D_t^{2\alpha}u(x, y, t) + D_t^{\alpha}u(x, y, t) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x, y, t) + 2\sin u - 2\sin[e^{-t}(1-\cos\pi x)(1-\cos\pi y)] + \pi^2e^{-t}[\cos\pi x + \cos\pi y - 2\cos\pi x\cos\pi y] = 0, \quad (16)$$

$$0 < \alpha \leq 1$$

subject to initial conditions

$$u(x, y, 0) = (1-\cos\pi x)(1-\cos\pi y),$$

$$u_t(x, y, 0) = -(1-\cos\pi x)(1-\cos\pi y), \qquad (x, y) \in \Omega, \quad (17)$$

The exact solution is: $u(x, y, t) = e^{-t}(1-\cos\pi x)(1-\cos\pi y)$.

Note that if $\alpha = 1$, then the 2-D time fractional sine-Gordon equation becomes the 2-D sine-Gordon equation.

**(a). Solution by STHPM:** First of all apply the Sumudu transform on each side of (16), then apply the differential property of the Sumudu transform and initial condition (17), after applying the inverse Sumudu transform

$u(\text{x}, \text{y}, \text{t}) = e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y}) + \mathbb{S}^{-1}\left\{\frac{u^{2\alpha}}{1+u}\mathbb{S}\left[\left(\frac{\partial^2}{\partial\text{x}^2} + \frac{\partial^2}{\partial\text{y}^2}\right)u(\text{x}, \text{y}, \text{t}) - 2\sin u + 2\sin[e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y})] - \pi^2 e^{-\text{t}}[\cos\pi\text{y} + \cos\pi\text{y} - 2\cos\pi\text{x}\cos\pi\text{y}]\right]\right\},$

Apply homotopy perturbation method

$$\sum_{\text{n}=0}^{\infty} p^{\text{n}} u_{\text{n}} = \mathbb{S}^{-1}\frac{u^{2\alpha}}{1+u}\mathbb{S}\left[\left(\frac{\partial^2}{\partial\text{x}^2} + \frac{\partial^2}{\partial\text{y}^2}\right)\sum_{\text{n}=0}^{\infty} p^{\text{n}} u_{\text{n}} - 2\sum_{\text{n}=0}^{\infty} p^{\text{n}} H_{\text{n}}(u) + 2\sin[e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y})] - \pi^2 e^{-\text{t}}[\cos\pi\text{x} + \cos\pi\text{y} - 2\cos\pi\text{x}\cos\pi\text{y}]\right],$$

where$H_{\text{n}}(u)$ is He's polynomial. Here some He's polynomials are

$$H_0 = \sin u_0, \quad H_1 = u_1\cos u_0, \quad H_2 = -\left(\frac{{u_1}^2}{2}\right)\sin u_0 + u_2\cos u_0,$$

$$H_3 = -\left(\frac{{u_1}^3}{6}\right)\cos u_0 + u_1 u_2\sin u_0 + u_3\cos u_0,$$

And so on. Comparing the like powers of $p$ on both sides,

$$p^0 = u_0 = e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y}),$$

$$p^1 = u_1 = \mathbb{S}^{-1}\frac{u^{2\alpha}}{1+u}\mathbb{S}\left[(u_0)_{\text{xx}} + (u_0)_{\text{yy}} - 2H_0 + 2\sin[e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y})] - \pi^2 e^{-t}[\cos\pi\text{x} + \cos\pi\text{y} - 2\cos\pi\text{x}\cos\pi\text{y}]\right],$$

$$\Rightarrow u_1 = 0,$$

$$p^1 = u_2 = \mathbb{S}^{-1}\frac{u^{2\alpha}}{1+u}\mathbb{S}\{(u_1)_{\text{xx}} + (u_1)_{\text{yy}} - 2H_1\} = 0,$$

$$u_3 = 0, u_4 = 0, \ldots,$$

The series solution is given by

$$u(\text{x}, \text{y}, \text{t}) = \sum_{\text{n}=0}^{\infty} u_{\text{n}} = e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y}).$$

which is required exact solution.

**(b). Solution by VIM:** The solution of 2D TFKGE by variational iteration method.

The initial guess $u_0 = e^{-\text{t}}(1-\cos\pi\text{x})(1-\cos\pi\text{y})$,

This question yields the same type of solution as the previous questions.

For $\alpha = 2$, the exact solution is: $u(x, y, t) = e^{-t}(1 - \cos \pi x)(1 - \cos \pi y)$.

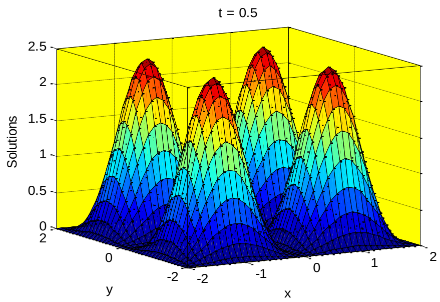


Fig. 5

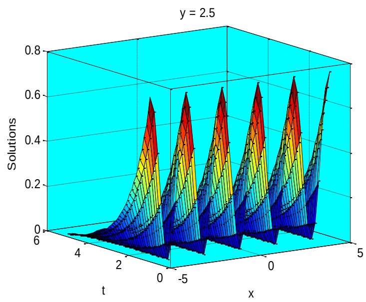


Fig. 6

## VII. Results and discussion

The findings demonstrate that the Sumudu Transform Homotopy Perturbation Method (STHPM) yields quicker convergence and requires simpler computations for time fractional PDEs, making it more efficient than the computationally intensive Variational Iteration Method (VIM). Several diagrams have been generated utilizing the Sumudu Transform Homotopy Perturbation Method. Figure 1 and Figure 2 show the physical and dynamic behavior of the solutions of example 1 at different values of $t$. Figure 3 and Figure 4 show the physical and dynamic behavior of the solutions of example 2 at different $t$. Figure 5 and Figure 6 show the physical and dynamic behavior of the solutions of example 3 at different values of $t$.

## VIII. Conclusion

In this study, we employed the sumudu transform-homotopy perturbation method (STHPM) to solve the 2-D time fractional Klein-Gordon and the 2-D time fractional sine-Gordon equations. By tackling the challenges posed by the time fractional components, our approach contributes to a deeper understanding and more effective analytical handling of these problems.

Our findings indicate that STHPM delivers precise and efficient solutions, accurately reflecting the physical phenomena described by these equations. This method simplifies the computational process while ensuring convergence to the exact solution, highlighting its potential for broader applications in the field of fractional differential equations.

Future research could explore the application of STHPM to higher-dimensional fractional differential equations and more complex coupled systems. Comparative studies with other numerical and analytical methods could further validate and enhance the robustness of the Sumudu transform-homotopy perturbation method.

In conclusion, the application of STHPM to the 2D-TFKGE and 2D-TFSGE demonstrates its effectiveness and opens new pathways for its application in solving fractional differential equations.

**Conflict of interest:**

There is no conflict of interest regarding this paper.

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