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EXTENSION OF LAPLACE - ARA TRANSFORM OF DIFFERENTIAL EQUATIONS

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Abstract

To solve differential equations, we utilize an extended Laplace-ARA transform result that we offer in this work to verify the existence of other pertinent theorems.

Keywords: ARA transform, Laplace transform, Triple Laplace-ARA transform, Volterra Integral equation, Volterra-integrodifferential equation, and integro-partial differential equation

I. Introduction

Partial differential equations can be solved using integral transforms. Applications to numerous phenomena in the fields of mathematics, physics, engineering, and other sciences. The differential equation is used in the writing of this kind of application [I-VI] [XII-XV]. Integral transforms can be equally utilized to solve both integral and integral differential equations [VII-XI] [XVI-XX]. Therefore, this triple integral transform gives us a quick and effective way to convert an integral-differential problem into an algebraic equation and get precise answers. Integral, partial, and integral-partial differential equations have all recently been solved using this technique.

II. n^{th} order ARA Transform:

A generalization of n^{th} order ARA transform of the continuous function f(t) on the interval $[0,\infty)$ is defined as

$$G_n[f(t)] = Q(n,s) = s\int_o^{\infty} t^{n-1}e^{-st}f(t)dt\,, s > o, n = 1,2,3 \tag{1}$$

and its inverse

$$G_{n+1}^{-1}[G_{n+1}(f(t)] = \frac{(-1)^{2n}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}Q(s)ds = f(t) \tag{2}$$

Some basic results of ARA Transform of order n:

(i). $G_n[1] = \frac{\Gamma(n)}{s^{n-1}}$

(ii). $G_n[t^n] = \frac{\Gamma(m+n)}{s^{m+n-1}}$

(iii). $G_n[e^{at}] = \frac{s\Gamma(\mathrm{n})}{(s-a)^n}$

(iv). $G_n[Sin(at)] = \left(1+\frac{a^2}{s^2}\right)^{-\frac{n}{2}} s^{1-n}\Gamma(\mathrm{n})\mathrm{Sin}(ntan^{-1}\left(\frac{a}{s}\right))$

(v). $G_n[Sinh(at)] = \frac{s}{2}\Gamma(\mathrm{n})\frac{1}{s^{\mathrm{n}}}\left[\frac{1}{\left(1-\frac{a}{s}\right)^{\mathrm{n}}}+\frac{1}{\left(1+\frac{a}{s}\right)^{\mathrm{n}}}\right]$

III. Triple Laplace–ARA Transform of order one (TL-ARAT)

Here a new integral transform TL-ARAT that combines the Laplace transform and the ARA transform is introduced. We present some basic results concerning the existence conditions. We first recall that the ARA transform of order one of a piecewise continuous function f(t) on $[o,\infty)$ defined as

$$G_1[f(t)] = Q(s) = s\int_o^\infty e^{-st}f(t)dt,\ s > o \tag{3}$$

On the other hand, the Laplace transform of the function $f(x)$ is defined as

$$L[f(x)] = F(s) = \int_o^\infty e^{-sx}f(x)dx,\ s > o \tag{4}$$

In the following, we introduce some basic results of the Laplace transform and ARA transform.

Let$f(x)$ $and g(x)$ be two continuous functions on the interval$[o,\infty)$.

Then

$$L[af(x) \pm bg(x)] = aL[f(x)] \pm bL[g(x)] \tag{5}$$
$$L[f^{'}(x)] = sF(s) - f(o) \tag{6}$$
$$L(x^n) = \frac{n!}{s^{n+1}} \tag{7}$$
$$L[e^{ax}] = \frac{1}{s-a} \tag{8}$$

Similarly, let $f(t)$ and $g(t)$be two continuous functions on the interval $[o,\infty)$, then ARA transform results are defined

$$G[af(t) \pm bg(t)] = aG[f(t)] \pm bG[g(t)] \tag{9}$$

$$G[f^{'}(t)] = sQ(s) - sf(0)] \tag{10}$$

$$G[t^n] = \frac{n!}{s^n} \tag{11}$$

$$G[e^{at}] = \frac{s}{s-a} \tag{12}$$

Where$a and b$ are nonzero constants,$n = 0,1,2,3$

Definition:

Let $u\ (x, y, t)$ be a continuous function and three positive variables $x, y\ and\ t$ .Then TL-ARAT of $u\ (x, y, t)$ is defined as

$$LG[u(x,y,t)] = Q(p,q,s) = qs\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}[u(x,y,t)]dxdydt\ p,q,s > 0 \qquad (13)$$

Clearly, the TL-ARAT is a linear integral transform as shown below:

$$LG[Au(x,y,t) + Bw(x,y,t)]$$
$$= qs\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}[Au(x,y,t) + Bw(x,y,t)]dxdydt$$
$$= Asq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}[u(x,y,t)dxdydt$$
$$+\ Bsq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}[w(x,y,t)]dxdydt$$

=ALG [u(x, y,t)]+BLG[w (x,y,t)]

Where A and B are constants and its inverse of the TL-ARAT is defined as

$$L^{-1}[G^{-1}[Q(p,q,s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}dp\left(\frac{1}{2\pi i}\right)\int_{d-i\infty}^{d+i\infty} e^{qy}dq\left(\frac{1}{2\pi i}\right)\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s}Q(p,q,s)ds = u(x,y,t)(14)$$

TL-ARAT of Some Basic Functions:

(i). Let $u(x,y,t) = 1$, $x > 0$, $y > 0, t > 0$ then

$$LG[1] = qs\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}dxdydt =$$
$$\int_0^\infty e^{-(px)}dx.\,q\int_0^\infty e^{-(qy)}dy.\,s\int_0^\infty e^{-st}dt = 1,\ \ Re(s), Re(p), Re(q) > 0\ \ (15)$$

(ii). Let$u(x,y,t) = x^\alpha y^\beta t^\gamma,\ \ x > 0, y > 0, t > 0\ and \alpha, \beta, \gamma$ are constants. Then

$$LG\left[x^\alpha y^\beta t^\gamma\right] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left[x^\alpha y^\beta t^\gamma\right]dxdydt$$
$$= \int_0^\infty e^{-px}[x^\alpha]dxq\int_0^\infty e^{-qx}\left[y^\beta\right]dys\int_0^\infty e^{-st}[t^\gamma]dt = L[x^\alpha]L[y^\beta]G[t^\gamma]\ (16)$$

Using equations (7) and (14) we get

$$LG\left[x^\alpha y^\beta t^\gamma\right] = L[x^\alpha]L\left[y^\beta\right]G[t^\gamma] = \frac{\alpha!\,\beta!\,\gamma!}{p^{\alpha+1}q^{\beta+1}s^\gamma}\ , Re(\alpha) > 1, Re(\beta) > 1, and Re(\gamma) > 0.$$

(iii). Let $u(x,y,t) = e^{(\alpha x+\beta y+\gamma t)},\ x > 0,\ y > 0,\ t > 0\ and \alpha, \beta, \gamma$ are constants. Then

$$LLG = \left[e^{\alpha x+\beta y+\gamma t}\right] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{(\alpha x+\beta y+\gamma t)}[e^{-(px+qy+st)}]\,dxdydt$$
$$= \int_0^\infty[e^{\alpha x}]e^{-px}dx\,q\int_0^\infty\left[e^{\beta y}\right]e^{-qy}dy\,s\int_0^\infty[e^{\gamma t}]e^{-st}dt = L[e^{\alpha x}]L[e^{\beta y}]G[e^{\gamma t}]$$

From the above equation (8) and equation (12) we get

$$\mathrm{LLG}\left[e^{\alpha x+\beta y+\gamma t}\right]=\frac{s}{(p-\alpha)(q-\beta)(s-\gamma)}\ ,Re(\alpha)+Re(\beta)+Re(s)>0,$$

Similarly,

$$\mathrm{LLG}\left[e^{i(\alpha x+\beta y+\gamma t)}\right]=\frac{s}{(p-i\alpha)(q-i\beta)(s-i\gamma)}$$

Using the property of complex analysis, we have

$$\mathrm{LLG}\left[e^{i(\alpha x+\beta y+\gamma t)}\right]=\frac{s(p+i\alpha)(q+i\beta)(s+i\gamma)}{(p^2+\alpha^2)(q^2+\beta^2)(s^2+\gamma^2)}$$

$$=\frac{s(pq-ip\beta+i\alpha q-\alpha\beta)(s+i\gamma)}{(p^2+\alpha^2)(q^2+\beta^2)(s^2+\gamma^2)}$$

$$=\frac{s(pqs+ipq\gamma+ip\beta s-p\beta\gamma+i\alpha qs-\alpha q\gamma-\alpha\beta s-i\alpha\beta\gamma)}{(p^2+\alpha^2)(q^2+\beta^2)(s^2+\gamma^2)}$$

$$=\frac{(pqs^2+p\beta\gamma s-\alpha qs\gamma-\alpha\beta\gamma s^2)+i(pqs\gamma+p\beta s^2+\alpha qs^2-\alpha\beta\gamma s)}{(p^2+\alpha^2)(q^2+\beta^2)(s^2+\gamma^2)}$$

Using the Euler's formulas:

$$Sinx=\frac{e^{ix}-e^{-ix}}{2}\,and\cos x=\frac{e^{ix}+e^{-ix}}{2}$$

and the formulas:

$$Sinhx=\frac{e^{x}-e^{-x}}{2}\,and\cos hx=\frac{e^{x}+e^{-x}}{2}$$

Therefore, we conclude with the following results:

$$\mathrm{LLG}[\sin(\alpha x+\beta y+\gamma t)]=\frac{s(pq\gamma+p\beta s+\alpha qs-\alpha\beta\gamma)}{(p^2+\alpha^2)(q^2+\beta^2)(s^2+\gamma^2)}$$

$$\mathrm{LLG}[\cos(\alpha x+\beta y+\gamma t)]=\frac{s(pqs+p\beta\gamma-\alpha q\gamma-\alpha\beta s)}{(p^2+\alpha^2)(q^2+\beta^2)(s^2+\gamma^2)}$$

$$\mathrm{LLG}[\sin h(\alpha x+\beta y+\gamma t)]=\frac{s(pq\gamma+p\beta s+\alpha qs-\alpha\beta\gamma)}{(p^2-\alpha^2)(q^2-\beta^2)(s^2-\gamma^2)}$$

$$\mathrm{LLG}[\cos h\,(\alpha x+\beta y+\gamma t)]=\frac{s(pq\gamma+p\beta s+\alpha qs-\alpha\beta\gamma)}{(p^2-\alpha^2)(q^2-\beta^2)(s^2-\gamma^2)}$$

$$\text{(iv)}LLG[\mathrm{J}_0(\lambda\sqrt{\mathrm{xt}})=\mathrm{qs}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}[\mathrm{J}_0(\lambda\sqrt{\mathrm{xt}})]\,dt$$

$$=\int_0^\infty[\mathrm{J}_0(\lambda\sqrt{\mathrm{xt}})]\mathrm{e}^{-\mathrm{px}}\,dxq\int_0^\infty[\mathrm{J}_0(\lambda\sqrt{\mathrm{xt}})]\mathrm{e}^{-\mathrm{qy}}\mathrm{dy}\ \ \mathrm{s}\int_0^\infty\mathrm{J}_0(\lambda\sqrt{\mathrm{xt}})]\mathrm{e}^{-\mathrm{st}}\mathrm{dt}$$

$$=\mathrm{qs}\int_0^\infty\mathrm{e}^{\frac{-\lambda^3}{\mathrm{sq}}}\,te^{-st}dt$$

From equation (12) we get

$$\mathrm{LLG}[J_0(\lambda\sqrt{xt})] = \frac{8sq}{8pqs + \lambda^3}$$

(v). Let $u(x, y, t) = f(x)\, g(t), x > 0, t > 0.$ Then

$$\mathrm{LLG}\,[f(x)g(x)h(x)] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-px+qy+st}[f(x)g(x)h(x)]\,dxdy\,dt$$

$$= \int_0^\infty e^{-px}[f(x)]dx.\, q\int_0^\infty e^{-qy}[g(x)]dy.\ s\int_0^\infty e^{-st}[h(x)]dt$$

$$= L[fex)\ L[g(x)]\ G[h(x)]$$

IV. Existence of TL-ARAT:

Let $u(x, y, t)$ be a function of exponential order $\alpha, \beta,$ and γ.Then it is defined as $x \to \infty, y \to \infty andt \to \infty if there exists a + ve N such that \forall x > X, \forall y > Y\ andt > T,$ we have $|u(x, y, t)| \le Ne^{\alpha x+\beta y+\gamma t}$.

We can write$u(x, y, t) = o(e^{\alpha x+\beta y+\gamma t})\ asx \to \infty, y \to \infty andt \to \infty,\ p > \alpha, q > \beta ands > \gamma.$

Theorem (1): Let(x, y, t) be a continuous function of exponential order $\alpha, \beta$and γ. Then LLG $[u(x, y, t)]$ $exists for p, q and s provided\ Re(p) > \alpha, Re(q) > \beta and Re(s) > \gamma$

Proof: Using the definition of TL-ARAT, we have

$$|Q(p,q,s)| = \left|sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}dxdydt\right|$$

$$\le sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}|u(x,y,t)|dxdydt$$

$$\le N\int_0^\infty e^{-(p-\alpha)x}dx\, q\int_0^\infty e^{-(q-\beta)y}dys\int_0^\infty e^{-(s-\gamma)t}dt$$

$= \frac{Nqs}{(p-\alpha)(q-\beta)(s-\gamma)} Re(p) > \alpha, Re(q) > \beta and Re(s) > \gamma$.

Theorem2: Let $u(x, y, t)$be a continuous function and LLG $\left[\frac{\partial u(x,y,t)}{\partial t}\right] = Q(p, q, s)$.

Then LLG$\left[e^{(\alpha x+\beta y+\gamma t)}u(x,y,t)\right] = \frac{sq}{(q-\beta)(s-\gamma)}Q(p-\alpha, q-\beta, s-\gamma)$

Proof: We have

$$\mathrm{LLG}\left[e^{\alpha x+\beta y+\gamma t}u(x,y,t)\right]$$

$$= sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(p-\alpha)x-(q-\beta)y-(s-\gamma)t}[u(x,y,t)]dxdydt$$

$$= \frac{sq}{(q-\beta)(s-\gamma)}Q(p-\alpha, q-\beta, s-\gamma)$$

Theorem (3): Let $LLG[u(x,y,t)]$ exists where u (x,y,t) is a periodic function of periods $\alpha, \beta and \gamma such that$

$$u(x+\alpha, y+\beta, t+\gamma) = u(x,y,t) \forall x,y,t$$

Proof: **U**sing the definition of TL-ARAT, we get

$$LLG[u(x,y,t)] = sq\int_0^\infty \int_0^\infty \int_0^\infty e^{-(px+qy+st)}[u(x,y,t)]dxdydt \tag{17}$$

Using the property of improper integral of the above equation (17) can be written as

$$LLG[u(x,y,t)] = sq\int_0^\alpha \int_0^\beta \int_0^\gamma e^{-(px+qy+st)}[u(x,y,t)]dxdydt$$

$$+sq\int_\alpha^\infty \int_\beta^\infty \int_\gamma^\infty e^{-(px+qy+st)}[u(x,y,t)]dxdydt \tag{18}$$

Putting $x = \alpha + \rho, y = \beta + \varphi and t = \gamma + \tau$ on the second integral in equation (18) we get

$$Q(p,q,s) = sq\int_0^\alpha \int_0^\beta \int_0^\gamma e^{-(px+qy+st)}[u(x,y,t)]dxdydt +$$

$$sq\int_0^\infty \int_0^\infty \int_0^\infty e^{-(p(\alpha+\rho)+q(\beta+\varphi)+s(\gamma+\tau)}\left(u(\alpha+\rho, \beta+\varphi,\ \gamma+\tau)\right)d\rho d\varphi d\tau \tag{19}$$

Using the periodicity of the function u(x, y, t) the above equation (19) can be written as

$$Q(p,q,s) = sq\int_0^\alpha \int_0^\beta \int_0^\gamma e^{-(px+qy+st)}[u(x,y,t)]dxdydt +$$

$$e^{-(p\alpha+q\beta+s\gamma)}sq\int_0^\infty \int_0^\infty \int_0^\infty e^{-(p\rho+q\varphi+s\tau)}[u(\rho,\varphi,\tau)]d\rho d\varphi d\tau \tag{20}$$

Using the definition of TL-ARAT we have

$$Q(p,q,s) = sq\int_0^\alpha \int_0^\beta \int_0^\gamma e^{-(px+qy+st)}[u(x,y,t)]dxdydt +$$

$$e^{-(p\alpha+qy+st)}Q(p,q,s) \tag{21}$$

Then the equation (21) can be simplified into

$$Q(p,q,s) = \frac{1}{1-e^{-(p\alpha+q\beta+s\gamma)}}\left(sq\int_0^\alpha \int_0^\beta \int_0^\gamma e^{-(px+qy+st)}[u(x,y,t)]dxdydt\right).$$

Theorem (4). Let LLG $[u(x,y,t)]$ exist and LLG $[u(x,y,t)] = Q(p,q,s)$ then

$$\text{LLG}\,[u(x-\delta, y-\rho, t-\xi)] = e^{-(p\delta+q\rho+s\xi)}Q(p,q,s) \tag{22}$$

Where

$H(x-\delta, y-\rho, t-\zeta)$ is the Heaviside unit step function defined as

$$H(x-\delta, y-\rho, t-\zeta) = \begin{cases} 1\,, x > \delta, t > \zeta \\ 0\,, otherwise \end{cases}$$

Proof: Using the definition of TL-ARAT, we get

$$L_xL_yG_t[u(x-\delta,y-\rho,t-\xi)] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\,(u(x-\delta,y-\rho,t-\xi)\;H(x-d,y-\rho,t-\xi))dxdydt$$

$$= sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left(u(x-\delta,y-\rho,t-\xi)\right)dxdydt \qquad (23)$$

Putting $x-\delta = \rho,\; y-\rho = \varphi andt-\xi = \tau$ in the above equation (23) we get

$$L_xL_yG_t[u(x-\delta,y-\rho,t-\xi)H(x-\delta,y-\rho,t-\xi)] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-p(\delta+\rho)-q(\rho+\varphi)-s(\tau+\xi)}\,u(\rho,\varphi,\tau)d\rho,d\varphi,d\tau \qquad (24)$$

Then (24) can be simplified into

$$L_xL_yG_t[u(x-\delta,y-\rho,t-\xi)H(x-\delta,y-\rho,t-\xi)]$$
$$= e^{-p\delta-q\varphi-s\xi}sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-p\rho-q\varphi-s\xi}\left(u(\rho,\varphi,\tau)\right)d\rho,d\varphi,d\tau$$
$$= e^{-p\delta-q\varphi-s\xi}Q(p,q,s).$$

Theorem (5):

Let$L_xL_yG_t[u(x,y,t)]$ and $L_xL_yG_t[w(x,y,t)]$are exists and

$$L_xL_yG_t[u(x,y,t)] = Q(p,q,s)$$
$$L_xL_yG_t[w(x,y,t)] = W(p,q,s)$$

Then

$$L_xL_yG_t[u ** w(x,y,t)] = \frac{1}{sq}Q(p,q,s)W(p,q,s) \qquad (25)$$

$$\Rightarrow u ** w(x,y,t) = \int_0^x\int_0^y\int_0^t u(x-\rho,y-\varphi,t-\tau)\,w(\rho,\varphi,\tau)d\rho,d\varphi,d\tau$$

Where the symbol ** denotes the triple convolution $w.r.tox,yandt$

Proof: Using the definition of TL-ARAT, we get

$$L_xL_yG_t[u *** w(x,y,t)] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left(u *** w(x,y,t)\right)dxdydt$$

$$= sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}(\int_0^\infty\int_0^\infty\int_0^\infty u(x-\rho,y-\varphi,t-\tau)\,w(\rho,\varphi,\tau)dxdydt \qquad (26)$$

Using the Heaviside unit step function the above equation (26) can be written as

$$L_xL_yG_t[u *** w(x,y,t)] = \int_0^\infty\int_0^\infty\int_0^\infty W\,(\rho,\varphi,\tau)d\rho,d\varphi,d\tau$$

$$(sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-p(x+\rho)-q(y+\varphi)-s(t+\tau)}\,u(x-\rho,y-\varphi,t-\tau))H(x-\rho,y-\varphi,t-\tau)\,dxdydt$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty W\,(\rho,\varphi,\tau)d\rho,d\varphi,d\tau(e^{(-px-qy-st)}Q(p,q,s))$$

$$=Q(p,q,s)\int_0^\infty\int_0^\infty\int_0^\infty e^{-p\rho-q\varphi-s\tau}\,w(\rho,\varphi,\tau)d\rho,d\varphi,d\tau=\frac{1}{sq}Q(p,q,s)\,W(p,q,s)$$

Theorem (6):

Let $u(x,y,t)$ be a continuous function and

$L_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial t}\right]=Q(p,q,s)$. Then we get the following result:

(i). $\mathcal{L}_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial t}\right]=sQ(p,q,s)-sL\,[u(x,y,0)]$

(ii). $L_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial x}\right]=pQ(p,q,s)-G\,[u(o,y,t)]$

(iii). $L_xL_yG_t\left[\frac{\partial^2 u(x,y,t)}{\partial t^2}\right]=s^2Q(p,q,s)-s^2L[u(x,y,o)]-sL\left[\frac{\partial u(x,y0t)}{\partial t}\right]$

(iv). $L_xL_yG_t\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right]=p^2Q(p,q,s)-pG[u(o,y,t)]-G\left[\frac{\partial u(o,y,t)}{\partial x}\right]$

(v). $L_xL_yG_t\left[\frac{\partial^3 u(x,y,t)}{\partial x\partial y\partial t}\right]=pqsQ(p,q,s)-pqsG[u(o,y,t)]-pqsL[u(x,y,o)]+sG[u(o,o,o)]$

Proof (i). Now

$$L_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial t}\right]=sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left[\frac{\partial u(x,y,t)}{\partial t}\right]dxdydt$$

$$=\int_0^\infty e^{-px}dx\,.\;q\int_0^\infty e^{-qy}dy\,.\;s\int_0^\infty e^{-st}\left[\frac{\partial u(x,y,t)}{\partial t}\right]dt \qquad (27)$$

Using the integration by parts we get

Let $u=e^{-st}\Rightarrow du=-se^{-st}dt$

$$dp=\left[\frac{\partial u(x,y,t)}{\partial t}\right]dt\Rightarrow p=u(x,y,t)$$

Thus

$$s\int_0^\infty e^{-st}\left(\frac{\partial u(x,y,t)}{\partial t}\right)dt=s(-u(x,y,o)+s\int_0^\infty e^{-st}u(x,y,t)dt$$

$$\therefore L_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial t}\right]=sQ(p,q,s)-sL[u(x,y,o)] \qquad (28)$$

(ii). $L_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial x}\right]=sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left[\frac{\partial u(x,y,t)}{\partial x}\right]dxdydt$

$$=s\int_0^\infty e^{-st}\,dt.\;q\int_0^\infty e^{-qy}dy\,.\int_0^\infty e^{-px}\left[\frac{\partial u(x,y,t)}{\partial x}\right]dx$$

Using integrating by parts, we get

Let $u = e^{-px} \Rightarrow du = -pe^{-px}dx$

$$dv = \frac{\partial u(x,y,t)}{\partial x}dx \Rightarrow v = u(x,y,t)$$

Then

$$\int_0^\infty e^{-px}\left(\frac{\partial u(x,y,t)}{\partial x}\right)dx = (-u(o,y,t) + p\int_0^\infty e^{-px}\,u(x,y,t)dx$$

$$\therefore L_xL_yG_t\left[\frac{\partial u(x,y,t)}{\partial x}\right] = pQ(p,q,s) - G[u(o,y,t)] \tag{29}$$

(iii). $L_xL_yG_t\left[\frac{\partial^2 u(x,y,t)}{\partial t^2}\right] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left[\frac{d^2u(x,y,t)}{dt^2}\right]dxdydt$

$$= \int_0^\infty e^{-px}dx.\ q\int_0^\infty e^{-qy}dy\, s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,y,t)}{\partial t^2}\right)dt$$

Using integrating by parts, we get

$$Letu = e^{-st} \Rightarrow du = -se^{-st}dt$$

$dp = \frac{\partial^2 u(x,y,t)}{\partial t^2}dt \Rightarrow p = \frac{\partial u(x,y,t)}{\partial t}$. Thus

$$s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,y,t)}{\partial t^2}\right)dt = s\left[-\frac{\partial u(x,y,o)}{\partial t}\right] + s\int_0^\infty e^{-st}\left(\frac{\partial u(x,y,t)}{\partial t}\right)dt)$$

Using equation (28) we have

$$L_xL_yG_t\left[\frac{\partial^2 u(x,y,t)}{\partial t^2}\right] = s^2Q(p,q,s) - s^2L[u(x,y,o)] - sL\left[\frac{\partial u(x,y,o)}{\partial x}\right] \tag{30}$$

(iv). $L_xL_yG_t\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left[\frac{\partial^2 u(x,y,t)}{\partial t^2}\right]dxdydt$

$$= s\int_0^\infty e^{-st}dt.\ q\int_0^\infty e^{-qy}dy\int_0^\infty e^{-px}\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right]dx$$

Using integrating by parts, we get

Let $u = e^{-px} \Rightarrow dx = -pe^{px}dx$

$$dv = \frac{\partial^2 u(x,y,t)}{\partial x^2}dx \Rightarrow v = \frac{\partial u(x,y,t)}{\partial x}$$

Thus

$$\int_0^\infty e^{-px}\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}\right]dx = \left(-\frac{\partial u(o,y,t)}{\partial x} + p\int_0^\infty e^{-px}\left(\frac{\partial u(x,y,t)}{\partial x}\right)dx\right.$$

Using the equation (29) in above, we have

$$L_xL_yG_t\left[\frac{\partial^2u(x,y,t)}{\partial x^2}\right] = p^2Q(p,q,s) - pG[u(o,y,t)] - G\left[\frac{\partial u(o,y,t)}{\partial x}\right] \quad (31)$$

(v). $L_xL_yG_t\left[\frac{\partial^3u(x,y,t)}{\partial x\partial y\partial t}\right] = sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left[\frac{\partial^3u(x,y,t)}{\partial x\partial y\partial t}\right]dxdydt$

$$= s\int_0^\infty e^{-st}\left(\frac{\partial^3u(x,y,t)}{\partial x\partial y\partial t}dtq\int_0^\infty e^{-qy}\left(\frac{\partial^3u(x,y,t)}{\partial x\partial y\partial t}\right)dy\int_0^\infty e^{-px}dx\right.$$

Using integrating by parts, we get

$$sq\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left(\frac{\partial^3u(x,y,t)}{\partial x\partial y\partial t}\right)dxdydt$$
$$= (-\int_0^\infty e^{-st}\left(\frac{\partial u(o,y,t)}{\partial t}\right)dt + \int_0^\infty e^{-qy}\left(\frac{\partial u(x,0,t)}{\partial y}\right)dy + pqs\int_0^\infty\int_0^\infty\int_0^\infty e^{-(px+qy+st)}\left(\frac{\partial u(x,y,t)}{\partial t}\right)dxdydt)$$

and using equation (28) and (10) we have

$$L_xL_yG_t\left[\frac{\partial^3u(x,y,t)}{\partial x\partial y\partial t}\right] = pqsQ(p,q,s) - psqG[u(o,y,t)] - pqsL[u(x,y,o)] + sG[u(o,o,o)] \quad (32)$$

V. Conclusion

This type of work introduces a new triple transform method termed Laplace – ARA transform. The theory and the properties of the TL-ARAT are disclosed. This innovative approach has shown remarkable efficiency in solving integral equations and achieving a much higher degree of integral equations.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding this article.

References:

I. AL-Qmari S. K. Q. (2020), Estimation of a modified integral associated with a special function kernel of fox's h-function type, Commun. Korean Math. Soc. , No. 1, pp. 125–136.

II. BhatterSanjay(2022), On Certain New Results of Fractional Calculus Involving Product of Generalized Special Functions, Int. J. Appl. Comput. Math. 10.1007/s40819-022-01253-0.

III. Choi J. H.(2013), Applications of multivalent functions associated with generalized fractional integral operator, Scientific Research, Advances in Pure Mathematics, 3, 1 – 5.

IV. Debnath, L., Bhatta, D.: Integral Transforms and Their Applications, 3rd edn. Chapman Haubold H. J.(2009), Mittag-Leffler Functions and their applications, Journal of Applied Mathematics**,** arXiv:0909.0230v2 .

V. Gupta K.(2011), A Study of modified H-transform and generalized fractional integral operator of weyl type, International Journal of Pure and Applied Sciences and Technology, **7(1) (2011),** pp. 59-67, ISSN 2229-6107.

VI. Gupta K. and Vandana Agrawal (2010). A Theorem Connecting the H-Transform and Fractional Integral Operators Involving the Multivariable H-Function*, Tamsui Oxford Journal of Mathematical Sciences, Aletheia University, 26(4) , 383-395.

VII. Gupta K. C. and Gupta T. (2005), On unified Eulerian type integrals having general arguments, Soochow Journal of Mathematics, Volume - 31, No. 4, pp. 543-548.

VIII. Haubold H. J.(2009). Mittag-Leffler Functions and their applications, Journal of Applied Mathematics, arXiv:0909.0230v2 .

IX. Jangid K.(2020),Fractional calculus and integral transforms of the product of a general class of polynomial and incomplete Fox–Wright functions , Advances in Difference Equations , 2020:606 https://doi.org/10.1186/s13662-020-03067-0

X. Koul C. L(1971), On fractional integral operators of functions of two variables, Proc. Nat. Acad. Sci.India,41A,233-240.

XI. Kumar D (2013), Generalized Fractional Differentiation of the–Function Involving General Class of Polynomials, Int. J. Pure Appl. Sci. Technol., 16(2) , pp. 42-53, 2229 – 6107.

XII. Kumar D.(2013), New Fractional-Calculus Results Involving General Class of Multivariable Polynomials and Multivariable H-function, International Journal of Modern Mathematical Sciences. 7(1): 55-64,2166-286X.

XIII. Kiryakoya V.(2010),The special functions of fractional calculus as generalized fractional calculus operators of some basic functions, Comp& mathematics with applications, 59(3) ,1128-1141. 10.1016/j.camwa.2009.05.014.

XIV. Oparnica L (2001), Generalized fractional calculus with applications in mechanics, МАТЕМАТИЧКИ ВЕСНИК, 53, 151 – 158.

XV. Singh S.K. (2018), Integral Transform and the Solution of Fractional Kinetic Equation Involving Some Special Functions, International Journal of Mathematics Trends and Technology (/IJMTT) –V (55), pp. 5- 16.

XVI. Satyanarayana B. and Kumar Pragathi(2011). Some finite integrals involving multivariable polynomials, H-function of one variable and H-function of 'r' variables, African Journal of Mathematics and Computer Science Research Vol. 4(8), pp. 281-285.

XVII. Sharma B. L(1961), On a generalized function of two variables, I. Ann. Soc. Sci., Bruxellers Ser. I 79(1965), 26-40.Fox C. The G and H-functions as symmetrical Fourier kernel, Amer. Math. Soc. Transl., 98 , pp.395-429

XVIII. Srivastava H. M. (1983). The Weyl fractional integral of a general class of polynomials, Boll. Un. Mat. Italy 602-B , pp.219-228.

XIX. Srivastava H. M.(1976), Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math., 283/284, 265-274.

XX. Thakur A.K, S.K. Sahani and J.K. Kushwaha (2020), Some applications of Quadraple Hypergeometric functions in functions Spaces, Vol 17(12) 894 -903| 10.5281/zenodo.7451049.