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DETERMINING THE DOMINANT METRIC DIMENSION FOR VARIOUS GRAPHS

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Abstract

In this paper, we examine the dominating metric dimension of various graph types. A resolving set is a subset of vertices that uniquely identifies each vertex in the graph based on its distances to others, and the metric dimension is the minimum size of such a set. A dominating set ensures each vertex is adjacent to at least one vertex in the set. When a set is both resolving and dominating, it forms a dominating resolving set, and the smallest such set defines the dominating metric dimension, denoted as $Ddim(G)$. *We calculate the dominating metric dimension for the splitting graph of* $K_{1,n}$ *book graph, globe graph, tortoise graph, and* $C_4@W_n$ *graph.*

Keywords: Distance, Dominant Metric Dimension, Dominant Resolving Set, Metric Dimension, Resolving Set,

I. Introduction

One of the topics in graph theory is the metric dimension. In 1975, Slater [XXVII] initially proposed the problem of investigating the metric dimension. All graphs used in this paper are finite, undirected, and simple.

Assume that the connected graph $G = (V, E)$ has the vertex set V and the edge set E. The length of the shortest path between any two vertices, indicated by $d(x, y)$more conveniently, represents the distance between them. The metric representation of v concerning W, that is, $r(u|W) \neq r(v|W)$, where $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ for an ordered subset $W = \{w_1, w_2, w_3, \dots, w_k\}$of vertices in a connected graph G and a vertex $v \in V(G)$. The metric dimension of G is the lowest cardinality among its resolving sets, and it is represented by $dim(G)$. A subset $S \subseteq V(G)$ is referred to as a dominating set of G if at least one vertex $u \in S$

exists such that $x \sim u$ for every vertex x in $V(G)\backslash S$. A dominating number of G is the set of dominating sets with the lowest cardinality and it is denoted by $\gamma(G)$ [XII].

Metric dimension is a parameter that has been used in numerous graph theory applications, including those in pharmaceutical chemistry, network discovery and verification [XXVIII], robot navigation [X, XXIX], pattern recognition and image processing problems [XXVIII], coin weighing problems [II], mastermind game strategies[XXVII] and combinatorial optimization [XI]. Many research investigations on the idea of the metric dimension of graphs have been conducted by Mohamed et al. [VI], Nazeer et al. [XXV], Singh et al. [XX], Mohaisen et al. [V], Siddiqui et al. [XIII], Deng et al. [III], Amin et al. [IX] and Wijaya et al. [XVII].

Both the problem of the dominant set and the problem of the metric dimension are NP-complete [XI, XIX]. As a result, finding whether $Ddim(G) \leq K$ for a given graph G and input K is a typical NP-complete problem for the dominating metric dimension of G. Wireless communication networks, electrical networks, economic networks, and chemical structures all apply the dominance hypothesis [I, XVI]. To overcome the problem of uniquely locating an intruder in a network, a minimal resolving set of a graph has been introduced in [XXI]. The concept of the smallest resolving set of a graph serving as the metric basis and its cardinality number serving as the metric dimension were independently introduced by the authors in [XI].

In [XVIII], the dominant metric dimension is investigated. Path graph P_n, cycle graph C_n, star graph S_n, complete graph K_n, and complete bipartite graph $K_{m,n}$ all have their dominating metric dimensions theorized in [XVIII]. It has been demonstrated that $Ddim(P_n)$, $n = 1, 2$ is 1, $Ddim(P_n)$, $n > 4$ is $\left\lceil \frac{n}{3} \right\rceil$, $Ddim(C_n)$, $n \geq 7$ is $\left\lceil \frac{n}{3} \right\rceil$, $Ddim(S_n), n \geq 2$ is $n-1$, $Ddim(K_n), n \geq 2$ is $n-1$ and $Ddim(K_{m,n}), m, n \geq 2$ is $m+n-2$. When H is a path graph P_n, cycle graph C_n, complete bipartite graph $K_{m,n}$, complete graph K_n or star graph S_n, the dominant metric dimension of the corona product graph of G and H is studied in [XXX]. In [VII], the dominant numbers of the twig network T_m, double fan network $F_{2,n}$ bistar network $B_{n,n}$ and linear kc_4- snake networks are theoretically determined.

In this paper, we determine the exact value of the domination metric dimension of some particular classes of graphs, such as the middle graph, tortoise graph, globe graph, open diagonal ladder graph, and splitting graph of $K_{1,n}$, book graph and $C_4@W_n$ graph. We first review a few results concerning the dominant number and metric dimension of several well-known graphs. The proofs and more information can be found in [XII, XXXI].

1. For path P_n and cycle C_n, we have
   $$\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil, dim(P_n) = 1, dim(C_n) = 2, Ddim(P_n) = \gamma(P_n).$$
2. For complete graph K_n, we have
   $$(K_n) = 1, dim(K_n) = n-1, Ddim(K_n + K_m) = dim(K_n) + m.$$

3. For star S_n, we have
$$(S_n) = 1, (S_n) = n - 2, D(G) = n- 1,$$
for all $n \geq 2$.

4. For complete bipartite graph Km,n, we have
$$\gamma\left(K_{m,n}\right) = 2, dim\left(K_{m,n}\right) = m + n - 2, Ddim(K_{m,n}) = m + n - 2,$$
for every $m, n \geq 2$.

5. For connected graph G, $dim(G) = 1$ if only if $G = P_n$ and $Ddim(G) = 1$ if only if $G \cong P_n$, $n = 1, 2$.

Since every dominant resolving set is a resolving set, $dim(G) \leq Ddim\ (G)$ for all connected graphs G. To illustrate this notion, consider the graph G in Fig 1. Further information might be found in the literature [IV, VIII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX, XXI]. An example of the metric dimension and dominant metric dimension is given in Fig 1.

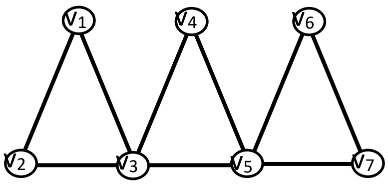

Fig 1. Δ_3-Snake Graph

The Δ_3 - snake graph is given in Fig 1. The set $W = \{v_1, v_6\}$ is a minimal resolving set but not a dominating set of Δ_3-snake graph since v_4 not adjacent to vertices in W. The set $\overline{W} = \{v_1, v_4, v_6\}$ is a minimal dominant resolving set of Δ_3-snake graph. Thus,
$$dim(\Delta_3 - snake\ graph) = 2 \text{ and } Ddim(\Delta_3 - snake\ graph) = 3.$$

II. Main Results

The exact values of resolving domination numbers of graphs are presented in this section.

Theorem 1. Let middle graph $M(G)$ with k blocks and n vertices, then $Ddim\ M(G) = \frac{n+1}{2}$ as shown in Fig 2.

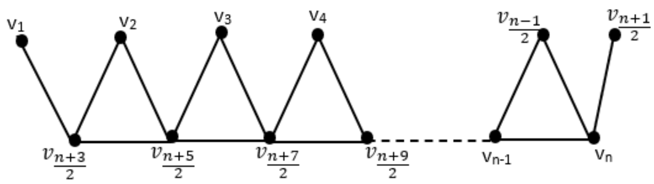

Fig 2. Middle Graph $M(G)$.

Proof. We label $M(G)$ as shown in Fig 1. It is clear that the number of vertices is $n = 2k + 3$ such that k is the number of blocks of $M(G)$. Let $W = \{v_1, v_2, \ldots, v_{\frac{n+1}{2}}\}$.

$$r(v_1|W) = \left(0, 2, 3,4, \dots, \frac{n-1}{2}, \frac{n+1}{2}\right)$$
$$r(v_2|W) = \left(2, 0, 2,3, \dots, \frac{n-3}{2}, \frac{n-1}{2}\right)$$
$$r(v_3|W) = \left(3, 2, 0,2, \dots, \frac{n-5}{2}, \frac{n-3}{2}\right)$$
$$\vdots$$
$$r\left(v_{\frac{n+1}{2}}|W\right) = (i, i-1, 2,0,2, \dots, 2,0)$$
$$r\left(v_{\frac{n+3}{2}}|W\right) = \left(1,1,2,3, \dots, \frac{n-3}{2}, \frac{n-1}{2}\right)$$
$$r\left(v_{\frac{n+5}{2}}|W\right) = \left(2,1,1,2, \dots, \frac{n-5}{2}, \frac{n-3}{2}\right)$$
$$\vdots$$
$$r(v_n|W) = \left(i - \frac{n+1}{2}, i - \frac{n+3}{2}, i - \frac{n+5}{2}, \dots, 2,1,1\right).$$

Theorem 2. Let G is the Tortoise graph T_n with k blocks and n vertices, then $Ddim\ (T_n) = \frac{n+1}{2}$ as shown in Fg 3.

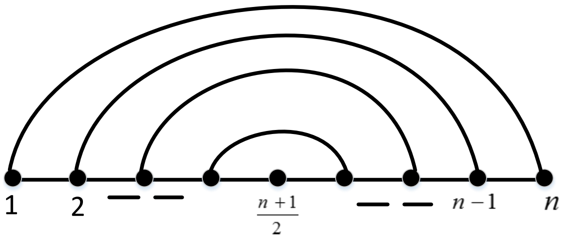

Fig 3. Tortoise Graph T_n.

Proof. We label T_n as shown in Fig 2. It is clear that the number of vertices is $n = 2k + 1$ such that k is the number of blocks of T_n. Let $W = \{v_1, v_2, \dots, v_{\frac{n+1}{2}}\}$.

$$r(v_1|W) = (0, 1, 2, \dots, k)$$
$$r(v_2|W) = (1,0, 1, 2, \dots, k-1)$$
$$r(v_3|W) = (2,1,0, 1, 2, \dots, k-2)$$
$$\vdots$$
$$r\left(v_{\frac{n-1}{2}}|W\right) = (i-1, i-2, \dots,1,0,1)$$
$$r\left(v_{\frac{n+1}{2}}|W\right) = (i-1, i-2, \dots,1,0)$$
$$r\left(v_{\frac{n+3}{2}}|W\right) = (i-2, i-3, \dots,1,1)$$
$$\vdots$$

$$r(v_{n-2}|W) = (3,2,1,2,\dots,i-k-1,i-k)$$
$$\vdots$$
$$r(v_n|W) = \left(n+1-i, n-i, n-1+i, \dots, \frac{n-1}{2}, \frac{n+1}{2}\right).$$

Theorem 3. Let G is a globe graph (Gl_n) with k blocks and n vertices, then $Ddim(Gl_n) = K+1 = n-2$, see Fig 4.

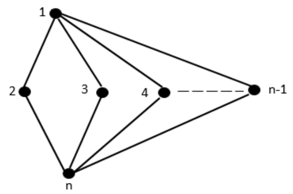

Fig 4. Globe Graph (Gl_n).

Proof. We choose a subset $W = \{v_1, v_2, \dots, v_{n-2}\}$, and we must show that $Ddim(Gl_n) = n-2$, for $n \geq 4$. We got the representations of vertices in graph Gl_n concerning W are

$$r(v_1|W) = (0,1,1,\dots,1,1)$$
$$r(v_2|W) = (1,0,2,\dots,2,2)$$
$$r(v_3|W) = (1,2,0,2,\dots,2,2)$$
$$\vdots$$
$$r\left(v_{\frac{n}{2}+1}|W\right) = (1,2,0,2,\dots,2,2)$$
$$r(v_{n-1}|W) = (1,2,\dots,2,2)$$
$$r(v_n|W) = (2,1,\dots,1,1)$$

From above, the representations of vertices in the graph Gl_n are distinct. This implies that W is the resolving set, but it is not necessarily the lower bound. Thus, the upper bound is $Ddim(Gl_n) \leq n-2$. For the Globe graph Gl_n there is no dominant resolving set that the cardinality is one. Thus, the lower bound is $dim(Gl_n) \geq n-2$. Obtained that $dim(Gl_n) \leq n-2$ and $dim(Gl_n) \geq n-2$, therefore we can say that $dim(Gl_n) = n-2$.

Theorem 4. If G is an open diagonal ladder graph $O(DL_n)$ of order $n \geq 6$, then $Ddim\ (O(DL_n)) = \frac{n}{2}$, see Fig 5.

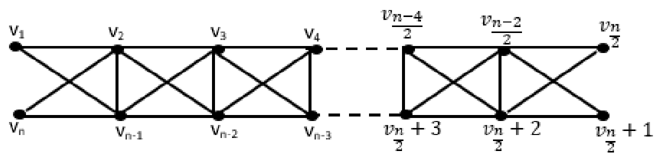

Fig. 5. Open Diagonal Ladder Graph $O(DL_n)$.

Proof. Consider the set $W = \left\{v_1, v_2, .., v_{\frac{n-2}{2}}, v_{\frac{n}{2}}\right\}$ be a minimum $Ddim(O(DL_n))$, $n \geq 6$, $n = 2k + 6$ and the representations of vertices $v_i \in V(O(DL_n))$ concerning W are as follows:

$$\boldsymbol{r}(v_i|W) = \begin{cases} (0,1,2,\dots,k+2), & i = 1 \\ (i-1, i-2, \dots,0,1,\dots,2k-i-2, 2k-i-1), & 2 \leq i \leq k+2 \\ (i-1, i-2, \dots,1,0), & i = k+3 \\ (i-2, i-3, \dots,1,2), & i = k+4 \\ (n-i, n-i-1, \dots,1,1,1,\dots, i-2k), & k+5 \leq i \leq n-1 \\ (2,1,2,3,\dots,k+2), & i = n \end{cases}$$

It is obvious that every vertex's representation with regard to W is unique, and it is demonstrated that

$$Ddim\big(O(DL_n)\big) = K + 3 = \frac{n}{2}.$$

Theorem 5. If G is a splitting graph of $K_{1,n}(Spl(k_{1,n}))$ of order $n \geq 6$, then $Ddim(Spl(k_{1,n})) = n - 2$, see Fig 6.

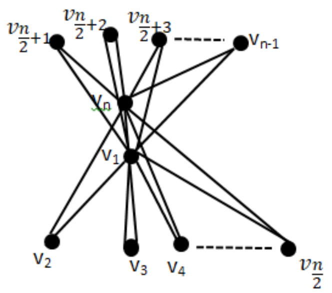

Fig 6. $Spl(k_{1,n})$ Graph.

Proof. Let the set $W = \{v_1, v_2, \dots, v_{n-3}, v_{n-2}\}$ be a minimum $Ddim(Spl(k_{1,n}))$, $n \geq 6$, and the representations of vertices $v_i \in V(Spl(k_{1,n}))$ concerning W are as follows:

$$r(v_1|\, W) = (0, 1,1, \dots, 1, 1)$$
$$r(v_2|W) = (1,0,2, \dots, 2, 2)$$
$$r(v_3|W) = (1,2,0,2, \dots, 2, 2)$$
$$r(v_4|W) = (1,2,2,0,2, \dots, 2, 2)$$
$$\vdots$$
$$r(v_{n-2}|W) = (1,2,2, \dots,2, 2,0)$$
$$r(v_{n-1}|W) = (1,2,2,2, \dots, 2, 2)$$
$$r(v_n|W) = (2,1,1, \dots,1, 1,1)$$

The representations of vertices in the splitting graph of $k_{1,n}$ are distinct, as seen above. This implies that W is the dominant resolving set, but it is not necessarily the lower bound. Thus, the upper bound is $Ddim(Spl(k_{1,n})) \leq n-2$. Now, we demonstrate that $Ddim(Spl(k_{1,n})) \geq n-2$. Let $W = \{v_1, v_2, .., v_{n-3}, v_{n-2}\}$ is a dominant resolving set which is $|W| = n-2$. Assume that W_1 is another minimum dominant resolving set or indicate $|W_1| < n-2$. If we choose an ordered set $W_1 \subseteq W - \{v_i\}$, for which i is odd and there are two vertices $v_i, v_{i+1} \in Spl(k_{1,n})$ such that $r(v_i|W) = r(v_{i+1}|W) = (1,2,2,2,\ldots,2,2)$. Also, W_1 is not a dominant resolving set, a contradiction with assumption. Thus the lower bound is $Ddim(Spl(k_{1,n})) \geq n-2$. From the above proof, we conclude that $Ddim(Spl(k_{1,n})) = n-2$.

Theorem 6. If G is a book graph of B_n of order $n \geq 6$, then $Ddim(B_n) = \frac{n}{2}$, see Fig 7.

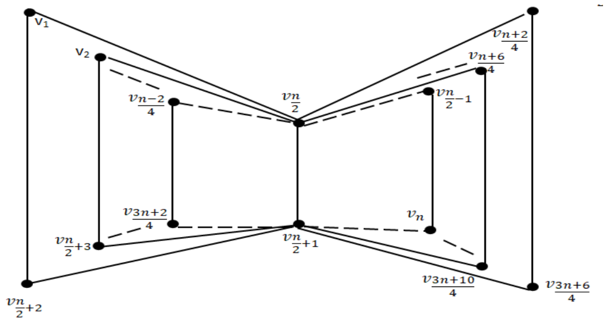

Fig. 7. Book Graph of B_n.

Proof. Consider the set $W = \left\{v_1, v_2, .., v_{\frac{n}{2}},\right\}$ be a minimum $Ddim(B_n)$, $n \geq 6$, and the representations of vertices $v_i \in V(B_n)$ with respect to W are as follows:

$$r(v_1|W) = (0,2,2,\ldots,2,1)$$
$$r(v_2|W) = (2,0,2,\ldots,2,1)$$
$$r(v_3|W) = (2,2,0,2,\ldots,2,1)$$
$$r(v_4|W) = (2,2,2,0,2,\ldots,2,1)$$
$$\vdots$$
$$r\left(v_{\frac{n-2}{2}}|W\right) = (2,2,2,\ldots,2,2,0,1)$$
$$r\left(v_{\frac{n}{2}}|W\right) = (1,1,\ldots,1,1,0)$$
$$r\left(v_{\frac{n+2}{2}}|W\right) = (2,2,2,\ldots,2,2,1)$$
$$r\left(v_{\frac{n+4}{2}}|W\right) = (1,3,3,\ldots,3,3,2)$$
$$r\left(v_{\frac{n+6}{2}}|W\right) = (3,1,3,\ldots,3,3,2)$$
$$r\left(v_{\frac{n+8}{2}}|W\right) = (3,3,1,3,\ldots,3,3,2)$$
$$\vdots$$
$$r(v_n|W) = (3,3,3,\ldots,3,1,2)$$

The representations of vertices in the book graph of B_n are distinct, as seen above. This implies that W is the dominant resolving set, but it is not necessarily the lower bound. Thus, the upper bound is $Ddim(B_n) \leq \frac{n}{2}$. Now, we demonstrate that $Ddim\,(B_n) \geq \frac{n}{2}$. Let

$$W = \left\{v_1, v_2, \ldots, v_{\frac{n}{2}}\right\}$$

be a dominant resolving set which is $|W| = \frac{n}{2}$. Assume that W_1 is another minimum dominant resolving set or indicate $|W_1| < \frac{n}{2}$. If we choose an ordered set $W_1 \subseteq W - \{v_i\}$, i is odd, so that there are two vertices $v_i, v_{i+1} \in (B_n)$ such that

$$r(v_i|W) = r(v_{i+1}|W) = (2, 2, 2, \ldots, 2,1).$$

Note that W_1 is not dominant resolving set, a contradiction with assumption. Thus the lower bound is $Ddim\,(B_n) \geq \frac{n}{2}$. From the above proving, we conclude that $Ddim(B_n) = \frac{n}{2}$.

Theorem 7. If G is $C_4@W_n$ graph of order $n \geq 2$, then $Ddim(C_4@W_n) = n - 4$, see Fig 8.

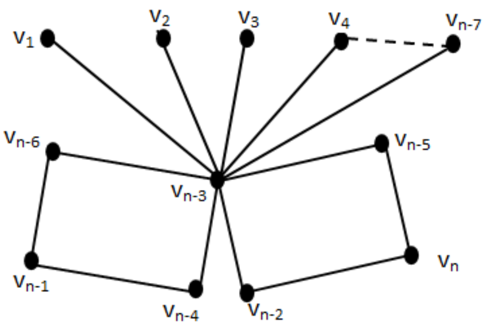

Fig 8: C_4 @ W_n graph**.**

Proof. Consider the set $W = \{v_1, v_2, .., v_{n-4}\}$ is a minimum $Ddim(C_4@W_n)$, $n \geq 2$, and the representations of vertices $v_i \in V(C_4@W_n)$ concerning W are as follows:

$$r(v_1|W) = (0, 2,2, \ldots, 2)$$
$$r(v_2|W) = (2,0,2, \ldots, 2)$$
$$r(v_3|W) = (2,2,0,2, \ldots, 2)$$
$$r(v_4|W) = (2,2,2,0,2, \ldots, 2)$$
$$\vdots$$
$$r(v_{n-4}|W) = (2,2,2, \ldots,2, 2,0)$$
$$r(v_{n-3}|W) = (1,1, \ldots,1,1)$$
$$r(v_{n-2}|W) = (2,2,2, \ldots,2, 2)$$
$$r(v_{n-1}|W) = (3,3,3, \ldots,3, 1,3,1)$$
$$r(v_n|W) = (3,3,3, \ldots,3, 1,3)$$

The representations of vertices in the $C_4@W_n$ are distinct, as seen above. This implies that W is the dominant resolving set, but it is not necessarily the lower bound. Thus, the upper bound is $Ddim(C_4@W_n) \leq n-4$. Now, we demonstrate that $Ddim(C_4@W_n) \geq n-4$. Let $W = \{v_1, v_2, .., v_{n-4}\}$ is a dominant resolving set which is $|W| = n-4$. Assume that W_1 is another minimum dominant resolving set or indicate $|W_1| < n-4$. If we choose an ordered set $W_1 \subseteq W - \{v_\text{i}\}$, i is odd, so that there are two vertices $v_\text{i}, v_{\text{i}+1} \in (C_4@W_n)$ such that

$$r(v_\text{i}|W) = r(v_{\text{i}+1}|W) = (2,2,2,\ldots,2).$$

Note that W_1 is not a dominant resolving set, a contradiction with assumption. Thus the lower bound is $Ddim(C_4@W_n) \geq n-4$. From the above proof, we conclude that $Ddim(C_4@W_n) = n-4$.

V. Conclusion

In this paper, we have studied the dominant metric dimension of the middle graph, tortoise graph, globe graph open diagonal ladder graph, and splitting graph of $K_{1,\text{n}}$. Based on our results, we have obtained

$$dim\big(M(G)\big) = 2, \qquad Ddim\big(M(G)\big) = \frac{n+1}{2}, \qquad dim(T_n) = 2,$$
$$Ddim(T_n) = \frac{n+1}{2}, \qquad dim(Gl_n) = n-2, \qquad Ddim(Gl_n) = n-2,$$
$$dim\big(O(DL_n)\big) = \frac{n}{2}, \qquad Ddim\big(O(DL_n)\big) = \frac{n}{2}, \qquad dim\left(Spl\big(K_{1,\text{n}}\big)\right) = n-3,$$
$$Ddim\left(Spl\big(K_{1,\text{n}}\big)\right) = n-2, \qquad Ddim(B_n) = \frac{n}{2}, \qquad Ddim(C_4@W_n) = n-4.$$

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Conflicts of interest

All authors declare that they have no conflicts of interest.

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