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ANALYSIS OF SH-WAVES IN ANISOTROPIC FIBER-REINFORCED MEDIUM OVER LINEARLY VARYING INHOMOGENEOUS SUBSTRATE UNDER NON-LOCAL ELASTICITY

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Abstract

An analytical technique employing the variable separable method has been used in an attempt to precisely grasp and assess the impact of the property non-local elasticity upon the wave transmission response of an anisotropic fiber-reinforced material embedded over a semi-infinite, inhomogeneous medium that changes linearly. As depth increases, the stiffness and density of a semi-infinite substrate are believed to alter linearly. Availing the Whittaker function, a relation about dispersion has been acquired to analyze the response of SH waves. The visual representation depicts a significant influence of non-local elasticity on the propagation of SH-wave modes. Special cases have been found assessing the concurrency of the model with the original form equation of Love wave. The effects of non-locality, fiber-reinforcement parameters, and inhomogeneity parameters carry implications in designing and gradation of material characteristics some important parameters on the wave characteristics of the studied model.

Keywords: Fiber-reinforced, Inhomogeneous; Love wave, Non-local elasticity.

I. Introduction

Seismology, a foundational domain within scientific inquiry, diligently probes the depths of seismic activity, offering profound insights into earthquakes and the intricate movements of seismic waves coursing through the Earth. Through meticulous analysis of seismograms, which meticulously record the Earth's mechanical vibrations, seismologists unravel the complex tapestry of seismic events, illuminating the Earth's structure and seismic behavior.

Seismologists, the vanguards of this discipline, employ sophisticated instruments such as seismometers to discern the arrival and characteristics of seismic waves. Beyond theoretical exploration, seismology finds practical utility across diverse domains. In Civil Engineering, it informs the design of earthquake-resistant infrastructure, while Rock Mechanics relies on seismic insights for subsurface exploration. Furthermore, Geophysical Prospecting benefits from seismic principles to identify subterranean structures and resources.

In essence, seismology serves as a beacon of understanding, offering invaluable knowledge to comprehend and mitigate the impacts of seismic hazards, fostering resilience and informed decision-making in the face of Earth's dynamic forces. The inner structure of the earth is assumed as onion which is assumed to have layered structure in spherical shells of certain thickness and carries several types of properties such as physical, chemical, and mechanical properties. Earth contains four main layers, such as the crust, mantle, outer core, and inner core. The crust exhibits significantly more heterogeneity compared to other mediums. Earthquakes have mostly occurred in the outer layer i.e., crust of the Earth. For understanding the physical damage caused by earthquakes, knowledge of seismic waves is very useful. Sudden displacement along a fault will emerge different types of waves. These waves can travel through the inner surface of the earth as well as along its surface. As waves propagate from their source to the Earth's surface, they traverse through a diverse array of geological layers, each characterized by unique compositions and properties. These waves act as probes, penetrating through solid rock, liquid, and even molten materials, providing valuable insights into the intricate configuration of the Earth's interior. By examining the features and actions of these waves about various geological layers, researchers can extract vital insights into the makeup, density, and arrangement of subterranean formations. Understanding the interior structure of the earth as well as the precious materials—such as metals, hydrocarbons, minerals, water, and petroleum—that are buried under the surface is made possible by this knowledge.

An innovative idea was first presented in 1911 by a mathematician of British origin named Augustus Edward Hough Love: the notion of a surface acoustic wave, or "Love-wave [XXVI]. Love waves can be identified from more conventional seismic waves like P-, S-, and Rayleigh waves by their distinct way of propagating along a medium's surface. Love waves, in contrast to their counterparts, are unable to move through a homogeneous half-space.

Love waves are known for their speed as they race across the ground, causing horizontal motion in a plane that is perpendicular to their direction of travel but parallel to the Earth's surface. Many scientists throughout the world have been fascinated by this remarkable phenomenon, which has sparked in-depth research into its characteristics and behavior [IX, XIV, XVIII, XXV, XXVII, XXXI-XXXIII, XXXV, XXXVIII]. The investigation of material inhomogeneity and its effect on the propagation of Love waves is very important. Researchers have discovered the complex interaction between wave properties and the varied structure of geological substrates through painstaking investigation and analysis. These discoveries not only broaden our knowledge of Love waves but also have applications in a variety of fields, including engineering and geoscience. Science opens the door to better earthquake detection, greater seismic monitoring, and more resilient engineering solutions by clarifying the intricacies of Love wave propagation and its sensitivity to environmental conditions. As a result, Augustus Love's ground-breaking work continues to inspire research and development in the field of surface wave phenomena.

The spread of love waves through various mediums has been considered one of the interesting subjects of numerous research investigations. The dispersion equation for Love waves was formulated by Chattopadhyay [XI], who made a substantial contribution. This formula provides important information about the behavior of Love waves as they proceed through heterogeneous geological structures by illuminating the dissimilarity in the depth of the non-homogeneous crustal layer. The movement of Love waves in the ground that had been moistened with water and encircled by an elastic media with heterogeneous properties was identified by Chakraborty and Dey [XII]. Dey et al. [XV], and Gupta S. et al. [XXI] established the idea of Love wave propagation in a heterogeneous crust over a heterogeneous mantle. Dey et al. [XVI] investigated the propagation of Love waves in an elastic material having void pores. Abd-Alla and Ahmed [I] also looked upon the Love waves propagation in a non-homogeneous orthotropic elastic layer beneath initial tension covering semi-infinite media. Understanding how anisotropy impacts Love waves in a self-reinforced material was provided by Pradhan et al. [XVII]. Their work clarifies the complex interactions that occur in such contexts between material properties and wave propagation. In the same way, Abd-Alla and Abo-Dahab [II] examined Rayleigh waves through a magneto thermal-visco elastic solid while taking thermal relaxation durations into account. Their research advances our knowledge of wave behavior in complicated materials under a range of environmental circumstances. Kalyani et al. have finished modelling seismic wave propagation in the monoclinic medium using finite differences [XXII]. Ahmed and Abo-Dahab [III] investigated the propagation of Love waves in an orthotropic granular layer under initial stress covering a semi-infinite granular medium. The research of fiber-reinforced layers is discovered to be extremely important in light of the issues with elastic stability for anisotropic media. When additional features are introduced, a medium may exhibit an induced anisotropy for small deformations in the vicinity of the initial stress state, even if it may be isotropic for small deformations or naturally anisotropic.

Considering the phenomenon of internal instability, additional research is required to examine the issues with anisotropic media in contrast to isotropic media. The influence of reinforcement on surface wave propagation in semi-infinite media has been the subject of investigation by several researchers, including Ahmed. S. M et. Al. [VIII], Manna et al. [XXX], Gupta and Ahmed [XX], Kundu [XXIII], Chattaraj and Samal [XIII], Pradhan et al. [XXXIV] and Kundu et al. Singh et al. [XXIV] also investigated the effects of homogeneous reinforcement and corrugated boundary surfaces on the dynamics wave propagation of Love-type. Their combined efforts have aided in the advancement of disciplines such as structural engineering and seismology by providing a greater understanding of the ways in which different factors, including material properties and boundary conditions, interact to shape the behavior of surface waves in diverse environments.

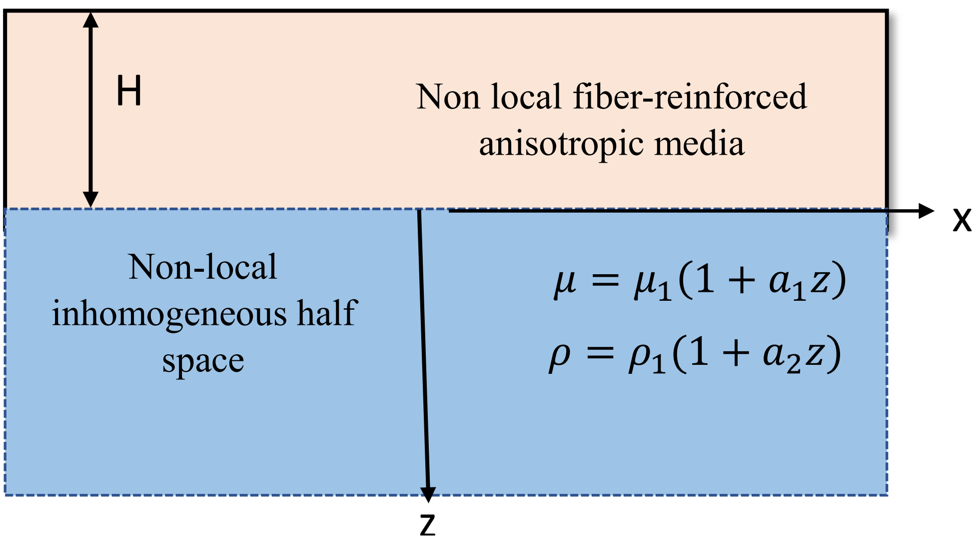
The fundamental idea of classical (local) continuum mechanics is to isolate the effect of nearby strain fields on a specific reference point. This idea is fundamental to comprehending how materials behave in many circumstances. But some events, referred to as neighbouring effects or non-local effects, go beyond the boundaries of this traditional paradigm. Non-local elasticity theory or non-local continuum theory is used to handle these nuances. These ideas offer an expanded viewpoint, taking into account the impact of strain that is dispersed throughout the medium as opposed to concentrating only on the strain in one particular area. Non-local elasticity theory essentially states that the stress felt at any given location inside the medium is governed by both the localized strain at that location and the collective strain dispersed throughout the entire material. This comprehensive method improves our comprehension of material behavior and enables more precise predictions in various scientific and engineering applications.

By ignoring strain effects that extend beyond a certain point in the medium, classical theory simplifies. On the other hand, non-local elasticity theory provides a more thorough method that takes into account the impact of strain that is dispersed throughout the material [XIV]. This more comprehensive viewpoint is especially helpful in engineering applications where precise modelling of the bending, buckling, and vibration behaviours of materials is necessary to design structures and systems that adhere to performance and safety requirements. This technique has reportedly been used for problems involving fracture mechanics, wave dispersion, dislocation, and other related concerns, according to Karlicic et al. and Eringen and Wegner [VII]. Because it can conceptually bridge the gap between the atomic theory of lattice dynamics and classical continuum theory, the elasticity theory of non-local has become an attractive framework. Scholars looking to further their knowledge of material behavior are very interested in this distinctive property. The theory's importance in mechanics has been widely explored since it was first put forth and clarified by Edelen et al. [V] and Eringen and Edelen [IV]. Eringen has added to our understanding of the theory's applications and consequences. Eringen [VI] has demonstrated that the use of non-local elasticity has several benefits, especially in resolving singularity problems that arise in classical theory when solving screw dislocation problems. In contrast to traditional methods, non-local elasticity offers a continuum mechanics framework that connects macroscopic and microscopic processes. This primary goal emphasizes how important it is to understand the intricate interactions between material behavior at various scales. Moreover, non-local elasticity is helpful in solving problems related to elastic wave propagation, providing insights into wave dynamics that are difficult for classical theories to explain. Its strong theoretical basis—first developed by Eringen and then later developed by other researchers—upholds its legitimacy and lends itself to a variety of scientific and engineering settings. Accepting non-local elasticity allows scientists to investigate a wider range of material behaviours, which promotes progress in a variety of disciplines, including materials science and solid mechanics. Using the novel approach in the presence of non-local elasticity might raise questions about the existence and uniqueness of solutions. The boundary value problem of Altan [XXXVI, XXXVII] helps to make these issues more understandable. Furthermore, Narasimhan and McCay [XXIX] explore surface wave dispersion analysis in the context of non-local elasticity, which advances our knowledge of wave behavior in materials with non-local interactions.

The dispersive characteristics of Love waves type propagating through a complex medium consisting of an anisotropic, non-local, fiber-reinforced layer on top of an inhomogeneous half-space is examined in this work. The stiffness and density of the semi-infinite medium vary quadratically in the z-direction, indicating its inhomogeneity. The displacement components of the media are computed independently for the layer and the semi-infinite medium in order to examine the propagation of Love-type waves in this layered system. The displacement component of Love-type waves is derived in the study by utilizing the Bessel differential equation and taking into account their interaction with the structural complexity of the medium. Furthermore, usual boundary conditions are set in order to guarantee the precision and suitability of the model, enabling a thorough comprehension of Love-type wave propagation in a range of material configurations with consequences for different geophysical and engineering uses.

II. Statement of the problem

In this study, we took up the Cartesian coordinate system (x, y, z) with the origin O placed precisely at the interface between the uppermost layer and the lowermost half-space, facilitating an in-depth examination of Love-like wave propagation across both medium at the confluence of z = 0. The axis is strategically positioned at the intersection of the layer and half-space, aligning with the direction of propagation of the Love-type wave, while the z-axis extends positively downwards, as illustrated in Figure 1. The uppermost medium is characterized by a non-local fiber-reinforced anisotropic structure with uniform thickness H, while the lower layer comprises a non-local inhomogeneous half-space, introducing complexity to the wave propagation dynamics under investigation.



III. Governing Equation and its Solution of the Upper Layer

The reinforcement direction $\vec{a} = (a_1, a_2, a_3)$is considered in the upper part of the model, where $a_1^2 + a_2^2 + a_3^2 = 1$. The vector $\vec{a}$ may be a function of position. The principal equation for a non-local elastic anisotropic medium reinforced with fibers is provided by Spencer and Eringen, formulated as follows:

$$(1-\varepsilon_1^2\nabla^2)\tau_{ij} = \lambda e_{kk}\delta_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + a_i a_j e_{kk}) + 2(\mu_L - \mu_T)(a_k a_i e_{kj} + a_k a_j e_{ki}) + \beta(a_k a_m e_{km} a_i a_j), i, j, k, m = 1,2,3 \quad (1)$$

Where τ_{ij} are the stress components, δ_{ij} are the Kronecker delta and

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2)$$

The parts of infinitesimal strain are indicated with indices ranging from 1 to 3, adhering to the summation convention for notation. Also α, β are anisotropic reinforced parameters, μ_L is shear modulus in longitudinal shear moduli and μ_T can be picked out as the shear moduli in transverse shear in the desired direction $\vec{a}$ respectively. In this scenario, λ denotes Lame's constant for elastic solid. The components of displacement of Love-type waves within the media are represented by u_i. The Laplacian operator ∇^2 is utilized, with the comma indicating partial differentiation.

Within this paper, our focus lies on the propagation of Love waves along the x-direction. Specifically, we have examined the components of the upper fiber-reinforced layer's mechanical displacement vector denoted as (u_1, v_1, w_1) relative to the x, y, and z dimensions.

In his work, Biot [X] has delineated the equation of motion in Cartesian coordinates, elucidating the propagation of surface waves in various directions, particularly in the non-appearance of body forces as,

$$\left.\begin{aligned}
\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} &= \rho\frac{\partial^2 u_1}{\partial t^2}\\
\frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} &= \rho\frac{\partial^2 v_1}{\partial t^2}\\
\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z} &= \rho\frac{\partial^2 w_1}{\partial t^2}
\end{aligned}\right\} \quad (3)$$

As the particle's motion is perpendicular to the wave direction in this instance and the Love waves are generated along the x-direction, the displacement components of the vectors in the layer take the following form:

$$u_1 = 0 = w_1 \text{ and } v_1 = v_1(x, z, t) \quad (4)$$

Using Eq.(4) into Eq.(3) we have,

$$\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=\rho \frac{\partial^2 v_1}{\partial t^2} \tag{5}$$

In our scenario, given that the uppermost layer under consideration is transversely isotropic, the directional unit vector will adopt the following format:

$(a_1, 0, a_3)$ with $a_1^2 + a_3^2 = 1$

Now using the above-mentioned criteria in Eq. (1) and taking help from both Eq. (2) and (4), for the propagation of Love waves, the stress components in the fiber-reinforced medium are

$$\left.\begin{aligned}
&(1-\varepsilon_1^2\nabla^2)\tau_{xx}=0\\
&(1-\varepsilon_1^2\nabla^2)\tau_{yx}=\mu_T\left(P\frac{\partial v_1}{\partial x}+Q\frac{\partial v_1}{\partial z}\right)\\
&(1-\varepsilon_1^2\nabla^2)\tau_{yy}=0\\
&(1-\varepsilon_1^2\nabla^2)\tau_{yz}=\mu_T\left(R\frac{\partial v_1}{\partial z}+Q\frac{\partial v_1}{\partial x}\right)\\
&(1-\varepsilon_1^2\nabla^2)\tau_{zz}=0\\
&(1-\varepsilon_1^2\nabla^2)\tau_{zx}=0
\end{aligned}\right\} \tag{6}$$

Where $P=1+\left(\frac{\mu_L}{\mu_T}-1\right)a_1^2, Q=\left(\frac{\mu_L}{\mu_T}-1\right)a_1a_3, R=1+\left(\frac{\mu_L}{\mu_T}-1\right)a_3^2$

After applying Equation (6) to Equation (5) and simplifying, the equation of motion transforms into:

$$P\frac{\partial^2 v_1}{\partial x^2}+2Q\frac{\partial^2 v_1}{\partial z\partial x}+R\frac{\partial^2 v_1}{\partial z^2}=\frac{\rho(1-\varepsilon_1^2\nabla^2)}{\mu_T}\frac{\partial^2 v_1}{\partial t^2} \tag{7}$$

Let us consider the solution of Eq.(7) as

$$v_1(x,z,t)=V_1 e^{ik(x-ct)} \tag{8}$$

Where c is the phase velocity and k is the wave number of the propagating harmonic wave semi-infinite medium, respectively.

Using Eq.(8), Eq.(7) takes the form

$$\left[R-\frac{c^2}{c_T^2}(\varepsilon_1^2k^2)\right]\frac{d^2V_1}{dz^2}+2Qik\frac{dV_1}{dz}+k^2\left[\frac{c^2}{c_T^2}(1+\varepsilon_1^2k^2)-P\right]V_1=0 \tag{9}$$

Where $c_T = \sqrt{\frac{\mu_T}{\rho}}$ and $\varepsilon_1 k, P, Q, R$ are all quantities with zero dimension.

Therefore the above solution of Eq.(9) is written as,

$$V_1(z) = A_1 e^{-ik\lambda_1 z} + B_1 e^{-ik\lambda_2 z} \tag{10}$$

Where λ_1 and λ_2 are respectively,

$$\lambda_1 = \frac{Q + \sqrt{Q^2 + (R - \frac{c^2}{c_T^2}(\varepsilon_1^2 k^2))(\frac{c^2}{c_T^2}(1+\varepsilon_1^2 k^2) - P)}}{(R - \frac{c^2}{c_T^2}(\varepsilon_1^2 k^2)}$$

$$\lambda_2 = \frac{Q - \sqrt{Q^2 + (R - \frac{c^2}{c_T^2}(\varepsilon_1^2 k^2))(\frac{c^2}{c_T^2}(1+\varepsilon_1^2 k^2) - P)}}{(R - \frac{c^2}{c_T^2}(\varepsilon_1^2 k^2)}$$

So, Eq.(8) finally can be written as,

$$v_1(x,z,t) = (A_1 e^{-ik\lambda_1 z} + B_1 e^{-ik\lambda_2 z}) e^{ik(x-ct)} \tag{11}$$

Equation (11) is the required displacement equation that is fundamental to Love-type wave propagation propagating in a fiber-reinforced non-local transversely isotropic layer, with layer thickness represented by H.

IV. Governing Equation and Solution for Inhomogeneous Half-Space

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = \frac{\partial^2}{\partial t^2}(\rho v_2) \tag{12}$$

where u_2, v_2, w_2 are the displacements along the x-, y-, and z- axis respectively. In this scenario, horizontally polarized surface waves propagate along the x-direction. Consequently, the displacement components of vectors within the layer assume the following form:

$u_2 = 0 = w_2$ and $v_2 = v_2(x,z,t)$

where τ_{ij} are the stress components in the half-space and ρ is the density of the material of the half-space.

Now, the inhomogeneity in medium is taken as,

$$\mu = \mu_1(1+a_1 z) \text{ and } \rho = \rho_1(1+a_2 z) \tag{13}$$

Where μ_1 and ρ_1 are the values of μ and ρ at $z = 0$ and a_1, a_2 are inhomogeneous parameters. Given that Love waves propagate along the x-direction while particle displacement manifests along the y-direction, only the e_{12} e_{23} shear strain components remain significant, rendering all other strain components null.
Hence the stress-strain relationship gives

$$\tau_{yx} = 2\mu_1(1+a_1 z)e_{xy} \text{ and } \tau_{yz} = 2\mu_1(1+a_1 z)e_{yz} \tag{14}$$

Where $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

Therefore $e_{xy} = \frac{1}{2}\frac{\partial v_2}{\partial x}$ and $e_{yz} = \frac{1}{2}\frac{\partial v_2}{\partial z}$

By employing Eq. (13) and Eq. (14), we can express the equation of motion from Eq. (12) as follows:

$$\frac{\partial}{\partial x}\left[\mu_1(1+a_1 z)\frac{\partial v_2}{\partial x}\right] + \frac{\partial}{\partial z}\left[\mu_1(1+a_1 z)\frac{\partial v_2}{\partial z}\right] = \rho_1(1+a_2 z)\frac{\partial^2 v_2}{\partial t^2}$$

which can be simplified as,

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + \left(\frac{a_1}{1+a_1 z}\right)\frac{\partial v_2}{\partial z} = \frac{\rho_1(1+a_2 z)}{\mu_1(1+a_1 z)}\left(\frac{\partial^2 v_2}{\partial t^2}\right) \tag{15}$$

Let's contemplate the solution of Eq.(15) as

$$v_2(x,z,t) = V_2 e^{ik(x-ct)}$$

Then Eq.(15) can be transformed as

$$\frac{d^2 V}{dz^2} + \left(\frac{a_1}{1+a_1 z}\right)\frac{dV}{dz} + \left[\frac{\rho_1(1+a_2 z)}{\mu_1(1+a_1 z)}\omega^2 - k^2\right]V = 0 \tag{16}$$

Now, let $V(z) = \frac{\phi(z)}{\sqrt{(1+a_1 z)}}$ to eliminate the term $\frac{dV}{dz}$, we get

$$\phi''(z) + \left[\frac{a_1^2}{4(1+a_1 z)^2} - k^2\left\{1 - \frac{c^2(1+a_2 z)}{c_1^2(1+a_1 z)}\right\}\right]\phi(z) = 0 \tag{17}$$

Where $c_1 = \sqrt{\frac{\mu_1}{\rho_1}}$,

we put

$$\gamma_1 = \sqrt{[1 - \frac{c^2 a_2}{c_1^2 a_1}]}$$

and $\eta = \frac{2\gamma_1 k(1+a_1 z)}{a_1}$, $\omega = kc$

in Eq.(17) and we have,

$$\frac{d^2\phi}{d\eta^2} + \left[\frac{R}{2\eta} + \frac{1}{4\eta^2} - \frac{1}{4}\right]\phi(\eta) = 0 \tag{18}$$

Where

$$R = \frac{\omega^2(a_1 - a_2)}{c_1^2 a_1^2 \gamma_1 k}$$

$$\phi(\eta) = C_1 W_{\frac{R}{2},0}(\eta) + C_2 W_{-\frac{R}{2},0}(-\eta)$$

The conclusion to Whittaker's equation (18) is given by,

$$\phi(\eta) = C_1 W_{\frac{R}{2},0}(\eta) + C_2 W_{-\frac{R}{2},0}(-\eta)$$

Where C_1 and C_2 are arbitrary constants and $W_{\frac{R}{2},0}(\eta)$ is the Whittaker's function.

The solution of Eq. (18) satisfies the condition $\lim z \to \infty$ when $V(z) \to 0$ i.e., $\lim \eta \to \infty$ when $\phi(\eta) \to 0$

Therefore, the solution takes the form,

$$\phi(\eta) = C_1 W_{\frac{R}{2},0}(\eta) \tag{19}$$

Hence, the displacement component v_2 in the lower inhomogeneous medium is given by

$$v_2(x,z,t) = V_2 e^{ik(x-ct)}$$

$$= \frac{C_1 W_{\frac{R}{2},0}(\eta)}{\sqrt{1+a_1 z}} e^{ik(x-ct)} \tag{20}$$

Enlarge Whittaker's function to linear terms, Eq.(20) reduces to

$$v_2(x,z,t) = C_1 e^{-\frac{\gamma_1 k(1+a_1 z)}{a_1}} \left\{\sqrt{\frac{2\gamma_1 k}{a_1}}\right\}.\left[1+(1-R)\frac{\gamma_1 k}{a_1}(1+a_1 z)\right] e^{ik(x-t)} \quad (21)$$

V. Boundary conditions:

1. The top surface of the fiber-reinforced layer is considered to be devoid of mechanical traction., i.e.

$$(1-\varepsilon_1^2\nabla^2)\tau_{yz} = 0 \text{ at } z = -H \quad (22)$$

2. $v_1(x,z,t) = v_2(x,z,t)$ at $z = 0$ (23)

3. $(1-\varepsilon_1^2\nabla^2)\tau_{yz} = 2\mu_1(1+a_1 z)e_{23}$ at $z = 0$ (24)

Using the above boundary conditions, we get the relation as follows,

$$\left.\begin{aligned} A_1 e^{-ik\lambda_1 H}(Q-\lambda_1 R) + B_1 e^{-ik\lambda_2 H}(Q-\lambda_2 R) = 0 \\ A_1 + B_1 - C_1 e^{\frac{\gamma_1 k}{a_1}}.\sqrt{\frac{2\gamma_1 k}{a_1}}[1+(1-R)\frac{\gamma_1 k}{a_1}] = 0 \\ A_1(Q-\lambda_1 R) + B_1(Q-\lambda_2 R) - \frac{\mu_1 C_1}{\mu_T ik}\sqrt{\frac{2\gamma_1 k}{a_1}}[e^{-\frac{\gamma_1 k}{a_1}}.\gamma_1 k\{(-a_1) \\ \{1+\frac{(1-R)\gamma_1 k}{a_1}\} + e^{(1+a_1 z)}(1-R)\}] = 0 \end{aligned}\right\} \quad (25)$$

Equation (25) outlines a series of homogeneous equations. To find a meaningful solution, it's essential for the determinant of the coefficient matrix pertaining to the constants A_1, B_1, C_1 to be set to zero. Thus, we proceed to derive the indispensable dispersive equation that governs the propagation of Love waves in a non-local elastic medium fiber-reinforced, situated over an inhomogeneous half-space.

VI. Special cases to validate the model:

Case I

Take into account the scenario in which the uppermost layer, reinforcement is negligible, i.e., $a_1 = 1, a_2 = a_3 = 0$. Then $P = \frac{\mu_L}{\mu_T}, Q = 0$ and $R = 1$

Then from Equation (25), we can write the determinant as,

$$\begin{vmatrix} -\lambda_1 e^{-ik\lambda_1 H} & -\lambda_2 e^{-ik\lambda_2 H} & 0 \\ 1 & 1 & -L \\ -\lambda_1 & -\lambda_2 & -M \end{vmatrix} = 0$$

Where,

$$L = e^{\frac{\gamma_1 k}{a_1}} \sqrt{\frac{2\gamma_1 k}{a_1}}$$

$$M = \frac{\mu_1}{\mu_T ik} \sqrt{\frac{2\gamma_1 k}{a_1}} [e^{-\frac{\gamma_1 k}{a_1}} (\gamma_1 k)(-a_1)]$$

Solving we get,

$$\tan(bkH) = \frac{i\{M(\lambda_2 - \lambda_1)}{M(\lambda_1 + \lambda_2) + 2L\lambda_2\lambda_1}$$

Where,

$$b = \frac{\sqrt{Q^2 + (R - \frac{c^2}{c_T^2}(\varepsilon_1^2 k^2))(\frac{c^2}{c_T^2}(1 + \varepsilon_1^2 k^2) - P)}}{(R - \frac{c^2}{c_T^2}(\varepsilon_1^2 k^2)}$$

Case II

In this case, we have considered $\varepsilon_1 = 0$ i.e., upper medium is local elastic. Subsequently, the dispersion equation governing the propagation of Love-type waves within a fiber-reinforced medium over an inhomogeneous half-space is formulated as follows:

$$tan(bkH) = \frac{-iMR(\lambda_1 + \lambda_2)}{(LQ - LR\lambda_1 - M)(Q - \lambda_2 R) + (LQ - LR\lambda_1 - M)(Q - \lambda_1 R)}$$

Where,

$$L = e^{\frac{\gamma_1 k}{a_1}} \sqrt{\frac{2\gamma_1 k}{a_1}} [1 + (1 - R)\frac{\gamma_1 k}{a_1}]$$

$$M = \frac{\mu_1}{\mu_T ik} \sqrt{\frac{2\gamma_1 k}{a_1}} [e^{-\frac{\gamma_1 k}{a_1}} (\gamma_1 k)\{(-a_1)\{1 + \frac{(1-R)\gamma_1 k}{a_1}\} + e^{1 + a_1 z}(1 - R)\}]$$

Case III

In this case, when the upper layer and lower are homogeneous i.e.,

$$a_1 = 1, a_2 = a_3 = 0, a = 0\,,$$

$$P = \frac{\mu_L}{\mu_T}, Q = 0, R = 1$$

Then we have,

$$\tan[kH(\sqrt{\frac{c^2}{c_T^2}}-1)]=\frac{\mu_1}{\mu_T}\frac{\sqrt{1-\frac{c^2}{c_T^2}}}{\sqrt{\frac{c^2}{c_T^2}-1}}$$

VII. Numerical calculation and graphical discussion:

This study utilizes MATHEMATICA software tools for numerical analysis, specifically focusing on exploring Love wave propagation within a fiber-reinforced layer with non-local properties situated above an inhomogeneous semi-infinite substrate with non-local properties. The upper layer exhibits anisotropic properties defined by its distinctive mechanical characteristics. (cf. Markham [XXVIII]) are given as: $\mu_T = 2.46\times10^9\, Nm^{-2}$, $\mu_T = 5.66\times10^9\, Nm^{-2}$ and $\rho = 7.5\times10^3 kgm^{-3}$. Furthermore, the lower layer consists of a non-local inhomogeneous half space, and its mechanical parameters, as detailed by Gubbins [XIX], are provided as follows: $\mu_1 = 6.34\times10^{10}\, Nm^{-2}$ and $\rho_1 = 3364\, kgm^{-3}$.

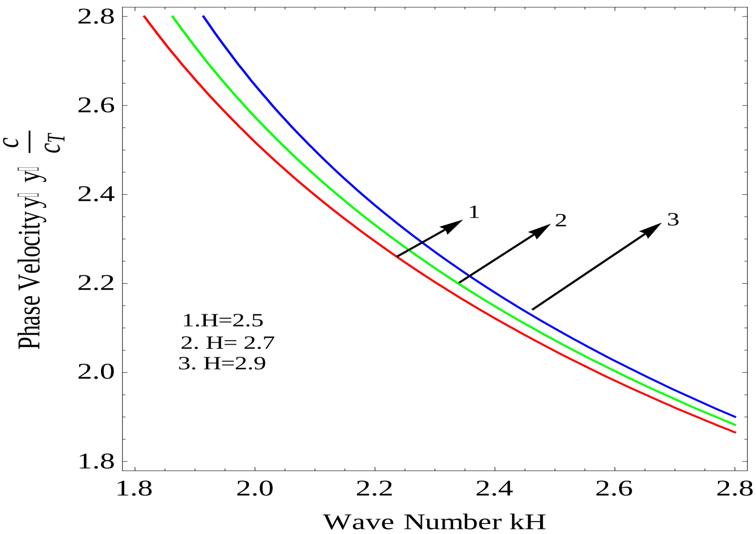


Fig. 2. Illustrates the connection between phase velocity and wave number for varying depths of the layer, with all other parameters held constant. Specifically, the values of H are set at 2.5, 2.7, and 2.9. It is observed from this graph that as the values of H increase, the phase velocity also increases.

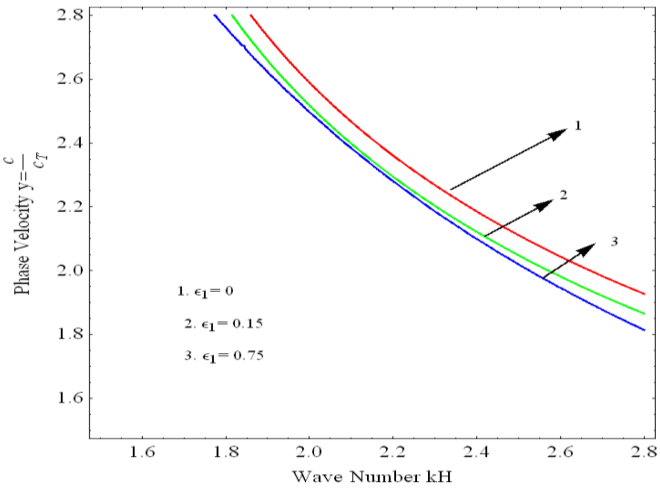


Fig. 3. Yields a graphical representation showcasing the relationship between phase velocity and wave number across different values of the inhomogeneity parameter ε_1, with all other parameters held constant. The amount of inhomogeneity parameter for the upper part of the model for the curves are taken as 0, 0.15, 0.75. From this graph, we can make out the conclusion that with the increasing values of ε_1, phase velocity decreases accordingly.

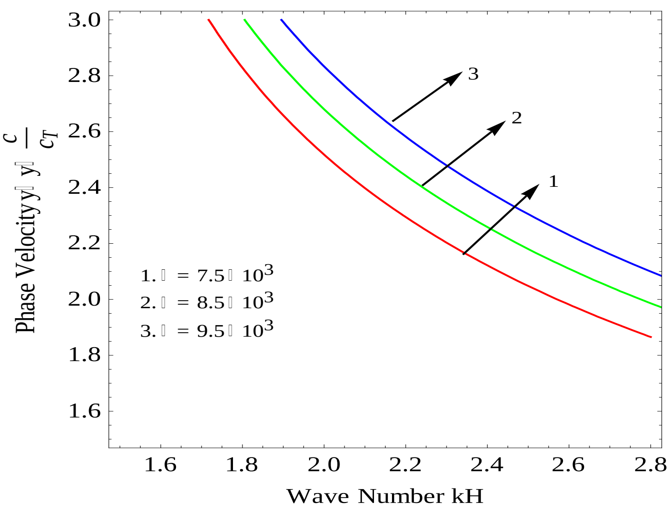


Fig. 4. is constructed to elucidate the relationship between phase velocity and wave number across a range of values for the upper section of the model, maintaining consistency in other parameters. This comprehensive analysis aids in comprehending the intricate dynamics of wave propagation within the model. Here the values of density are treated as $7.5*10^3$, $8.5*10^3$ and $9.5*10^3$. After observing the graph, we see that with the increase values of ρ, phase velocity increases.

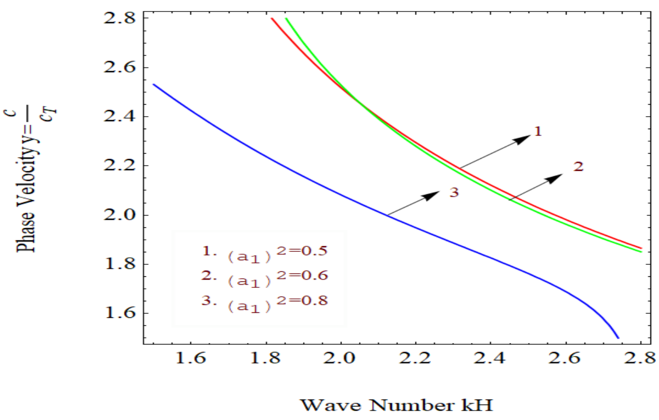


Fig. 5. vividly showcases how varying directions of fiber reinforcement intricately impact the dispersive characteristics exhibited by Love waves, providing valuable insights into their propagation dynamics. Different values a_1^2 can be taken as 0.5, 0.6, 0.8. After observing the above graph, we can say that, with the increase of the values of a_1^2, phase velocity decreases.

VIII. Conclusions

This paper provides an in-depth analysis of shear horizontal (SH) wave propagation within a complex layered system composed of a non-local fiber-reinforced layer resting above a non-local inhomogeneous semi-infinite substrate. The main focus is to explore the intricate interactions and transmission dynamics that occur within this configuration, offering insights into the behavior of Love-type waves. Love waves, a specific type of surface seismic wave, are characterized by horizontal shear motion and typically propagate through layered media, making them particularly relevant in geophysical and engineering contexts.

The study describes these wave characteristics through the use of general dispersion equations, which play a pivotal role in defining the relationship between various parameters governing wave propagation. To derive meaningful solutions, the method of separation of variables is applied, allowing the authors to develop precise mathematical expressions for both the fiber-reinforced layer and the underlying inhomogeneous substrate. This approach helps in understanding how different physical conditions within the layers influence wave behavior.

Furthermore, the paper represents a comprehensive graphical analysis that highlights the relationship between phase velocity and wave number. Many key factors, including the depth of the layer, the inhomogeneity parameter, the direction of fiber reinforcement, and the non-local elastic parameter, are shown to significantly affect wave propagation. Changes in these variables lead to observable shifts in phase velocity, illustrating the complex nature of wave transmission in such systems. These insights are critical for advancing the understanding of wave behavior in non-local, fiber-reinforced, and inhomogeneous materials, with potential applications in seismic wave analysis and material design.

Conflicts of interest

All authors declare that they have no conflicts of interest.

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