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A WEAK FORM FOR EXTENDING ACTS

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Abstract

Important areas for study in the field of act theory include expansions for the notion for extending acts. There has already been an introduction to the idea of extending acts. Our focus recently has been on semi-extending acts as a means for investigating one of these generalizations. The condition that every sub-act for an S-act M_S has been semi ⋂-large in a retracted from M_S has been what defines it as a semi-extending S-act. Based on this, we provide several characteristics for semi-extending acts. Also included are examples that show how this idea works. We prove relationships between semi-extending acts and related conceptions utilizing a fully essential notion.

Keywords: Semi ⋂-large sub-acts, ⋂-large sub-acts, strongly closed sub-acts, Closed sub-acts, fully ⋂-large acts.

I. Introduction

Exploring the concept of semi-extending acts has been the main focus of this research. In this work, the commutative monoid S and the unitary right S-act M_S have been defined, unless otherwise specified. It has been assumed that all S-acts in this paper have prime sub-acts. The annihilator for M in S has been represented by $J_{(\Theta)} = j = \{s \in S: Ms = (\Theta)\}$ for S, where M has been considered faithful if and only if J=(Θ)[I]. The set {s∈S: Ms⊆A} has been known as the related ideal for A. It becomes an ideal for S if A has been a sub-act for a right S-act M. This ideal will be denoted as JA [I]. If, for every element a in M and every element b in S, the given condition $aSb \subseteq N$ implies that a∈N or b $\in J_N$ sets, then a certain type of sub-act for a right S-act M has been considered prime. If every element b in M, when multiplied with any element x in S, has been contained in N, this means $bxSx \subseteq N$ and implies that βx has been also in N, then N has been a semiprime sub-act for M. If the zero sub-act for M, denoted as Θ, has been a prime or semiprime sub-act, then the S-act M has been also referred to as prime or semiprime. Similarly, S has been considered a prime or semiprime if for S_S, the zero ideal (0) has been also prime or semiprime as sub-act [I]. Let's define a nonzero sub-act N for M as semi-⋂-large if A⋂P≠ Θ for every nonzero prime sub-act P for M_S. In simpler terms, a sub-act A for an S-act M_S has been considered semi-⋂-large when the condition $A \cap P = \Theta$ implies that $P = \Theta$ for each prime sub-act P for M_S. If an S-epimorphism $\alpha: M^I \to N$ for some index set I exists, then a sub-act N for M_S has been considered as a M_S-generated. The name given to N_S when I have been finite has been Finitely M_S-generated for M_S. As stated in the source [VII], on page 63, when an S-act A_S has been formed by a single element, it has been referred to as the principal act and denoted as $A_S = < u >$, with u being an element for A. This means that $A_S = uS$. The multiplication module in [II] can be extended to an S-act M_S if all of its sub-acts follow the pattern MI, where I represents a right ideal I for S[X]. This has been the form that every principal sub-act takes this form. It has been widely understood that Z_Z represents a form for the multiplication act and that every multiplication can be represented as a duo [X]. These concepts have coincided with projective S-act, but this has been not always the case in general S-act [X]. We will now delve into the fundamental concepts for extending acts. If a subset A for M_S does not contain any proper ⋂-large in M_S, we say that A has been closed. In M_S, a sub-act H has been considered strongly closed if it does not have any proper semi-⋂-large elements. Put simply, if M_S has a sub-act K, then the only possible solution to the equation $H \hookrightarrow^{semi-\cap l} K \hookrightarrow_{\neq} M_S$ has been H=K. For an S-act M_S to be considered as an extending act, or CS-act, all of its sub-acts must be ⋂-large in a retract for M_S. For more information on injective acts and related concepts, readers can consult the following references: [III, IV, V, VIII, IX, XI, XII, XIV, XV].As a continuation for extending acts, the present study explores a new category for acts known as semi-extending acts. A particular type for S-act M_S has been classified as semi-extending if every sub-act for M_S has been a semi-intersection large for M_S.

II. Semi-extending acts

The main goal for this section has been to establish a new category for extending acts known as semi-extending acts. The study also explores the hereditary characteristics of semi-extending acts between S-act M_S and S itself. In addition, an example was provided to demonstrate how the semi-extending act has been a broader concept than the extending act.

Definition (2.1): A nonzero sub-act A of M is referred as semi-⋂-large, if $A \cap P \neq \Theta$ for each nonzero prime subact P of M_S. In other words, a subact A of an S-act M_S is termed as semi-⋂-large if whenever $A \cap P = \Theta$, implies that $P = \Theta$ for each prime sub-act P of M_S.

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Definition (2.2): If every sub-act for an S-act M_S has been a semi-intersection large in a retract for M, then we say that M_S has been semi-extending. If an S-act S has been semi-extending, then the monoid S has been also considered semi-extending.

A semi-simple S-act M_S has been characterized by the property that all of its sub-acts have been either union for simple sub-acts or retracts [XIII].

Definition (2.3): Any non-zero sub-act for a non-zero S-act M_S that has been semi-large (or semi-intersection-large) has been called semi-reversible (or semi ⋂ -reversible).

Examples with Remarks (2.4):

1. Z_6 has been a semi-extending, as all of its sub-acts exhibit a semi-⋂-large property in a retract of Z_6

2. The Extending Act has been semi-extending in every situation.

Proof for (2): It can be deduced from the given reference [XIII]

3. The reverse of (2) is not true as usual. Consider the Z-act $M_S = Z_8 \dot{\cup} Z_2$ as an illustration. All sub-acts for M_S have been represented as follows

$\{<(\bar{0},\bar{0})>, <(\bar{1},\bar{0})>, <(\bar{0},\bar{1})>, <(\bar{1},\bar{1})>, <(\bar{2},\bar{0})>, <(\bar{2},\bar{1})>, <(\bar{4},\bar{0})>, <(\bar{4},\bar{1}),$

$<(\bar{0},\bar{1}),(\bar{4},\bar{0})>, <(\bar{2},\bar{0}),(\bar{4},\bar{1})\}$ and M_S. As every strongly closed sub-acts in M_S can be represented as $<(\bar{0},\bar{1})>, <(\bar{4},\bar{1})$, and M_S, and both for them have been retracting (direct summand) for M_S. In fact, M_S can be expressed as the union for <(1̅,1̅)> and <(0̅,1̅)>, and M_S can also be expressed as the disjoint union as in the following $M_S = <(\bar{1},\bar{1})> \dot{\cup} <(\bar{0},\bar{1})>$, and also $M_S = <(\bar{1},\bar{0})> \dot{\cup} <(\bar{4},\bar{1})>$, thereafter M_S is semi-extending act. However, this act does not extend the act since the sub-act $<(\bar{2},\bar{1})>$ has been closed in M_S, which has been not a retracted sub-act in M_S.

4. Z_6, Z_3, Z_2 Have been examples for semi-extending Z-acts because they have been all semi-simple acts. This means that all semi-simple acts are semi-extending

Proof for (4): As stated in remark (2.3) (2) in [XIII], it can be observed that every semi-simple act has been an extending act. We can infer the result from equation (2).

5. A semi-extending act has been defined as being ⋂-reversible. In other words, each ⋂-reversible act is semi-extending.

Proof for (5): Let's consider M_S as an intersection-reversible S-act. Since every sub-act for ∩-reversible act has been considered intersection large in M_S, it can be concluded that M_S has been an extending act. Based on (2) M_S can be considered a semi-extending act.

6. In most cases, the reverse for (5) has been not true. For example: The Z-act Z_{36} has been a semi-extending act, but it lacks the property of being ∩-reversible. This has been because there has been a sub-act for Z_{36} which is $\overline{(18)}$ that has been not ∩-large for Z_{36}. In fact, the sub-act $\overline{(12)}$ for Z_{36} has zero result for the intersection of it's with the sub-act $\overline{(18)}$ (this means, $\overline{(18)} \cap \overline{(12)} = (\Theta)$).

7. Every act that has been semi ∩-reversible has been also a semi-extending act.

Proof for (7): Consider M_S as a semi ∩-reversible S-act Since every sub-act for a semi ∩-reversible act has been a semi ∩-large in M_S, it can be concluded that M_S has been an extension for an act. Based on (2), M_S can be classified as a semi-extending act.

8. The reverse of (7) is not true at all. For example, Z_{24} is semi-extending Z-act, but it lacks the property of being semi ∩-reversible. It is clear that $\overline{(8)}$ has not been semi ∩-a large sub-act for Z_{24}

9. In most cases, a semi-∩-reversible act cannot be an indecomposable, for example, a semi -∩-reversible Z-act Z_{36} failed to be an indecomposable act. Actually, $Z_{36} = (\bar{4})\dot{\cup}(\bar{9})$.

10. It is not necessary that semi-∩-large sub-act for a semi-extending act will be semi-extending act, as in the following: if M_S is S-act and not semi-extending act, then, since the injective envelope for M_S is injective, thereafter it will be semi-extending act. Hence, every sub-act is semi-∩-large for E(M). This means that a semi-∩-large sub-act for a semi-extending act, cannot be semi-extending act.

11. When M_1 is semi-extending act and M_2 is isomorphic to M_1, then, M_2 is also semi-extending act.

Proposition (2.5): When a nonzero S-act M_S has been both semi-extending and indecomposable, then M has been a semi-intersection-reversible act.

Proof: Let's consider anon zero sub-act A for M_S.Assuming that M_S has been a semi-extending act. This suggests that A has been semi-∩-large sub-act for M_S and then will be for B, where B has been a retract sub-act for M_S. However, M_S has been an indecomposable act, then there have been only two possibilities: either B has been equal to zero or B has been equal to M_S. But, as A has been not equal to Θ, we can deduce that B has been equal to M_S, which has been the desired result.

Definition (2.6): A nonzero S-act M_S has been defined as fully-⋂-large if each nonzero semi-⋂-large sub-act for M_S has been ⋂-large sub-act for M_S.

Keep in mind that a prime sub-act A for a right act M_S has been characterized by its ability to meet a specific condition. In this case, if certain conditions have been met, it can be determined that either one element has been included in a particular subset or another element has been included in a different subset. This means that when x belongs to M_S and y belongs to S, then we have $xSy \subseteq A$ which is implied to $x \in A$ or $y \in J_A$.

Definition (2.7): For S-act M_S has been considered fully prime when all of its proper sub-acts have been prime sub-acts.

Lemma (2.8): Every fully prime act has been also fully-⋂-large.

Proof: Let's assume that A has been a sub-act for S-act M_S that has been semi-intersection-large. If $A=\Theta$ and given that M_S has been a fully prime act, it can be concluded that A has been a prime sub-act. According to the definition of prime sub-act, it has been ensured that there will always be a non-empty intersection between any sub-act for M_S and A. This suggests that A has been an intersection large sub-act.

Corollary (2.9): Think for M_S as a semi-extending and indecomposable act. If M_S has been a fully -⋂-large act, then it has been also ⋂-reversible act.

Before diving into the properties for semi-extending acts, it has been crucial to establish a clear understanding of the concept of strongly closed.

Definition (2.10): Assume that a sub-act A for M_S has been considered to be strongly-closed in M_S if it does not contain any proper semi-⋂-large elements in M_S. In other words, if there has been a sub-act B for M_S, then the equation $A \hookrightarrow^{\text{semi}-\cap l} B \hookrightarrow_{\neq} M_S$ has only one solution which is A=B.

Proposition (2.11): Assume that A and B are two sub-acts for an S-act M_S. If A has been semi -⋂-large in B and the sub-act B is semi -⋂-large in M_S, then A will be semi -⋂-large in M_S

Proof: Assume that P be prime sub-act for M_S, where $A \cap P = \Theta$. Note that $\Theta = A \cap P = (A \cap P) \cap B = A \cap (P \cap B)$. However, P is a prime sub-act for M_S, so we have two cases. If $B \subseteq P$, then, $\Theta = A \cap (P \cap B) = A \cap B$. Thereby, $A \cap B = \Theta$. But $A \subseteq B$, so $A \cap B = A$ and this implies that A is equal to zero. But this is a contradiction with our assumption. Thereby, $B \nsubseteq P$ with $P \cap B$ has been a prime sub-act for B. But A has been a semi-⋂-large in B, hence $P \cap B = \Theta$. Since B is semi-⋂-large in M_S, then $P = \Theta$. This means that A is semi-⋂-large in M_S.

Proposition (2.12): Suppose that M_S has been an S-act and A has been a non-empty subset for M_S. In this case, there has been a sub-act B in M_S that has been strongly closed, and A has been semi-⋂-large in B.

Proof: Let us take the set ℂ, $\mathbb{C}=\{B|B \text{ is a subact of } M_S \text{ where A is semi} - \cap - \text{large in} B\}$. It has been clear that ℂ≠ Θ. Based on Zorn's Lemma, there exists a maximal element referred to as K. To prove that K has been a strongly closed sub-act in M_S. Let's consider that there has been a sub-act L for M_S, where K has been semi ⋂-large in L and L has been a sub-act for M_S. Based on a proposition (2.11), it can be inferred that there has been a relationship between A, K, and L, suggesting that A has been semi ⋂-large in K with K has been semi ⋂-large in L. Thereafter, A is semi ⋂-large in L. However, this contradicts the maximality for K, so we can infer that K=L. That has been K has been a sub-act in M_S that has been strongly closed, with A being a semi ⋂-large subset for K.

Theorem (2.13): In a semi-extending act, which we will denote as S-act M_S, each strongly-closed sub-act in M_S has been guaranteed to be a retract for M_S.

Proof: ⟹) Let's assume that A has been a strongly closed sub-act in M_S. Considering that M_S has been a semi-extending act, it follows that A has been a semi-⋂-large sub-act for B, with B being a retract for M_S. Based on the definition of a strongly closed sub-act, it can be concluded that A has been equal to B. This implies that A has been retract for M_S.

⟸) When H has been equal to the empty set, then it has been evident that H has been a semi intersecting large sub-act in a retract for M_S. If H has been not equal to Θ, then based on proposition (2.12) implies the existence for a strongly closed sub-act H in M_S where N has been semi-⋂-large in H. Given the assumptions, it can be concluded that H has been a retract for M_S, suggesting that M_S has been a semi-extending act.

Proposition (2.14): Consider A and B are two sub-acts for an S-act M_S. If A has been strongly closed in B with B has been strongly closed in M_S, then A has been strongly closed in M_S.

Proof: Let's assume that A has been a strongly closed sub-act for B, and B has been strongly closed a sub-act for M_S. Think of ℂ as a collection for all proper sub-acts for M_S that act as a semi ⋂-large extension for A in M_S. Then, it has been that ℂ is non-empty, due to the inclusion of B in ℂ. Based on Zorn's lemma, there exists a maximal semi ⋂-large extension C for A in M_S, which has been a strongly closed sub-act for

M_S. Then, given the maximality for A, it can be inferred that A has been equivalent to C and A has been a strongly closed sub-act for M_S.

Proposition (2.15): Assume that a strongly closed for semi-extending acts has been also considered as semi-extending.

Proof: Let's consider N as a strongly closed sub-act for the semi-extending act M_S. Think of A as a strongly closed sub-act for N. Based on proposition (2.14), it can be inferred that A has been strongly closed in M_S. Furthermore, M_S has been semi-extending. Thus, according to theorem (2.13), A can be considered a retract for M_S. In addition, A will be a retract sub-act for N. Thus, N can be seen as a semi-extending act.

Proposition (2.16): Take the sub-acts $A_1$and A_2be sub-acts for a semi-extending S-act M_S. If $A_1 \cap A_2$ is strongly closed in M_S, then $A_1 \cap A_2$ is a retraction of $A_1$and A_2.

Proof: Based on theorem (2.13), $A_1 \cap A_2$ can be shown as a retracted for M_S. However, $A_1 \cap A_2 \subseteq A_1$. Therefore, $A_1 \cap A_2$ is a retraction for A_1. Similarly, it can be demonstrated that the intersection for A_1 and A_2has been a retracted for A_2.

In the following theorem, we present a condition and provide a characterization for semi-extending acts.

Theorem (2.17): Assume that M_S is an S-act where the intersection for M_S and N has been a strongly extending act for every N that has been a retract for E(M). Then the following statements have been equivalent:

1. M_S has been a semi-extending act.

2. When N has been retracted for the injective envelope E(M) for M_S, then $N \cap M_S$ has been also retracted for M_S.

Proof: ⟹) Given the hypothesis, it can be deduced that $N \cap M_S$ has been strongly closed in M_S. Since N M_S has been semi-extending, thus, $N \cap M_S$ will be a retract for M_S.

⟸) Suppose that A be a sub-act for M_S and let B will be a relative complement for A in M_S. According to the proposition (2.6) [XIII]. $A \dot{\cup} B$ has been $\cap$-large in M_S. Based on the fact that M_S has been $\cap$-large sub-act in the E(M), we can utilize lemma (3.1) from [VI] $A \dot{\cup} B$ is $\cap$-large in $E(M)$. Therefore, $E(M) = E(A \dot{\cup} B)$ which is equal to $E(A) \dot{\cup} E(B)$. As $E(A)$ has been a retract of $E(M)$, thus by hypothesis $E(A) \cap M_S$ will be a retraction for M_S. A has been an $\cap$-large sub-act for E(A) and M_S has been an $\cap$-large sub-act for M_S. Thereby, $A = A \cap M_S$ has been $\cap$-large for $E(M) \cap M_S$. This implies that A has been semi-$\cap$-large in $E(M) \cap M_S$ which is a retraction for M_S.

Using two theorems, we can obtain the following outcome.

Proposition (2.18): Assume that M_S can be considered as an S-act where the intersection for N and M_S has been always strongly closed in M_S for every retract N for E(M). Then the following statements have

been equivalent:

1. M_S has been a semi-extending act.

2. Any sub-act that has been strongly closed in M_S can be viewed as a retract for M_S .

3. When N has been retracted for the injective envelope E(M) of M_S, then $N \cap M_S$ has been also retracted for M_S.

Proof: (1⟹2) If the first point has been true, then it logically follows that another point must also be true. Based on theorem (2.13).

(2⟹3) Take that the set E(M) is equal to the following: $E(M_S) = A\,\dot{\cup}\,B$ where B being a sub-act for E(M). Consider a situation where $A \cap M_S$ has been semi ∩-large in K, with K being a sub-act for M_S and also for E(M). Now, let's consider a value k that belongs to the set K. Then k belongs to E(M), which leads to that k has been either in A or in B (this means that k has been equal to a or k has been equal to b). Now, let's consider a situation where k does not belong to A and a non-zero k has been equal to b. Because that M_S has been semi ∩-large in E(M), thus there has been an element $\Theta \neq s \in S$ where $ks = bs \in M_S$. However, if $\Theta \neq b \in B$, then bs will also belong to B. Therefore, we have $bs \in M_S \cap B$. Moreover, $A \cap M_S$ has been regarded as semi ∩-large in K with B as semi ∩-large in B. Therefore, $A \cap M_S \cap B$ is semi ∩-large in $K \cap B$. However, $M_S \cap A \cap B = \Theta$, it can be concluded that the intersection for K and B has been also zero $K \cap B = \Theta$. Therefore, it has been necessary for as to be zero as well, resulting in a contradiction. Therefore, it has been shown that $A \cap M_S$ has been strongly closed for M_S and, as stated in (2), it has been a retracted for M_S.

(3 ⟹1) Based on a particular theorem (2.17).

This proposition demonstrates the hereditary property for the semi-extending between S-act M_S and a

monoid S itself. However, it has been important to have a prerequisite before progressing to the next theorem.

Proposition (2.19): Let's look at M_S as an S-act which consists of two sub-acts A and B. It's worth noting that A has been a subset for B and B has been a subset for M. If A has been considered prime in M_S and B has been also considered prime in M_S, then A has been also considered prime in B.

Proof: Given that x belongs to A and s belongs to S. Where "sx" belongs to set A. Now, to prove that A has been prime in B, we will show that either x has been a member for A or s has been a member for [A:B]. If A has been a prime in M_S, then either x has been a member for A or s has been a member for [A:M]. Let's suppose that s∈[A:M]. This suggests that sM ⊆ A. However, given that B has been a subset for M, it can be concluded that sB ⊆ A. Therefore, s∈[A:B]. Thus, A has been regarded as prime in B. If x has been an element for A, then there has been no need for further demonstration.

Theorem (2.20): Suppose that M_S has been an S-act, and there have been two semi-⋂-large sub-acts, N_1 and N_2 for M_S. Furthermore, let's assume that the intersection $N_2 \cap P$ has been a prime sub-act for M_S for every prime sub-act P for M_S. Based on the given information, it can be inferred that $N_1 \cap N_2$ has been also a semi-⋂-large sub-acts for M_S

Proof: Let's consider P as a prime sub-act for M_S where$(N_1 \cap N_2) \cap P = \Theta$.It can be inferred that $N_1 \cap (N_2 \cap P) = \Theta$. On the other hand, $N_2 \cap P$ has been seen as a prime sub-act for M_S, whereas N_1has been a semi-⋂-large sub-act for M_S. Thus, it can be inferred that$N_2 \cap P = \Theta$. Furthermore, since N_2has been a semi-⋂-large sub-act for M_S, it can be deduced that$P = \Theta$. Consequently, $N_1 \cap N_2$ is also qualifies as a semi-⋂-large sub-act for M_S.

Keep in mind that in the realm of algebra, an S-act M has been deemed a multiplication when there exists an ideal I for S for every sub-act A for M_S, where A has been equal to IM [X].

Keep in mind that an S-act M_S has been considered faithful if J=(Θ), where an ideal $J = \{s \in S | M_S = (\Theta)\}$ for S has been known as the annihilator for M in S[I].

Condition (*): If A has been a prime ideal for B, then AM has been a prime sub-act for BM for any two ideals A and B for S.

Proposition (2.21): Let's consider M_S as both a multiplication and faithful act that satisfies the condition (*), and assume that A and B have been ideals for S. If AM is semi-⋂-large in BM, then A is semi-⋂-large in B.

Proof: Let's assume that P has been a prime ideal for B with the property $A \cap P = \Theta$. Then, we have $(A \cap P)M = \Theta M$. Based on the fact that M_S is a faithful multiplication, therefore, $AM \cap PM = \Theta$. According to the condition (*), PM is a prime sub-act for JM. However, AM is semi-essential in PM, then PM = (Θ). As M is a faithful act, so P = (Θ), and thus A is semi-essential in P.

Proposition (2.22): For each sub-act N of a semi-extending act that possesses the property of being a retract for N (this means that each sub-act for N that intersection with any retract for M_S is retracted for N) has been classified as semi-extending.

Proof: Let's assume that N has been a sub-act for M_S and A has been a sub-act for N. Considering that M_S has been a semi-extending act, hence it can be inferred that there exists a retract B for M_S where A has been semi $\cap$-large in B. However, $A \subseteq B \cap N \subseteq B$. Consequently, according to the given proposition (2.11), it can be inferred that A has been semi $\cap$-large in $B \cap N$. In addition, based on the hypothesis, $B \cap N$ has been retracted for N. Therefore, N has been semi-expanding.

Proposition (2.23): For each sub-act that has been fully invariant in a semi-extending S-act has been also semi-extending.

Proof: We can consider M_S as a semi-extending S-act and N as a fully invariant sub-act for M_S. If A has been a sub-act for N, then A has been also a sub-act for M_S. Given the semi-extending M_S, then there exists a retract D for M_S where A has been semi $\cap$-large in D. It can be inferred that M_S has been equivalent to the following $M_S = D \dot{\cup} H$ with H representing any sub-act for M_S. Given that N is fully invariant, thus $N = (N \cap D) \dot{\cup} (N \cap H)$. It appears that the intersection for N and D can be considered a retract for N. Considering the given conditions, A has been semi $\cap$-large in D and also N has been semi $\cap$-large in N. it can be concluded that $A = N \cap A$ has been semi $\cap$-large in the intersection for N and D, denoted as $N \cap D$. Thereby, N has been semi-extending.

Proposition (2.24): Let's consider M_S as $M_S = M_1 \dot{\cup} M_2$, where both M_1 and M_2 have been semi-extending acts. After that, it has been true that M_S has been semi-extending if and only if each strongly closed sub-act N for M_S where $N \cap M_1 = \Theta$ or $N \cap M_2 = \Theta$ has been retracted for M_S.

Proof: ⟹) The importance has been clear based on the theorem (2.13).

⟸) Let's consider any strongly closed sub-act N for M_S where $N \cap M_1 = \Theta$ or $N \cap M_2 = \Theta$ which has been retracting for M_S. Suppose that A has been strongly closed sub-act for M_S. Thereafter, there has been a complement B in A where $A \cap M_2$ has been semi $\cap$-large in B, and because A has been strongly closed for M_S, hence B has been also strongly closed for M_S according to the proposition (2.14). When the intersection $(A \cap M_2) \cap M_1$ has been semi intersected large in $B \cap M_1$, it implies that M_1

has been semi intersect large in M_1.So, $B \cap M_1 = \Theta$ (as $A \cap (M_2 \cap M_1) = A \cap \Theta = \Theta$) that is implies that Θ has been semi ⋂-large in Θ. Based on the given assumption, M can be expressed as $M = B \dot{\cup} B^{/}$ where $B^{/}$ Sub-act for M_S. Additionally, it has been mentioned that B has been retracted for M_S. Now, we have the equation $A = A \cap M_S = A \cap (B \dot{\cup} B^{/}) = B \dot{\cup} (A \cap B^{/})$. Thereafter, $A \cap B^{/}$ has been strongly closed for M_S (because that $A \cap B^{/}$ has been strongly closed in A). In addition, $(A \cap B^{/}) \cap M_2 = \Theta$. It follows that $A \cap B^{/}$ has been a retract for M_S, and thus a retract for $B^{/}$(As, $A \cap B^{/} \subseteq B^{/}$). So, the equation has been

$B^{/} = (A \cap B^{/}) \dot{\cup} N$, with N being a sub-act for $B^{/}$. Then, $M_S = B \dot{\cup} B^{/} = B \dot{\cup} (A \cap B^{/}) \dot{\cup} N = (B \dot{\cup} (A \cap B^{/})) \dot{\cup} N = A \dot{\cup} N$. It can be inferred that A has been a retract for M_S. Thus, M_S has been a semi-extending act.

Proposition (2.25): [XIII] Assume that we have two S-acts, M_1 and M_2. In addition, M_S has been represented as $M_S = M_1 \dot{\cup} M_2$. M_1 has been M_2-injective if there has been a sub-act $M^{/}$ for M_S that satisfies certain conditions which are $M_S = M_1 \dot{\cup} M^{/}$ and $N \subseteq M^{/}$ for each sub-act N for M_S where $N \cap M_1 = \Theta$.

There has been a proposition that explains why the direct sum for semi-extending acts has been also semi-extending:

Proposition (2.26): Let's consider M_S as a finite direct sum for acts Mi which happen to be relatively injective, this means that $M_S = \dot{\cup}_{i=1}^{n} M_i$. Then, it has been true that M_S has been semi-extending if all Mi have been semi-extending.

Proof: ⟹) The importance has been evident (because retracting for semi-extending has been semi-extending).

⟸) Assuming that each Mi has been semi-extending and that all Mi have been relatively injective acts. When n=2, using induction on n has been enough to demonstrate that M_S has been semi-extending when n = 2. Suppose that $A \subseteq M_S$ has been strongly closed with $A \cap M_1 = \Theta$. According to the proposition (2.25), there has been exists a sub-act $M^{/}$ for M_S where $M_S = M_1 \dot{\cup} M^{/}$ and $A \subseteq M^{/}$.

It has been clear that $M^{/} \cong M_2$ and so $M^{/}$has been semi-extending. It has been evident that A has been strongly closed sub-act for $M^{/}$. Based on the semi-extending property for $M^{/}$, we can utilize theorem (2.13) to deduce that A has been retracted for $M^{/}$. As a result, A has been also retracted for M_S, when that $M^{/}$has been also retracted for M_S. Similarly, if there has been a strongly closed sub-act B for M_S where $B \subseteq M_S$ and

$B \cap M_2 = \Theta$, then B has been a retract for M_S. As a result, according to the proposition (2.24), M_S has been a semi-extending act.

III. Connection for semi-extending acts with other classes for injectivity:

This section explores the factors that determine if a semi-extending act can be classified as an extending act. Furthermore, we delve into the relationships between semi-extending acts and different types of acts.

Theorem (3.1): Let's consider M_S has been an S-act. It has been important to mention that any non-zero semi-essential extension for each sub-act of M_S would be considered as fully essential. Thus, M_S has been semi-extending iff M_S has been extending.

Proof: ⟹) It has been evident.

⟸) Let's assume that N has been a sub-act for M_S. Given that M_S as a semi-extending act, it follows that N has been a semi essential sub-act within the retract for M_S, which has been denoted as H. If H has been equal to Θ, then N must be zero. It has been clear that N has been essential sub-act in a retract for M_S.Otherwise, if we consider H as a fully essential act, then it logically implies that N has been an essential sub-act for H, resulting in M_S being an extending act.

Similarly, we can illustrate the following theorem.

Theorem (3.2): In the context for S-acts, if every nonzero retract of M_S has been regarded as a fully essential act, then M_S has been an extending act if and only if it has been a semi-extending act.

Theorem (3.3): Suppose that M_S has been a fully prime S-act. Thereafter, an act has been semi-extending, iff it has been an extending act.

Proof: ⟹) Let's assume that N has been a sub-act for M_S. If N has been equal to zero, it can be inferred that N has been an essential sub-act within the retract for M_S. If N has been not equal to Θ, and assuming that M_S has been a semi-extending act, then it can be concluded that N has been a semi-essential sub-act in a retract for M_S, which has been denoted as H. However, understanding that M_S has been fully prime, thus N has been a semi-essential sub-act for H. This implies that M_S has been extending act.

⟸) It has been evident.

Theorem (3.4): Consider that M_S has been an S-act. For each sub-act X for M_S, there has been a strongly closed sub-act H for M_S and contains X as an essential in H. After that, M_S has been a semi-extending act if it has been also an extending act.

Proof: ⟹) Consider the S-act M_S that has been a semi-extending act, and X has been a sub-act for M_S. Based on the given assumption, there has been a strongly closed

sub-act H for M_S, and X has been a essential sub-act within H. Considering that M_S has been a semi-extending act, it can be concluded that H has been a retract for M_S. Thereby, M_S has been extending the act.
⟸) Here has been the modified text: It has been clear. Keep in mind that a sub-act N for an S-act M_S has been considered fully invariant if $f(N) \subseteq N$ for each S-endomorphism f for M_S. An S-act M_S has been referred to as a duo if all sub-acts for M_S have been fully invariant. Before proceeding to the next proposition, it has been essential to establish the following definition:

Definition (3.5): A specific type for S-act, M_S, has been considered to be a FI-extending if certain conditions have been met. Specifically, if each fully invariant sub-act for M_S has been an essential in a retract for M_S. Given that each extending act has been FI-extending, we can put forth the following proposition.

Proposition (3.6): Suppose we have M_S as a duo S-act. M_S has been a semi-extending act if it meets the criteria for FI-extending.
Proof: Let's consider N that has been a sub-act of M_S. As M_S has been a duo act, it can be concluded that N has been a fully invariant sub-act of M_S. However, M_S has been an FI-extending act. It follows that N has been essential sub-act in A, where A has been a retract for M_S. It concluded that M_S has been semi-extending act.

The validity of Proposition (3.6) can be established by substituting the condition "duo act" with the stipulation that "every nonzero retract of M_S has been a fully essential act," as shown in the subsequent proposition.

Proposition (3.7): Consider M_S as an S-act, where any nonzero retract of M_S has been regarded as a fully essential act. If M_S has been a semi-extending act, then it logically follows that M_S has been a FI-extending act.

Proof: Given that M_S has been a semi-extending act, we can conclude, based on theorem (3.2), that M_S has been indeed an extending act. Thus, M_S has been an FI-extending act.

IV. Results and Discussion

We determined in proposition (2.22) that each retract of any sub-act N for a semi-extending act M is also semi-extending. But the sub-act of N will be retracted of N if it possesses the property that this sub-act of a N must be intersection with any retract of the act M.

Proposition (2.23) presents the important result for which the sub-act of the semi-extending act will be semi-extending and this will be if the sub-act is fully invariant

Besides, it has been illustrated that the union of two semi-extending acts will be semi-extending if it satisfies a condition and this condition exists in proposition (2.24). In the same context, proposition (2.26) generalized the result in proposition (2.24) such that it was explained the finite direct sum of the semi-extending acts will be semi-extending if it met specific criteria.

Also, we found the link between extending and semi-extending acts where they will equivalency under a certain condition, and this is clarified in theorem (3.1) and theorem (3.2).

Based on proposition (3.6) and proposition (3.7), we can see the relation between a semi-extending act and an FI-extending act but this relation was under certain conditions.

V. Conclusion

In this article, we introduced a novel concept called Semi-extending acts and achieved several intriguing findings, including various new characterizations for that notion. The link between the classes of Semi-extending acts with other classes of injectivity is adopted. In a related context, we found the conditions for equivalence of these classes.

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Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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