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SOME EXTENDED MUTUAL RELATIONSHIPS BETWEEN THE CONVOLUTIONS TRANSFORM

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Abstract

In this paper, we establish several interesting mutual relationships between two integral transforms of convolutions transform have been established.

Keywords: Convolution Theorem, Inverse Laplace Transform, Laplace Transform.

I. Introduction

Sometimes we need to determine the inverse Laplace transform of the product of two functions. As in differential and integral calculus, when the derivative and integral of the product of two functions do not yield the product of the derivative and integral respectively, nor does the inverse Laplace transform of the product yield the product of inverse Laplace transforms. The convolution theorem tells us how to calculate the inverse Laplace transform of the product of two functions. Integral transforms can be played to display colorful results in various ways. The mutual relationships between two integral transforms are no exceptions to it. These can be exploited to yield results of appreciable nature. We have enchased these in establishing recurrence relations in this paper. Several mutual relationships between two integral transforms have been established [I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII].

In this section, we have considered the transforms of the following types:

$$\int_0^x h_1(x-t)h_2(t)\,dt \tag{1.1}$$

$$\int_x^\infty h_1(t-x)h_2(t)\,dt \tag{1.2}$$

$$\int_y^x h_1(x-t)h_2(t-y)dt \tag{1.3}$$

Definition: The Convolution Theorem: Suppose that f(x) and g(x) are piecewise continuous on $[0, \infty)$ and both are of exponential order. Further, suppose that the Laplace transform of f(t) is F(s) and that of g(t) is G(s), Then

$$L^{-1}\{F(s).G(s)\} = L^{-1}\{L\{(f * g)(t)\}\} = \int_0^t (t-v)g(v)dv. \tag{1.4}$$

Here, $(f * g)(t) = \int_0^t (t-v)g(v)dv.$ is called the convolution integral.

II. Some Extended Mutual Relationships between the Convolutions Transform

In this section, we develop some extended results of mutual relationships between the convolution transform.

Theorem 1: If $g(x)$ is continuous in $o \leq x < \infty$

$$f_1(x) = \int_x^\infty e^{\overline{\frac{1}{2(\alpha+\beta)(t-x)}}}[D_{2\lambda+1}\{-2\frac{1}{2}(\alpha-\beta)^{\frac{1}{2}}(t-x)^{\frac{1}{2}}] - D_{2\lambda+1}[2^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}}(t-x)^{\frac{1}{2}}]\, g(t)\, dt \tag{2.1}$$

$$f_2(x) = \int_x^\infty (t-x)^{-\frac{1}{2}} e^{\frac{1}{2}(\alpha+\beta)(t-x)}\, [D_{2\lambda+1}\{-2\frac{1}{2}(\alpha - \beta)^{\frac{1}{2}}\left(t-x\right)^{\frac{1}{2}}\Big] + D_{2\lambda}[2^{\frac{1}{2}}\{2^{\frac{1}{2}}\left(\alpha-\beta\right)^{\frac{1}{2}}\left(t-x\right)^{\frac{1}{2}}\Big\}\Big]\, g(t)dt \tag{2.2}$$

then we get

$$f_1(x) = (-\lambda - \tfrac{1}{2})2^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}} \int_x^\infty e^{\beta(t-x)} f_2(t)dt.$$

Proof: We have

$$f_1(x) = \int_x^\infty e^{\frac{1}{2}(\alpha+\beta)(t-x)}[D_{2\lambda+1}\{-2^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}}(t-x)^{\frac{1}{2}}] - D_{2\lambda+1}[2^{\frac{1}{2}}\left(\alpha-\beta\right)^{\frac{1}{2}}(t-x)^{\frac{1}{2}}\Big]\, g(t)dt$$

$$f_2(x) = \int_x^\infty (t-x)^{\frac{1}{2}} e^{\frac{1}{2}(\alpha+\beta)(t-x)}[D_{2\lambda+1}\{-2^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}}\left(t-x\right)^{\frac{1}{2}}\Big\} + D_{2\lambda}[2^{\frac{1}{2}}\{2^{\frac{1}{2}}\left(\alpha-\beta\right)^{\frac{1}{2}}\left(t-x\right)^{\frac{1}{2}}\Big\}\Big]\, g(t)dt$$

Taking the Laplace transform of the above equations using the following Laplace results:

(i) $\int_x^\infty f_1(x-t)f_2(t)dt = \overline{g_1}(-p)g_2(p),$
where, $\overline{g_1}(-p) = \int_0^\infty f_1(t-x)e^{px}dx.$

(ii) $2^{-(\lambda+1)}\pi^{-1}\left[\left(\frac{1}{2} - \lambda\right)t^{-\frac{1}{2}}e^{\frac{1}{2}}(\alpha+\beta)tx\{\, D_{2\lambda}[-2^{\frac{1}{2}}\left(\alpha-\beta\right)^{\frac{1}{2}}t^{\frac{1}{2}}\Big] + D_{2\lambda}\left[2^{\frac{1}{2}}\left(\alpha-\beta\right)^{\frac{1}{2}}t^{\frac{1}{2}}\right]\right\} = (p-\alpha)^{\lambda}(p-\beta)^{-(\lambda+\frac{1}{2})}.$

(iii) $2^{-(\lambda+\frac{3}{2})}\pi^{-1}\left(\alpha-\beta\right)^{-\frac{1}{2}}\left[(-\frac{1}{2}-\lambda\right)e^{\frac{1}{2}(\alpha-\beta)t}[D_{2\lambda+1}[-2^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}}t^{\frac{1}{2}}]$ $-D_{2\lambda+1}[2^{\frac{1}{2}}\left(\alpha-\beta\right)^{\frac{1}{2}}t^{\frac{1}{2}}\Big]\} = (p-\alpha)^{\lambda}(p-\beta)^{-(\lambda+\frac{3}{2})}.$

Finally equation (2.1) and (2.2) becomes

$$\Rightarrow F_1(p) = \frac{2^{\lambda+\frac{3}{2}}\pi(\alpha-\beta)^{\frac{1}{2}}(-1)^{\frac{-3}{2}}}{\Gamma(-\frac{1}{2}-\lambda)}(p+\alpha)^{\lambda}(p+\beta)^{-\left(\lambda+\frac{3}{2}\right)}G(p)$$

and

$$F_2(p) = \frac{-2^{\lambda+1}\pi(-1)^{\frac{-1}{2}}}{\Gamma(-\frac{1}{2}-\lambda)}(p+\alpha)^{\lambda}(p+\beta)^{-\left(\lambda+\frac{1}{2}\right)}G(p)$$

where, $f_n(x) \doteqdot F_n(p) n = 1,2\ and g(x) \doteqdot G(p)$ on eliminating G (p) from their equations.

Using the invention of the Laplace results:

$\int_x^{\infty} f_1(x-t)\, f_2(t)dt = \bar{g}_1(-s).\, g_2(s)$, where $\bar{g}_1(-s) = \int_0^{\infty} f_1(t-x)\, e^{sx}dx$

and

$$t^{\nu-1}e^{-\alpha t},\ Re\ \nu > 0 = \lceil(\nu).(s+\alpha)^{-\nu},\ Re\ s > -Re\ \alpha.$$

We get the required results:

$$F_1(P) = \ (-\lambda-\tfrac{1}{2})\ )2^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}}(-p-\beta)^{-1}F_2(p).$$

Theorem 2: If $g(x) \in C\ in\ 0 \le x < \infty, Re\ \alpha > -1\ , Re\ \beta > -1$ and

$$f_1(x) = \int_x^{\infty}(t-x)^{\alpha}\, L_n^{\alpha}(t-x)g(t)dt \qquad (2.3)$$

$$f_2(x) = L_n^{\beta}(t-x)^{\beta}\, g(t)dt \qquad (2.4)$$

then we get

$$f_1\,(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\beta+n+1)\Gamma(\alpha-\beta)}\int_x^{\infty}(t-x)^{\alpha-\beta-1}f_2(t)dt.$$

Proof: Taking the Laplace transform of the above equation (2.3) and (2.4) and using the

known Laplace results:

$$t^{\nu}, Re\ \nu > -1 \Rightarrow \lceil(\nu+s)s^{-(\nu+1)}, \qquad Re\ s > 0.$$

$$t^{\alpha}L_n^{\alpha}(t)Re(\alpha) > -1 \Rightarrow \frac{\lceil(\alpha+n+1)}{n!} - \frac{(s-1)^n}{s^{(\alpha+n+1)}}\ , Re\ \ s > 0$$

Finally, the equation (2.3) and (2.4), becomes after simplifying

$$f_1\,(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\beta+n+1)\Gamma(\alpha-\beta)}\int_x^{\infty}(t-x)^{\alpha-\beta-1}f_2(t)dt.$$

Theorem 3: If a and b are real's, m and n, are non-negative integers

$Re\ \alpha > -1; Re(n+\alpha) > Re\ m$

$$f_1(x,y) = \int_y^x e^{b(x-t)}\,(x-t)^{\alpha})L_n^{\alpha}(t-x)g(t-y)dt. \tag{2.5}$$

$$f_2(x,y) = \int_y^x e^{b(x-t)}\,J_n\{a(x-t)\}g(t-y)dt \tag{2.6}$$

then

$$\int_y^x e^{(b+1)(x-t)}\,(x-t)^{n-1}f_1(t,y)dt$$

$$= \frac{\lceil(n+\alpha+1)}{n.\lceil(n+\alpha-m)}a^{-m}\sum_{r=0}^{m}\binom{m}{r}\int_y^x e^{b(x-t)}\,(x-t)^{n+\alpha-m-1}$$

$${}_1F_2\{\tfrac{1}{2}(r-m-1):\tfrac{1}{2}(n+\alpha-m),\tfrac{1}{2}(n+\alpha-m+1):-\tfrac{1}{4}a^2\,(x-t)^2\}f_2\,(t-y)dt.$$

Proof: Subjecting the transform (2.5) and (2.6) to the substitutions

$t-y = zandx - y = u$, we get

$$f_1(x,y) = \int_0^u e^{b(u-z)}(u-z)^{\alpha}\,L_n^{\alpha}(u-z)g(z)dz = \emptyset_1\text{ (u) (say)} \tag{2.7}$$

$$f_2(x,y) = \int_0^u e^{b(u-z)}J_m\{a(u-z)\}g(z)dz = \ \phi_2(u)(say) \tag{2.8}$$

Let $\phi_k(u) = \phi_k(s)fork = 1,2\ andg(x) = G$

Taking the Laplace transform of (2.7) and (2.8), using the Laplace results:

$$\int_0^t f_1(u).f_2(t-u)du = g_1(\beta),g_2(\beta),$$

$$t^{\alpha}e^{\lambda t}\Big|_n^{\alpha}(kt), Re\ \alpha > -1 = \frac{\lceil(\alpha+n+1)}{n!} - \frac{(p-k-\alpha)^n}{(p-\lambda)^{\alpha+n+1}}Re(p-\lambda) > 0.$$

and

$$J_n(\alpha t), Re\ v > -1 = r^{-1}\left(\frac{\alpha}{r}\right)^{v}, Re\ p > \left|\left[\begin{matrix}\alpha\\ m\end{matrix}\right|.r = (p^2+\alpha^2)^{\frac{1}{2}}, R = p + r$$

We get

$$\overline{\phi_1}(p) = \frac{\lceil(\alpha+n+1)}{n!}\frac{(p-b-1)^n}{(p-b)^{\alpha+n+1}}G(p)$$

and

$$\overline{\phi_2}(p) = \ a^m\,\{(p-b)^2+a^2\}^{\frac{1}{2}}[(p-b)+\{(p-b)^2a^2\}]^{-m}G(p)$$

The G (p), eliminating these two equations gives

$$\frac{\overline{\phi_1}(p)}{\overline{\phi_2}(p)} = \frac{\Gamma(\alpha+n+1)}{n!\ .\ a^m} \cdot \frac{(p-b-1)^n}{(p-b)^{\alpha+n+1}} \{(p-b)^2 a^2\}^{\frac{1}{2}} [(p-b) + \{(p-b)^2 + a^2\}^{\frac{1}{2}}]^m$$

$$\Rightarrow (s-b-1)^n \overline{\phi_2}(p) \frac{\Gamma(\alpha+n+1)}{n!\ .\ a^m} \frac{[(p-b)^2+a^2)^{\frac{m}{2}+\frac{1}{2}}}{(p-b)^{\alpha+n+1}} \left[1 + \frac{(p-b)}{\{(p-b)^2+a^3\}^{\frac{1}{2}}}\right]^m \overline{\phi_2}(p) = \frac{\Gamma(\alpha+n+1)}{n!\ .\ a^m} \frac{[(p-b)^2+a^2)^{\frac{m}{2}+\frac{1}{2}}}{(p-b)^{\alpha+n+1}}$$

$$\sum_{r=0}^{m} \binom{m}{r} \left[\frac{(p-b)}{[(p-b)^2+a^2]^{\frac{1}{2}}}\right]^r \overline{\phi_2}(p)$$

$$= \frac{\Gamma(\alpha+n+1)}{n!\ \ a^m} \sum_{r=0}^{m} \binom{m}{r} (p-b)^{-\alpha-n-1+r} \{(p-b)^2 + a^2)^{\frac{m}{2}+\frac{1}{2}-\frac{r}{2}} \overline{\phi_2}(p)$$

$$\Rightarrow (p-b-1)^{-n} \overline{\phi_1}(p) = \frac{\Gamma(\alpha+n+1)}{n!\ \ a^m} \sum_{r=0}^{m} \binom{m}{r} (p-b)^{-2-\frac{1}{2}(\alpha+n+1-r)} \{(p-b)^2 + a^2\}^{-\frac{1}{2}(r-m-1)} \overline{\phi_2}(p)$$

Using [III] results of integral transform;

$$t^{\nu-1} e^{-\alpha t} Re\ \nu > 0 = \Gamma(\nu)(p+\alpha)^{-\nu}, Re\ p > -Re\ \alpha$$

and

$$[\Gamma(2\lambda+2\nu)]^{-1} t^{2\lambda+2\nu-1} \times, F_2\ [\nu{:}\lambda+\nu,\ \lambda+\nu+\tfrac{1}{2};\ -\tfrac{1}{4}\alpha^2 t^2]$$
$$= s^{-2\lambda}(p^2+\alpha^2)^{-\nu}, Re(\nu+\lambda) > 0$$

This gives an invention results

$$\int_0^u e^{(b+1)(u-z)} (u-z)^{n-1}\, \phi_1(z) dz =$$

$$\Rightarrow (p-b-1)^{-n} \overline{\emptyset}_1(p) = \frac{\Gamma(\alpha+n+1)}{n!\ \ a^m} [(p-b)^2 + a^2]^{\frac{m+1}{2}}$$

$$\left[1 + \frac{(p-b)}{[(p-b)^2+a^2]^{\frac{1}{2}}}\right]^m \emptyset_2(p)$$

$$= \frac{\Gamma(\alpha+n+1)}{n!\ \ a^m} \cdot \frac{[(p-b)^2+a^2)]^{\frac{m+1}{2}}}{(p-b)^{\Gamma(\alpha+n+1)}} \sum_{r=0}^{m} \binom{m}{r} \left[\frac{(p-b)}{[(p-b)^2+a^2]^{\frac{1}{2}}}\right]^r \overline{\emptyset}_2(p)$$

$$= \frac{\lceil(\alpha+n+1)}{n!\ a^m}\sum_{r=0}^{m}\binom{m}{r}(p-b)^{-\alpha-n-1+r}[(p-b)^2+a^2]^{\frac{m+1-r}{2}}\overline{\emptyset}_2(p)$$

$$\Rightarrow (p-b-1)^{-n}\overline{\emptyset}_1(p)$$

$$= \frac{\lceil(\alpha+n+1)}{n!\ \ a^m}\sum_{r=0}^{m}\binom{m}{r}(p - b)^{-2+\frac{1}{2}(\alpha+n+1-r)}[(p-b)^2+a^2]^{-\frac{1}{2}(r-m-1)}\overline{\emptyset}_2(p)$$

On inversion, in the light of the results

$$t^{\nu-1}e^{-\alpha t}Re\ \nu > 0 = \lceil(\nu).(p+\alpha)^{-\nu}, Re\ p > -Re\ \alpha$$

and

$$\lceil(2 \curlywedge +2\nu)]^{-1}t^{2\curlywedge+2\nu} \times\ 1F2\left[\nu{:}\curlywedge+\nu{:}\ \ \curlywedge+\nu+\frac{1}{2};\frac{-1}{4}\alpha^2t^2\right]$$

$$= s^{-2\curlywedge}(p^2\alpha^2)^{-\nu}, Re(\nu+\curlywedge) > o$$

If gives the inversion

$$\int_o^u e^{(b+1)(u-z)}(u-z)^{n-1}\emptyset_1(z)dz$$

$$= \frac{\lceil(\alpha+n+1)}{\lceil(\alpha+n-m)}a^{-m}\sum_{r=0}^{m}\binom{m}{r}\int_o^u e^{b(u-z)}(u - z)^{n+\alpha-m-1}1F2\left[\begin{matrix}\frac{1}{2}(r-m-1);\frac{1}{2}(\alpha+n-m),\frac{1}{2}(n+\alpha-m+1)\\ ;-\frac{1}{4}a^2(u-z)^2\end{matrix}\right]\emptyset_2(z)dz \tag{2.9}$$

This on restoring original furnishes the required reset.

$$\Rightarrow \int_o^u e^{-(b+1)z}(u-z)^{n-1}\emptyset_1(z)dz = \frac{\lceil(\alpha+n+1)a^{-m}}{n.\ \lceil(\alpha+n-m)}\sum_{r=0}^{n}\binom{m}{r}$$

$$\int_o^u e^{-bz}(u-z)^{n+\alpha-m-1}1f_2\left[\frac{1}{2}(r-m-1);\frac{1}{2}(\alpha+n-m),\right]$$

$$\frac{1}{2}(n+\alpha-m+1);-\frac{1}{4}a^2(u-z)^2\emptyset_2(z)dz$$

Which on being streamed through n-times differentiation w.r.t. u by Leibnitz rule;

$$\emptyset_1(u) = \frac{e^{(b+1)u}.\ \lceil(\alpha+n+1)a^{-m}}{n!.\lceil(n+\alpha-m)}.\left(\frac{d}{du}\right)^n\sum_{r=0}^{m}\binom{m}{r}e^{-u}\int_o^u e^{-ba}(u-z)^{n+\alpha-m-1}$$

$${}_1F_2\left[\frac{1}{2}(r-m-1);\frac{1}{2}(\alpha+n-m),\frac{1}{2}(n+\alpha-m+1);-\frac{1}{4}a^2(u-z)^2\right]\emptyset_2(z)dz$$

$$\Rightarrow f_1(x,y) = \frac{e^{(b+1)(x-y)}\lceil(\alpha+n+1)a^m}{n!\ \lceil(n+\alpha-m)} D^n \sum_{r=0}^{m} \binom{m}{r} e^{-(x-y)}$$

$$\int_y^x e^{-b(t-y)} (x-t)^{n+\alpha-m-1} 1F_2\left[\frac{1}{2}(r-m-1); \frac{1}{2}(n+\alpha-m)\frac{1}{2}(n+\alpha-m+1); -\frac{1}{4}a^2(x-t)^2\right]\emptyset_2(z)dz$$

On restoring to original variables, where D $=\frac{d}{d(x-y)}$.

Theorem 4: If a and b are real's, m and n are non-negative integers

$$Re\ \ \alpha > -1, \qquad Re(n+\alpha) > Re\, m,$$

$$f_1(x,y) = \int_y^x e^{b(x-t)} (x-t)^{\alpha} L_n^{\alpha}(t-x) g(t-y)dt \tag{2.10}$$

$$f_2(x,y) = \int_y^x e^{b(x-t)} I_m(xa-ta) g(t-y)dt \tag{2.11}$$

Proof: The proof is quite similar to that of above Theorem 3, but the following operational results used

$$t^{\nu-1}e^{-\alpha t}, Re\ \nu > 0 = \lceil(\nu), (s+\alpha)^{-\nu}, Re\ p > -Re\ \alpha,$$

$$t^{\alpha}e^{\lambda t}\Big\lfloor_n^{\alpha}(kt), Re\alpha > -1 = \frac{\lceil(\alpha+n+1)}{n!} - \frac{(s-1)^n}{s^{(n+\alpha+1)}}, Re\ s > 0$$

$$I_\nu(\alpha t), Re\ \nu > -1 = a^{\nu}e^{-1}p^{-1} Re\ p > |Re\ \alpha|$$

$$[\lceil(2\lambda+2\nu)]^{-1}t^{2\nu+2\lambda-1} \times [\nu; \lambda+2\nu;]\, \frac{\lambda+\nu+1}{2};$$

$$\frac{-1}{4}\alpha^2 t^2] = s^{-2\lambda}(s^2+\alpha^2)^{-\nu}, Re(\nu+\lambda) > 0$$

$$\Rightarrow f_1(x,y) = \frac{e^{(b+1)(x-y)\lceil(n+\alpha+1)a^{-m}}}{n!\ \lceil(n+\alpha-M)} d^n \sum_{r=0}^{m} \binom{m}{r}$$

$e^{-(x-y)} \int_y^x e^{-b(t-y)} (x-t)^{(n+\alpha-m-1)} 1F2\left[\frac{1}{2}(r-m-1); \frac{1}{2}(\alpha+n-m), \frac{1}{2}(\alpha+n-m+1); \frac{1}{4}a^2(x-t)^2\right] f_2(t-y)dt$, which$D \equiv \frac{d}{d(x-y)}$.

Theorem 5: If n is non-negative integer, $Re\ \alpha > Re\ (2m) > -1$,,

$$f_1(x,y) = \int_y^x e^{b(x-t)} (x-t)^{\alpha} L_n^{\alpha}(t-x) g(t-y)dt \tag{2.12}$$

$$f_2(x,y) = \int_y^x e^{b(x-t)} (x-t)^{m} I_m(ax-at) g(t-y)dt \tag{2.13}$$

Then

$$f_1(x,y) = \frac{\lceil(\alpha+n+1)\pi^{\frac{1}{2}}}{(2a)^m \lceil(m+\frac{1}{2})} \sum_{r=0}^{m} \frac{(-1)^r}{r!\ (n-r)!} [\lceil(\alpha-2m+r)]^{-1}$$

$$\int_y^x e^{b(x-t)} (x-t)^{\alpha-2m+r-1} 1F2\left[-\left(m+\frac{1}{2}\right); \frac{1}{2}(\alpha-2m+r),\right.$$
$$\left.\frac{1}{2}\left(\alpha+r-1-2m\right); -\frac{1}{4}a^2(x-t\right)^2\Big] f_2(t,y)dt$$

Proof: The proof is quite similar to that of above Theorem 4, but the following operational results used

$$t^{\alpha} e^{\leftthreetimes t} \Big\lfloor_{n}^{\alpha} (kt), Re\alpha > -1 = \frac{\lceil(\alpha+n+1)}{n!} \frac{(s-k-\leftthreetimes)^n}{(s-\leftthreetimes)^{\alpha+n+1}}, Re(s-\leftthreetimes) > 0$$

$$t^{\nu} J_{\nu}(\alpha t), Re\nu > \frac{-1}{2} = \ 2^{\nu}\pi^{\frac{-1}{2}} \left\lceil\left(\nu+\frac{1}{2}\right)\alpha^{\nu} r^{-(2\nu+1)} Res > |I_{m\alpha}|\right.$$

and

$$[\lceil(2\leftthreetimes+2\nu)]^{-1} t^{2\leftthreetimes+2\nu-1} \times = s^{-2\leftthreetimes}(s^2+\alpha^2)^{-\nu}, Re(\nu+\leftthreetimes) > 0$$

$${}_1F_2\left[\nu{:}\leftthreetimes+\nu,\ \leftthreetimes+\nu+\tfrac{1}{2}; \tfrac{-1}{4}\alpha^2 t^2\right]$$

$$\Rightarrow f_1(x,y) = \frac{\lceil(\alpha+n+1).\pi^{\frac{1}{2}}}{(2a)^m \lceil(m+\frac{1}{2})} \sum_{r=0}^{m} \frac{(-1)^r}{r!\ \ (n-r)!} [\lceil(\alpha-2m+r)]^{-1}$$

$$\int_y^x e^{b(x-t)}(x-t)^{\alpha-2m+r-1} 1F2\left[-\left(m+\frac{1}{2}\right); \frac{1}{2}(\alpha-2m+r),\right]$$
$$-\frac{1}{2}(\alpha+r+1-2m); -\frac{1}{4}a^2(x-t)^2 f_2(t,y)dt.$$

Theorem 6: If a and b are reals, n is a non-negative integer,

$$Re\alpha > Re(2m) > -1.$$

$$f_1(x,y) = \int_y^x e^{b(x-t)}(x-t)^{\alpha} \Big\lfloor_{n}^{\alpha}(x-t), g(t-y)dt \tag{2.14}$$

$$f_2(x,y) = \int_y^x e^{b(x-t)}(x-t)^m I_m(ax-at)g(t-y)\,dt \tag{2.15}$$

$$f_1(x,y) = \frac{\lceil(\alpha+n+1).\ \pi^{\frac{1}{2}}}{(2a)^m \lceil(m+\frac{1}{2})} \sum_{r=0}^{n} \frac{(-1)^r}{r!\ \ (n-r)!} [\lceil(\alpha-2m+r)]^{-1}$$

$$\int_y^x e^{b(x-t)}(x-t)^{\alpha-2m+r-1} \cdot {}_1F_2\left[-\left(m+\frac{1}{2}\right); \frac{1}{2}(\alpha+r-2m), \frac{1}{2}(\alpha+r+1\right.$$
$$\left.-\,2m); \frac{1}{4}a^2(x-t)^2\right]. f_2(t,y)dt.$$

Proof: Here the proof is traditional and the operational results exploited are

$$t^{\alpha}e^{\lambda t}\left\lfloor_{n}^{\alpha}(kt), Re\ \alpha > -1 = \frac{\lceil(\alpha+n+1)}{n!} - \frac{(p-k-\lambda)^{n}}{(p-\lambda)^{\alpha+n+1}}\right., Re(p-\lambda) > 0$$

$$t^{\nu}J_{n}(\alpha t), Re\nu > \frac{-1}{2} = 2^{\nu}\pi^{\frac{-1}{2}}\left\lceil(\nu+\frac{1}{2})\ \alpha^{\nu}r^{-(2\nu+1)}Rep > |I_{m}\alpha|\right.$$

and

$$[\lceil(2\lambda+2\nu)]^{-1}t^{2\lambda+2\nu-1} \times = ._{1}F_{2}\left[\nu;\lambda+\nu;\ \lambda+\nu+\frac{1}{2};\frac{-1}{4}\alpha^{2}t^{2}\right]$$

$$= s^{-2\lambda}(p^{2}+\alpha^{2})^{-\nu}, Re(\nu+\lambda) > 0$$

$$\Rightarrow f_{1}(x,y) = \frac{\lceil(\alpha+n+1).\ \pi^{\frac{1}{2}}}{(2a)^{m}\left\lceil(m+\frac{1}{2})\right.}\sum_{r=0}^{n}\frac{(-1)^{r}}{r!\ \ (n-r)!}[\lceil(\alpha-2m+r)]^{-1}$$

$$\int_{y}^{x}e^{b(x-t)}(x-t)^{\alpha-2m+r-1}._{1}F_{2}\left[-\left(m+\frac{1}{2}\right);\frac{1}{2}(\alpha+r-2m),\frac{1}{2}(\alpha+r+1 - 2m);\frac{1}{4}a^{2}(x-t)^{2}\right]f_{2}(t-y)dt.$$

Theorem 7: If a is real, n is a positive integer, n

$$Re\ \alpha > -1\ , \qquad Re\ m, > -1, \qquad g(x) \in c\ \ in\ 0 \le x < \infty$$

$$f_{1}(x,y) = \int_{y}^{x}(x-t)^{\alpha}\lfloor_{n}^{\alpha}(x-t)g(t-y)dt \tag{2.16}$$

$$f_{2}(x,y) = \int_{y}^{x}(x-t)^{\frac{m}{2}}I_{m}(x-t)g(t-y)dt \tag{2.17}$$

then

$$\int_{y}^{x}e^{(x-t)}(x-t)^{n-1}f_{1}(t-y)dt = \frac{\lceil(\alpha+n+1)}{n.\ \ a^{\frac{1}{2}}\ \ (n+\alpha+k-1)}$$

$$\int_{y}^{x}(x-t)^{\frac{1}{2}(\alpha+n+s-1)}I_{n+\alpha+s-1}\left\{2\sqrt{(ax-at)}\right\}D^{k}f_{2}(t,y)dt$$

Where m = k-s, k being a non-negative integer and $0 \le s > 1\ and D \equiv \frac{d}{dt}$.

Proof: The transform (2.16) and (2.17) on being processed through the substitutions employed in the result

If$E_{n}(x^{a})$ a met tag-Lefflev;s function $a > 1, g(x) \in cin\ 0 \le x < \infty and$

$$\mathrm{f}(x,y) = \int_{y}^{x}E_{a}(x-t)^{a}g(t-y)dt$$

then

$$g(x-y) = df(x,y) - [\lceil(a-1)]^{-1}\int_{y}^{x}(x-t)^{a-2}f(t,y)dt$$

Where $D \equiv \frac{d}{dt}$, $u = (x - y)$

If the result

$$f_1(x,y) = \int_0^u (u-z)^{\alpha} \lfloor_n^{\alpha}(u-z) g(z) dr = \phi_1(u) say \tag{2.18}$$

and

$$f_2(x,y) = \int_0^u (u-z)^{\frac{m}{2}} I_m(2\sqrt{(au-az)} g(z) dz = \phi_2(u)\, d(say) \tag{2.19}$$

These transform in the light of the convolutions theorem and the operational results

$$t^{\alpha} \lfloor_n^{\alpha}(t), Re(\alpha) > -1 = \frac{\lceil(n+\alpha+1)}{n!} - \frac{(p-1)^n}{p^{(n+\alpha+1)}} , Re\ \ p > 0$$

and

$$t^{\frac{\nu}{2}} J_{\nu}\left(2\alpha^{\frac{1}{2}} + \frac{1}{2}\right) = \alpha^{\frac{\nu}{2}} \bar{p}^{(\nu+1)} e^{-\frac{\alpha}{s}}, Re\, p > 0.$$

$$\Rightarrow \phi_1(p) = \frac{\lceil(n+\alpha+1)}{n!} - \frac{(p-1)^n}{p^{n+\alpha+1}} G(p)$$

and

$$\phi_2(p) = a^{\frac{m}{2}} p^{-m-1} e^{\frac{-a}{p}} G(p), Re\, p > o.$$

The G(p) eliminates of these

$$(p-1)^n \phi_1(p) = \frac{\lceil(n+\alpha+1)}{n!\ a^{\frac{m}{2}}} e \frac{a}{p} p^{-n-\alpha-s} p^{-k} \phi_2(p) \tag{2.20}$$

Where k is the r least non-negative integer no less than m,

$0 \le p < 1$ and $m = k - p$.

Now the lowest degree of (u-z) in the expansion of $(u-z)^{\frac{m}{2}} I_m\left(2\sqrt{(au-az)}\right)$is m

Hence if$k > 0, \phi_2(u)$and its (k-1) d.c. w.r.t. u vanish at (u-z).

Also as $(x) \in c$. We have$\phi_2(u) \in c^k$.

$$D^k \phi_2(u) = \frac{k}{s} \phi_2(p)$$

The inversion(2.20)in the light of the result

$$t^{\frac{\nu}{2}} I_{\nu}\left(2\alpha^{\frac{1}{2}} + \frac{1}{2}\right), Re\, \nu - 1 = \alpha^{\frac{\nu}{r}} p^{-(\nu+1)} e^{\frac{\alpha}{p}}, Re\, p > 0, P = (p^2 - \alpha^2)^{\frac{1}{2}} S = s + p$$

$$\int_0^u e^{(u-z)} (u-z)^{n-1} \phi_1(z)\, dz = \frac{\lceil(n+\alpha+1)}{na^{\frac{m}{2}}} \int_0^u \frac{(u-z)^{\frac{1}{2}(n+\alpha+P-1)}}{a^{\frac{1}{2}}_{2}(n+\alpha+P-1)} I_n + \alpha + P - 1\left(2\sqrt{(au-az)}\right) D^k \phi_2(z)\, dz.$$

This required results in restoring to the original variables.

Theorem 8: If n is a positive integer,

$$Re\ \alpha > -1, \quad Re\ u > -1;\ a, b > 0, \quad g(x)\ \epsilon\ c\ \ in\ \ 0 \le x < \infty$$

$$f_1(x,y) = \int_y^x (x-t)^{\alpha} \lfloor_n^{\alpha}(x-t) g(t-y)dt \tag{2.21}$$

$$f_2(x,y) = \int_y^x (x-t)^{\frac{m}{2}} I_m \left(2\sqrt{(ax-at)}\, g(t-y)dt \right. \tag{2.22}$$

Then

$$\int_y^x e^{(x-t)}(x-t)^{n-1} f_1(t,y)dt = \frac{\lceil(n+\alpha+1)}{n\, a^{\frac{-1}{2}}\ \ (n+\alpha+k-1)}$$

$$\int_y^x (x-t)^{\frac{1}{2}(n+\alpha+p-1)} I_{n+\alpha+p-1} 2\sqrt{(ax-at)}\, D^k f_2(t,y)dt$$

Where $m = k - p$,k being a non-negative integer and

$$0 \le p < 1,\ \ \frac{d}{dt} = D.$$

The proof & is quite similar to that of the preview theorem,

Here we use the operational results that need

$$t^{\frac{\nu}{2}} I_\nu \left(2\alpha^{\frac{1}{2}} + t\frac{1}{2}\right),\ Re\ \nu > -1 = \alpha^{\frac{\nu}{s}} p^{-(\nu+1)} e^{\frac{\alpha}{s}},\ \ Re\ p > 0 \tag{2.23}$$

Where, $sp = (p^2 - \alpha^2)^{\frac{1}{2}}\ Rep\ \emptyset\ (s) = p + s$

$$t^{\alpha} \lfloor_n^{\alpha}(t), Re\alpha > -1 = \frac{\lceil(\alpha+n+1)}{\lfloor n} - \frac{(p-1)^n}{p^{(\alpha+n+1)}}, Re(p - \lambda) > 0 \tag{2.24}$$

$$t^{\frac{\nu}{2}} I_\nu \left(2\alpha^{\frac{1}{2}} + t\frac{1}{2}\right) = \alpha^{\frac{\nu}{2}} p^{-(\nu+1)} e^{\frac{\alpha}{p}}, Rep > 0\ Re\ \nu > -1,$$

We get the inversion result of (2.23) and (2.24) becomes

$$\int_y^x e^{(x-t)}(x-t)^{n-1} f_1(t,y)\ dt = \frac{\lceil(n+\alpha+1)}{n.\ e^{\frac{-1}{2}(n+\alpha+k-1)}} \int_y^x (x - t)^{\frac{1}{2}(n+\alpha+s-1)} I_{n+\alpha+s-1} \left(2\sqrt{(ax-at)}\, D^K f_2(t,y)\ dt. \right.$$

Theorem 9: If a and b are real, m and n are non-negative integers,

$$Re\ \alpha > -1, Re\ m\ > -1, \quad a > 0, \quad g(x) \in c\ in\ 0 \le x < \infty$$

$$f_1(x,y) = \int_y^x e^{b(x-t)}(x-t)^{\alpha} \lfloor_n^{\alpha}[K(x-t)]\ g(t-y)dt \tag{2.25}$$

$$f_2(x,y) = \int_y^x e^{b(x-t)}(x-t)^m I_m(ax-at) g(t-y)dt \tag{2.26}$$

then

$$\int_y^x e^{(b+k)(x-t)}(x-t)^{n-1}f_1(t,y)dt$$

$$= \frac{\lceil(\alpha+n+1)\alpha^{\frac{1}{2}}}{n\ e^m \left\lceil\left(m+\frac{1}{2}\right)\right.} \sum_{r=0}^{m+\frac{1}{2}} \binom{m+\frac{1}{2}}{r} \frac{a^{2r-m}}{\lceil(2r+\alpha+n-2m)}$$

$$\int_y^x e^{b(x-t)}(x-t)^{2r+\alpha+n-2m} f_2(t,y)dt$$

Proof: We have the operational results exploited here

$$t^{\nu-1}e^{\alpha t}, \quad Re(\nu) > 0 = \lceil(\nu).(p+\alpha)^{-\nu}, \quad Re(p) > -Re\ \alpha$$

$$t^{\alpha} \left\lfloor_n^{\alpha}(t), \quad Re\ \alpha > -1 = \frac{\lceil(n+\alpha+1)}{\lfloor n} - \frac{(p-1)^n}{p^{(n+\alpha+1)}}, \quad Re\ p > 0\right.$$

and

$$t^{\frac{\nu}{2}}I_\nu\left(2\alpha^{\frac{1}{2}}+\frac{1}{2}\right),\ Re\ \nu > -1 = e^{\frac{\nu}{2}}p^{-(\nu+1)}e^{-\frac{\alpha}{p}},\ Re\ p > 0 \tag{2.27}$$

Using this operation results, we get the inversion

$$\int_y^x e^{(b+k)(x-t)}(x-t)^{n-1}f_1(t,y)dt$$

$$= \frac{\lceil(\alpha+n+1)\alpha^{\frac{1}{2}}}{n\ e^m \left\lceil\left(m+\frac{1}{2}\right)\right.} \sum_{r=0}^{m+\frac{1}{2}} \binom{m+\frac{1}{2}}{r} \frac{a^{2r-m}}{\lceil(2r+\alpha+n-2m)}$$

$$\int_y^x e^{b(x-t)}(x-t)^{2r+\alpha+n-2m} f_2(t,y)dt$$

Theorem 9: `If Re n $> Re\ m > -1, \alpha\ real,\ g$(x) ∈ C in 0 ≤ x < α

$$f_1(x,y) = \int_y^x e^{\beta(x-t)}I_n[\alpha(x-t)]\ g(t-y)dt \tag{2.28}$$

$$f_2(x,y) = \int_y^x e^{\beta(x-t)}I_m\lceil\alpha(x-t)\ g(t-y)dt \tag{2.29}$$

then

$$f_1(x,y) = (n-m)\int_y^x e^{\beta(x-t)}(x-t)^{-1}I_{n-m}\lceil\alpha(x-t)f_2(t-y)dt$$

Proof: We have the following operational results

$$e^{\alpha t}f(t) = g(p+\alpha)$$

and

$$\mathrm{I}_\nu(\alpha t), \quad Re\ \nu - 1, = r^{-1}\left(\frac{\alpha}{r}\right)^\nu, \quad Re\ p > |I_{m\alpha}| r = (p^2+\alpha^2)^{\frac{1}{2}}\ R = p + r$$

Using the operational results, the inversion becomes

$$f_1(x,y) = (n-m)\int_y^x e^{\beta(x-t)}(x-t)^{-1}I_{n-m}[\alpha(x-t)]f_2(t,y)dt.$$

Conflict of Interest:

The authors declare that there are no conflicts of interest regarding this paper.

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