JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 Vol.-19, No.-8, August (2024) pp 60-66 ISSN (Print) 0973-8975

EXTENSION OF SOME INTEGRAL TRANSFORM BY THE METHOD OF MULTIPLE INTEGRALS

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https://doi.org/10.26782/jmcms.2024.08.00005

(Received: June 11, 2024; Revised: July 26, 2024; Accepted: August 09, 2024)

Abstract

Extension of some Integral Transform by the Method of Multiple Integrals by Lebesgue measurable and Lebesgue integrable.

Keywords: Fourier transform, Inverse Fourier transform, and Lebesgue measurable.

I. Introduction

The extension of some integral transform, we will express the result of the form

$$\int_E \int_F \int_G \alpha(x)\beta(y)\gamma\left(\sum_{i=1}^{k} u_i v_i z_i, xy\right)\rho(a,b,u,v)\,dxdyuddv \tag{1.1}$$

Here $G$is the path of integration, γ is the kennel of the integral transform and $z_i's$ and xy may be real or complex. We will want to interchange the order of integration of the equation of (1.1) we get

$$\int_G \beta(y)\int_F \alpha(x)\int_E \gamma\left(\sum_{i=1}^{k} u_i v_i z_i, xy\right)\rho(a,b,u,v)\,dxdyuddv$$

$$= \int_G \alpha(x)\int_F \beta(y)\rho(u_i v_i z_i, xy)\,dydx = \int_G \alpha(x)\gamma(a,b,z,x)\,dx,\ \mathrm{Re}(a), \mathrm{Re}(b) > 0.$$

Where γ is the k-variable analogue of the kernelϕ . The following lemma provides the general condition on α and γ which are sufficient for the validity of the interchange of order of integration [I-VI].

Lemma 1: Let $z_1, z_2, ...z_k$ be fixed complex numbers and let G be a connected subset of a straight line. Assume that the Complex-valued function $\alpha(x)$ is Lebesgue measurable on$G$, $\gamma(\sum_{i=1}^{k}(u_i v_i z_i, xy))$ is measurable and

$$\left|\gamma(\sum_{i=1}^{k}(u_i v_i z_i, xy) \le f(x) f(y)\right|$$ where$|\gamma|f$ is Lebesgue integrable [VII].

Then for

$$\int_F\int_G\int_H \alpha(x)\beta(y)\gamma(\sum_{i=1}^{k} u_i v_i z_i, xy)\rho(a,b,u,v)dxdyuddv$$

$$= \int_H \alpha(x)\int_G \beta(y)\int_F \gamma(\sum_{i=1}^{k}(u_i v_i z_i, xy))\rho(a,b,u,v)dudvdydx$$

$$= \int_H \alpha(x)\gamma(a,b,z,x,y)dx$$

Where γ is the k-variable analogue of α .

Proof: The function $\rho(a, b, u, v)$ is measurable on G and therefore, by the assumption, that the integrand of the inner integral is measurable, we have

$$\int_F\int_G\int_H |\alpha(x)\beta(y)\gamma(\sum_{i=1}^{k}(u_i v_i z_i, xy)\rho(a,b,u,v)|dxdyuddv$$

$$= \int_E |\rho(a,b,u,v)|\int_F |\alpha(x)|\int_G |\beta(y)||\gamma(\sum_{i=1}^{k}(u_i v_i z_i, xy))\rho(a,b,u,v)|dudvdydx$$

$$\le \int_E |\rho(a,b,z,x,y)|\int_F |\alpha(x)|\int_G |\beta(y)| f(y)dy \le \int_E |\rho(a,b,u,v)|dv\int_G |\alpha(x)| f(x)dx$$

$$\le \frac{B\{\mathrm{Re}(a),\mathrm{Re}(b)\}}{B(a)B(b)}\int_G \left|\alpha(x) f(x)dx\right|$$

$$< \infty, \mathrm{Re}(a), \mathrm{Re}(b) > 0.$$

Using Fubini's Theorem (1966), we may interchange the order of integration, we get,

$$\int_F\int_G\int_H \alpha(x)\beta(y)\gamma(\sum_{i=1}^{k} u_i v_i z_i, xy)\rho(a,b,u,v)dxdyuddv$$

$$= \int_E \alpha(x)\int_F \beta(y)\int_G \gamma(\sum_{i=1}^{k}(u_i v_i z_i, xy))\rho(a,b,u,v)dudvdydx$$

$$= \int_G \alpha(x)\gamma(a,b,z,x,y)dx, \quad \mathrm{Re}(a), \mathrm{Re}(b) > 0.$$

Now in the case of the Fourier transform of real variables, the ordinary Fourier transform of $\hbar$, defined as

$$\bar{\hbar}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hbar(t)\, exp(-ixt)dt, \; x \in R \tag{1.2}$$

Hence $\bar{\hbar}$ may be generalized to a function $\bar{H}$ of k real variables by the method of multiple integrals, we have

$\bar{H}(a,b,x,y) = \int_E \int_F \int_G \bar{\hbar}\left(\sum_{i=1}^{k}(u_i\, v_i z_i, x, y)\right)\rho(a,b,u,v)dudv\; , Re(a), Re(b) > 0$ (1.3)

where$x = (x_1, x_2 - - - x_k) \in R^k$ means the k-th cartesian power of R.

II. Main results:

Theorem (1):

Let$\hbar \in L$ of $Re(a), Re(b) > 0$ and $x \in R^k$, then

$\mathrm{H}(a,b,x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hbar\,(t) s(a,b,-ixt)dt,$

Where $s(a,b,-ixt) = s\;(a_1, - - -a_k, b_1, - - -b_k, -ix_1 t - - - ix_k t)$

is obtained by replacing $t\; by - it$ and $z\; by\; x$.

Proof: On replacing x by $\sum_{i=1}^{k} u_i\, v_i x_i y_i$

$$\bar{H}(a,b,x) = \frac{1}{\sqrt{2\pi}} \int_E \int_{-\infty}^{\infty} \hbar(t)\exp(-it\left(\sum_{i=1}^{k} u_i\, v_i x_i y_i\right)\rho(a,b,u,v)dtdudv$$

is measurable on R. The function $exp(-it \sum_{i=1}^{k} u_i\, v_i x_i y_i)$ is continuous on R.

Since $|exp(-it \sum_{i=1}^{k} u_i\, v_i x_i y_i = 1|$, then

$$\bar{H}(a,b,x) = \int_{-\infty}^{\infty} \left|\hbar(t)exp(-it \sum_{i=1}^{k} u_i\, v_i x_i y_i)\right| dt = \int_{-\infty}^{\infty} |\,\hbar(t)|dt < \infty.$$

In Lemma (1), we put E= $R and \gamma = R$ and observe that the exponential functionα is majored by$f(t) = 1$. Also the Fourier transform in the case of complex variables, if the assumption of the above lemma is imposed upon the function of $\hbar$, the Fourier transform of$\hbar$ function defined as

$$\bar{\hbar}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hbar\,(t)exp(-ist)dt, \qquad (1.4)$$

is analytic on either the lower half-plane. Then the analogue with the generalization result is defined as

$$\bar{H}(a,b,s) = \int_E \int_F \bar{\hbar}\;(\sum_{i=1}^{k} v_i v_i s_i)\;\rho(a,b,u,v)dudv, Re(a), Re(b) > 0 \quad (1.5)$$

is analytic on D^K But in the case of inversion of Fourier transform defined as

$$\hbar(\mathrm{t}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{\hbar}(s)exp(its)dt, \qquad t \in R \qquad (1.6)$$

and its generalization to a function H of k real variables, if $t = (t_1, t_2 - - t_k) \in R^k$,

$$H(a,b,t) = \int_E \int_F \hbar\;(\sum_{i=1}^{K} u_i v_i t_i)\rho(a,b,u,v)dudv \qquad (1.7)$$

Then we have

$$\bar{H}(a,b,x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hbar\,(t)\int_E\int_F exp(-it)\bar{\hbar}\,(\sum_{i=1}^{k} u_i v_i x_i y_i)\rho(a,b,u,v)dvdudt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hbar\,(t)s(a,b,-ixt)dt, \qquad Re(a), Re(b) > 0$$

Here we note that if $x_1 = x_2 = --= x_k = \xi and y_1 = y_2 = - - y_K = \eta$.

Then $\bar{H}(a,b,x,y) = \bar{\hbar}(\xi).\bar{\hbar}(\eta)$.

Also, since $\bar{\hbar}(x) and \bar{\hbar}(y)$ are continuous on R.

Therefore $\bar{H}(a,b,x,y)$ are continuous in x and y on R^k.

Theorem (2):

Let$\hbar$ satisfy the assumptions of lemma and let D be the corresponding analytic function of $\bar{\hbar}$. If $Re(a) > 0, Re(b) > 0$ then

$$\bar{H}(a,b,x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hbar\,(t)s(a,b,-ist)dt, \;\; Re(a), Re(b) > 0.$$

Proof: On replacing s by $\sum_{i=1}^{K} u_i v_i s_i$ and putting in (1.5) we get

$$\bar{H}(a,b,s) = \frac{1}{\sqrt{2\pi}}\int_E\int_F\int_{-\infty}^{\infty} \hbar\,(t)exp(-it)\sum_{i=1}^{k} u_i v_i s_i)\rho(a,b,u,v)dtdudv$$

$$= \frac{1}{\sqrt{2\pi}}\int_E\int_F\int_0^{\infty} \hbar\,(-t)exp\left(it\sum_{i=1}^{k} u_i v_i s_i)\rho(a,b,u,v)dtdudv\right)$$

$$+\frac{1}{\sqrt{2\pi}}\int_E\int_F\int_0^{\infty} \hbar\,(-t)exp\left(it\sum_{i=1}^{k} u_i v_i s_i)\rho(a,b,u,v)dtdudv\right)$$

The integrand of the inner integral in each of the above terms is measurable if

$$s_i = u_i + \tau_i, \qquad let m = \min_i(\tau_i)\; and m = \max_i(\tau_i)\,.$$

Then $\int_0^{\infty} |\hbar\,(-t)exp\left(it\sum_{i=1}^{K} u_i v_i s_i)|dt = \int_0^{\infty} |\hbar\,(-t)exp\left(-t\sum_{i=1}^{k} u_i v_i \tau_i)dt\right)\right)$

$$\leq \int_0^{\infty} |\hbar\,(-t)exp(-tm)dt,$$

and $\int_0^{\infty} \hbar\,(t)exp\left(-it\sum_{i=1}^{K} u_i v_i s_i\right)dt =$

$\Big||exp\left(-it\sum_{i=1}^{k} u_i v_i \tau_i\right)| \leq \int_0^{\infty} |\hbar\,(t)\Big|\, |exp(tm)|dt.$

Under the assumption of lemma, $h(-t) = 0, 0 < t < \infty \; and \; m < 0$, so each of the integrals is finite. Under the assumption h vanishes onside a finite interval so these integrals are finite.

Using lemma to each of these integrals $\alpha(t) = h(-t) \; and f(t) = \; exp(-tm)$ for the first integral and $f(t) = exp(tm)$ For the second integral [VIII-XI].

Then

$$\bar{H}(a,b,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h\,(t) \int_E \int_F exp(-it) \sum_{i=1}^{k} u_i v_i s_i) \rho(a,b,u,v) dv du dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hbar\,(t) s(a,b,-ist) dt, \quad Re(a), Re(b) > 0$$

Here we note that if $\quad s_1 = s_2 = --= s_k \;\; then \bar{H}(a,b,s) = \bar{h}(z)$.

Also since $\bar{h}$is analytic, then$\bar{H}(a,b,s)$ is analytic

Theorem (3): Let$\bar{h}$(s) satisfy the condition of the above lemma if $Re(a), Re(b) > 0$ and $t \in R^k$, then

$$H(a,b,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}\,(s) s(a,b,ist) ds.$$

Proof: On replacing t by $\sum_{i=1}^{k} u_i v_i t_i$ then the equation (1.7) becomes

$$H(a,b,t) = \frac{1}{\sqrt{2\pi}} \int_E \int_F \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}\,(s) exp\left(is \sum_{i=1}^{k} u_i v_i t_i \right) \rho(a,b,u,v) ds du dv$$

$$= \frac{1}{\sqrt{2\pi}} \int_E \int_F \int_{-\infty}^{\infty} \bar{h}\,(\sigma + i\gamma) exp[i(\sigma + i\gamma) \left(\sum_{i=1}^{k} u_i v_i t_i) \rho(a,b,u,v) d\sigma du dv \right).$$

Since $\bar{h}(s)$ was assumed to be integrable n the line $\mathrm{Im}(s) = \gamma$, then $\bar{h}(\sigma + i\gamma)$, considered on a function of the real variable σ , is integrable on R. Then the integrand of the inner integral is measurable. Let $\bar{t} = \max_i \{|t_i|\}$. Then

$$\int_{-\infty}^{\infty} |\bar{h}\,(\sigma + i\gamma) exp\left[i(\sigma + i\gamma) \sum_{i=1}^{k} u_i v_i t_i \,\right] d\sigma = \int_{-\infty}^{\infty} |\bar{h}\,(\sigma + i\gamma) \Big| \, exp - \gamma \sum_{i=1}^{k} u_i t_i \,] d\sigma$$

$$\leq e^{|\gamma|\bar{t}} \int_{-\infty}^{\infty} |\bar{h}\,(\sigma + i\gamma)| \, d\sigma < \infty.$$

Using the above lemma with $f(t) = e^{|\gamma|\bar{t}}$ we have,

$$H(a,b,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}\,(t) \int_E \int_F exp\left(is \sum_{i=1}^{k} u_i v_i t_i \right) \rho(a,b,u,v) du dv ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{h}\,(s) s(a,b,ist) ds, \quad Re(a), Re(b) > 0$$

If $t_1 = t_2 = --= t_k = \xi, then H(a,b,t) = \hbar(\xi)$. Also since $\hbar(t)$ is continuous on R, then $H(a,b,t)$ is continuous is t R^k .

III. Conclusion

In this study, we developed some extensions of some Integral Transform by the Method of Multiple Integrals by Lebesgue measurable and Lebesgue integrable with some useful theorems and lemma.

Conflict of Interest:

The authors declare that there are no conflicts of interest regarding this paper.

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