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ANISOTROPIC PICONE IDENTITIES FOR HALF LINEAR CONFORMABLE ELLIPTIC EQUATIONS

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Abstract

This study is devoted to investigating the anisotropic picone identities for half-linear Conformable elliptic equations and the Hardy-type inequality. Further, we provide some results for the nonlinear analogue to Picone identity.

Keywords: Anisotropic picone identities, Conformable elliptic equations, Half-linear Conformable elliptic equations, Hardy-type inequality

I. Introduction

The oscillatory theory of partial differential equations signifies its initiation by P. Hartman and A. Winter in 1955. The oscillatory behaviour of partial differential equations garnered considerable interest in numerous studies during the earlier 20 years [XXIII], [XXIV], [XXV]. In the beginning, physical phenomena like the vibration of stings, the conclusion of heat in materials, transport processes, etc. were mostly described using partial differential equations[XIV], [XXVII]. Over the past two centuries, there has been a significant advancement in the concept of elliptic partial differential equations. Similar to a general PDE, an elliptic PDE may be nonlinear and have a non-constant coefficient. It has evolved into one of the most richly expanded fields of mathematics, along with electrostatics, heat and mass diffusion, hydrodynamics, and many more applications; see references there [II], [IX].

The identity of Picone played a significant role in proving the Sturm Comparison Theorem, as is widely acknowledged [XXI]. In the nonlinear framework, Tyagi [XXVI] has recently demonstrated a generalized version of Picone's identity.

For bi-harmonic operators on the Heisenberg group Jagmohan Tyagi [XIII] has created a non-linear counterpart of Picone's identity. Regarding qualitative issues like stability criteria, Morse index, and Picone identity for bi-harmonic operators, G.Dwivedi J. Tyagi [XII] has gathered several insightful comments. By using a differential identity violating multidimensional formula Jaroslav Jaros [XIX] has obtained caccioppoli-type estimates for a class of nonlinear differential operators, which includes the P-Laplacian as a specific example.

However, as shown by several studies, the study of anisotropic issues has also drawn the attention of experts in partial differential equations [V], [X], [XI], [XXII]. A time-independent variant of their equations will be utilized as a statistical model to represent the transmission of a widespread illness [VII], [VIII], [XVIII]. In 2004, Khalil et al. [XX] introduced a unique derivative known as the Conformable derivative that is substantially similar to classical derivatives. The use of fractional derivatives in the fluid dynamics traffic models eliminates the deficiency resulting from the presumption of continuous traffic flow [I], [XV], [XVII].

The classical Picone identity proved the homogeneous linear several-order differential system in [IV], [VI]. Allegretto [III] established the oscillation theory of solutions and the Sturmian Comparison Principle.

As far as we are aware, there hasn't been any work on the "Anisotropic Picone identity for half-linear Conformable elliptic equations" topic. In this article, we deal with Conformable elliptic equations for anisotropic Picone identities, which are in the form of

$$\sum_{i=1}^{n} \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha u}{\partial x_i^\alpha}\right),\ p_i > 1 \tag{1}$$

where $\alpha \in (0,1], u(x)$ is an absolutely continuous function with α - fractional derivative.

If $\beta = 1$ then equation (1) becomes a Conformable anisotropic equations.

If $\beta = 1, p_i = \mathrm{p}$ then equation (1) is Conformable p - Laplacian equations.

If $\beta = 1, p_i = 2$ then equation (1) is called a Conformable Laplacian equation.

If $\beta, \alpha = 1, p_i = 2$ then equation (1) is named Classical Laplacian.

We will now provide an analogue of the α-operator for Picone identity as

$$T_\alpha\big((a_1)T_\alpha(u)\big) + a_2 u = 0,$$
$$T_\alpha\big((A_1)T_\alpha(v)\big) + A_2 v = 0,$$

then the Oscillation Theory of Solution and the Sturmian Comparison Principle

$$T_\alpha\left(\frac{u}{v}(a_1T_\alpha(u)v - A_1uT_\alpha(v))\right) = (A_2 - a_2)u^2 + (a_1 - a_2)(T_\alpha(u))^2 + A_1\left(T_\alpha(u) - T_\alpha(v)\frac{u}{v}\right)^2 \qquad (2)$$

Furthermore, we create an equation (2) to the α-Laplacian operator for the differential function of u and v, which is provided by

$$\begin{aligned}\left(\nabla^\alpha u - \frac{u}{v}\nabla^\alpha v\right)^2 &= |\nabla^\alpha u|^2 + \frac{u^2}{v^2}|\nabla^\alpha v|^2 - 2\frac{u}{v}\nabla^\alpha u\nabla^\alpha v \\ &= |\nabla^\alpha u|^2 - \nabla^\alpha\left(\frac{u^2}{v}\right)\nabla^\alpha v\end{aligned} \qquad (3)$$

Following that, we extend equation (3) to a p-Laplacian operator for the differential function of u and v, which is given by

$$\begin{aligned}&|\nabla^\alpha u|^p + (p-1)\frac{u^p}{v^p}|\nabla^\alpha v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla^\alpha u\nabla^\alpha v|\nabla^\alpha v|^{p-2} \\ &= |\nabla^\alpha u|^{p-1} - \nabla^\alpha\left(\frac{u^p}{v^{p-1}}\right)|\nabla^\alpha v|^{p-2}\nabla^\alpha v.\end{aligned} \qquad (4)$$

The following describes the paper's structure: Section 2, covers the basic definition, characteristics, and auxiliary lemmas that are utilized to establish the theorem. In section 3, we establish fundamental theorems with corollaries and stated lemmas. The nonlinear case is introduced in the final section along with a few theorems.

II. Preliminaries

Here, we list a few characteristics and terminologies that will help us create our major result.

Definition: 2.1. Given $u: [0,\infty) \to \mathbb{R}$. A conformable fractional derivative of u of order α is given by

$$T_\alpha(u)(x) = \lim_{\epsilon\to 0}\frac{u(x+\epsilon x^{1-\alpha}) - u(x)}{\epsilon},\ \forall x > 0,\ \alpha \in (0,1).$$

If u can be α-differentiable in some $(0,a), a > 0$ and $\lim_{x\to 0^+} u^\alpha(x)$ exists, then we define

$$u^\alpha(0) = \lim_{x\to 0^+} u^\alpha(x).$$

Definition: 2.2. $I_\alpha^a(u)(x) = I_1^\alpha(x^{\alpha-1})(u)$, here the integral is the standard Riemann improper integral.

Properties: 2.1. Let $\alpha \in (0,1]$ and at some point $x > 0$. u and v will eventually be α-differentiable. Then,

(1) $T_\alpha(a_1u + a_2v) = a_1T_\alpha(u) + a_2T_\alpha(v),\ \forall a_1, a_2 \in \mathbb{R}$

(2) $T_\alpha(uv) = uT_\alpha(v) + vT_\alpha(u)$

(3) $T_\alpha(x^p) = px^{p-\alpha},\ \forall p \in \mathbb{R}$

(4) $T_\alpha(a) = 0,\ u(x) = a$ for every constant function.

(5) $T_\alpha\left(\frac{u}{v}\right) = \frac{vT_\alpha(u)-uT_\alpha(v)}{v^2}$

(6) If u is differential, then $T_\alpha(u(x)) = x^{1-\alpha\frac{du(x)}{dx}}$.

Definition: 2.3. Let v be a function with m variable $y_1, \dots \dots, y_m$, and the conformable partial derivative of v of order $0 < \alpha \leq 1$ in y_i is defined as follows

$$\frac{\partial^\alpha}{\partial y_i^\alpha} v(y_1, \dots \dots, y_m) = \lim_{\epsilon \to 0} \frac{u\left(y_1, \dots, y_{i-1}, y_i + \epsilon y_i^{1-\alpha} \dots, y_m\right) - v(y_1, \dots \dots, y_m)}{\epsilon}$$

Lemma: 2.1. Let $C_0 > 0$ and $p > 1$ then the inequality

$$C_0 v^{p-1} \leq -\Delta_p^\alpha v \tag{4}$$

has no positive solution in $W_{\text{loc}}^{1,p}(G)$ where $G = \mathbb{R}^N$ or $G = \mathbb{R}_+^N$.

Proof : We contend by contradiction. Suppose that v is a nonnegative solution to the equations (4). Let us assume that $G = \mathbb{R}_+^N$. Taking $\mathbb{R} > 0$ and $x_0 \in \mathbb{R}_+^N$ such that $\overline{B_\mathbb{R}(x_0)} \subset \mathbb{R}_+^N$ and $\lambda_1\left(B_\mathbb{R}(x_0)\right) \geq C_0$. Denoting by φ_1 the positive eigenfunction associated with $\lambda_1\left(B_\mathbb{R}(x_0)\right)$. Since $w > 0$ or $B_\mathbb{R}(x_0)$. Testing (4) by $\frac{\varphi_1^p}{f(v)}$ and integrating by parts, we get

$$C_0 \int_{B_\mathbb{R}(x_0)} \varphi_1^p \leq \int_{B_\mathbb{R}(x_0)} \left(-\Delta_p^\alpha v\right) \frac{\varphi_1^p}{v^{p-1}}$$
$$= \int_{B_\mathbb{R}(x_0)} p|\nabla^\alpha v|^{p-2} \frac{\varphi_1^p}{v^{p-1}} \nabla^\alpha v \nabla^\alpha \varphi_1 - (p-1) \int_{B_\mathbb{R}(x_0)} \frac{\varphi_1^p}{v^p} |\nabla^\alpha v|^p,$$

the proceeding inequality and picone's identity indicate that

$$C_0 \int_{B_\mathbb{R}(x_0)} \varphi_1^p - \int_{B_\mathbb{R}(x_0)} |\nabla^\alpha \varphi_1|^p \leq -\int_{B_\mathbb{R}(x_0)} \Psi_1(\varphi_1, v) \leq 0.$$

It follows that

$$C_0 \leq \frac{\int_{B_\mathbb{R}(x_0)} |\nabla^\alpha \varphi_1|^p}{\int_{B_\mathbb{R}(x_0)} \varphi_1^p} = \lambda_1\left(B_\mathbb{R}(x_0)\right)$$

which contradicts the proof.

III. Main Results

In this section, we have derived some results for anisotropic Picone identity and anisotropic hardy inequality.

Theorem: 3.1. Let $u \geq 0$ and $v > 0$ represent two continuously α-differentiable functions in $\Omega \in \mathbb{R}^n$, denoted by

$$\Psi_1(u,v) = \sum_{i=1}^n \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} - \sum_{i=1}^n \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u^{\beta+p_i-1}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\frac{\partial^\alpha v}{\partial x_i^\alpha} \tag{5}$$

$$\Psi_2(u,v) = \sum_{i=1}^n \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} + \sum_{i=1}^n (\beta+p_i-2)\left(\frac{u^{\beta+p_i-1}}{v^{\beta+p_i-1}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-1}$$

$$-\sum_{i=1}^n (\beta+p_i-1)\left(\frac{u^{\beta+p_i-2}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\frac{\partial^\alpha v}{\partial x_i^\alpha}\frac{\partial^\alpha u}{\partial x_i^\alpha} \tag{6}$$

here $p_i > 1, i$ takes the value $1,2 \ldots n$. Then

i) $\Psi_2(u,v) = \Psi_1(u,v)$,
ii) $\Psi_2(u,v) \geq 0$,
iii) $\Psi_2(u,v) = 0$ iff a.e in Ω, here c is a constant.

Proof :

$$\Psi_1(u,v) = \sum_{i=1}^n \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} - \sum_{i=1}^n \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u^{\beta+p_i-1}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\frac{\partial^\alpha v}{\partial x_i^\alpha}$$

$$= \sum_{i=1}^n \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} - \sum_{i=1}^n \left[\frac{(\beta+p_i-1)u^{\beta+p_i-2}\frac{\partial^\alpha u}{\partial x_i^\alpha}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\frac{\partial^\alpha v}{\partial x_i^\alpha}}{v^{\beta+p_i-2}}\right.$$

$$\left. - \frac{(\beta+p_i-2)u^{\beta+p_i-1}\frac{\partial^\alpha v}{\partial x_i^\alpha}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\frac{\partial^\alpha v}{\partial x_i^\alpha}}{v^{\beta+p_i-1}}\right]$$

$$= \sum_{i=1}^n \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} - \sum_{i=1}^n (\beta+p_i-1)\frac{u^{\beta+p_i-2}}{v^{\beta+p_i-2}}\frac{\partial^\alpha u}{\partial x_i^\alpha}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\frac{\partial^\alpha v}{\partial x_i^\alpha}$$

$$+ \sum_{i=1}^n (\beta+p_i-2)\frac{u^{\beta+p_i-1}}{v^{\beta+p_i-1}}\left|\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-1}\right.$$

$$= \Psi_2(u,v).$$

Now to verify $\Psi_2(u,v) \geq 0$.

$$\Psi_2(u,v) = \sum_{i=1}^{n} \left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-1} - \sum_{i=1}^{n} (\beta+p_i-2)\left(\frac{u^{\beta+p_i-2}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-2}\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|$$
$$+\sum_{i=1}^{n} (\beta+p_i-2)\left(\frac{u^{\beta+p_i-1}}{v^{\beta+p_i-1}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-1} \tag{7}$$
$$+\sum_{i=1}^{n} (\beta+p_i-1)\left(\frac{u^{\beta+p_i-2}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\left\{\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right| - \frac{\partial^\alpha u}{\partial x_i^\alpha}\frac{\partial^\alpha v}{\partial x_i^\alpha}\right\}$$
$$= I + II$$

where

$$I = \sum_{i=1}^{n} (\beta+p_i-2)\left[\frac{\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1}}{\beta+p_i-1} + \frac{\beta+p_i-2}{\beta+p_i-1}\left(\left(\frac{u}{v}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|\right)^{\beta+p_i-2}\right)^{\frac{\beta+p_i-1}{\beta+p_i-2}}\right]$$
$$-(\beta+p_i-1)\left(\frac{u^{\beta+p_i-2}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-2}\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|$$
$$II = \sum_{i=1}^{n} (\beta+p_i-1)\left(\frac{u^{\beta+p_i-2}}{v^{\beta+p_i-2}}\right)\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3}\left\{\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right| - \frac{\partial^\alpha u}{\partial x_i^\alpha}\frac{\partial^\alpha v}{\partial x_i^\alpha}\right\}$$

Young's inequality has given us

$$m_1 m_2 \leq \frac{m_1^{p_i}}{p_i} + \frac{m_2^{q_i}}{q_i} \tag{8}$$

in which $p_i > 1$ and $q_i > 1 (i = 1,2 \ldots n)$. Therefore $\frac{1}{p_i} + \frac{1}{q_i} = 1$.

Here $m_1 = \frac{1}{m_2^{p_i-1}}$.we take a

$$m_1 = \sum_{i=1}^{n} \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right| \quad m_2 = \sum_{i=1}^{n} \left(\frac{u}{v}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|\right)^{\beta+p_i-2}$$

to obtain

$$\sum_{i=1}^{n} (\beta+p_i-1)\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|\left(\frac{u}{v}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|\right)^{\beta+p_i-2}$$
$$\leq \sum_{i=1}^{n} (\beta+p_i-1)\left[\frac{\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1}}{\beta+p_i-1} + \frac{\beta+p_i-2}{\beta+p_i-1}\left(\left(\frac{u}{v}\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|\right)^{\beta+p_i-2}\right)^{\frac{\beta+p_i-1}{\beta+p_i-2}}\right] \tag{9}$$

∴ I become ≥ 0 form (9).

similarly, II ≥ 0 by virtue of $\sum_{i=1}^{n}\left(\left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right| - \frac{\partial^\alpha u}{\partial x_i^\alpha}\frac{\partial^\alpha v}{\partial x_i^\alpha}\right) \geq 0$.

Therefore, from (7) $\Psi_2(u,v) \geq 0$.

iii)Because $\Psi_1(u,v)$ is positive, we know that $\Psi_1(u,v) = 0$ and that $\Psi_2(u,v) = 0$. Finally, $\nabla\left(\frac{u}{v}\right) = 0$ which yields u = cv, here c is constant.

Lemma: 3.1. If $k_i > 0$ is a constant and a function $l_i(x)(i = 1,2 \dots n)$ that is satisfied by a differentiable function $v > 0$ in Ω satisfies

$$-\sum_{i=1}^{n} \frac{\partial^\alpha u}{\partial x_i^\alpha}\left(\left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha v}{\partial x_i^\alpha}\right) \geq \sum_{i=1}^{n} k_i l_i(x) v^{\beta+p_i-2} \tag{10}$$

In the case of $0 \leq u \in C_0{}'(\Omega)$ we have

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} d_\alpha x \geq \sum_{i=1}^{n} k_i \int_\Omega l_i(x) u^{\beta+p_i-1} d_\alpha x \tag{11}$$

Theorem: 3.2. (Anisotropic Hardy type inequality) If $\in C_0{}'(E), 1 < p_i < n, i = 1,2 \dots n$, and $E = \{x \in \mathbb{R}^n / x_i \neq 0, i = 1,2 \dots n\}$. Following,

$$\sum_{i=1}^{n} \int \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} d_\alpha x \geq$$

$$\sum_{i=1}^{n} \left(\frac{\beta+p_i-2}{\beta+p_i-1}\right)^{\beta+p_i-1} \int \frac{(\alpha(\beta+p_i-1)-(\beta+p_i-2))|u|^{\beta+p_i-1}}{|x_i|^{\alpha(\beta+p_i-1)}} d_\alpha x \tag{12}$$

Proof : With no reduction in generality we consider $0 \leq u \in C_0{}^\infty$. By using the lemma, we will give the introduction of the auxiliary function

$$v = \prod_{j=1}^{n} |x_j|^{\gamma_j} = |x_i|^{\gamma_i} \bar{v}_1 \tag{13}$$

where $\gamma_j = \frac{\beta+p_i-2}{\beta+p_i-1}$ and $\bar{v}_1 = \prod_{j=1}^{n} |x_j|^{\gamma_j}$.

Therefore $\sum_{i=1}^{n} \frac{\partial^\alpha v}{\partial x_i^\alpha} = \gamma_i \bar{v}_1 |x_i|^{\gamma_i-2} x_i$

$$\sum_{i=1}^{n} \left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3} = \sum_{i=1}^{n} \gamma_i^{\beta+p_i-3} \bar{v}_1^{\beta+p_i-3} |x_i|^{\gamma_i\beta+\gamma_i p_i-3\gamma_i-\alpha\beta-\alpha p_i+3\alpha}$$

$$\sum_{i=1}^{n} \left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \left|\frac{\partial^\alpha v}{\partial x_i^\alpha}\right| =$$

$$\sum_{i=1}^{n} \gamma_i^{\beta+p_i-2} \bar{v}_1^{\beta+p_i-2} |x_i|^{\gamma_i\beta+\gamma_i p_i-3\gamma_i-\alpha\beta-\alpha p_i+3\alpha+\gamma_i-2} x_i^{2-\alpha}$$

And $\sum_{i=1}^{n} \int \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} d_\alpha x$

$$\geq \sum_{i=1}^{n} \left(\frac{\beta+p_i-2}{\beta+p_i-1}\right)^{\beta+p_i-1} \int \frac{(\alpha(\beta+p_i-1)-(\beta+p_i-2))|u|^{\beta+p_i-1}}{|x_i|^{\alpha(\beta+p_i-1)}} d_\alpha x \tag{14}$$

Taking $k_i = \frac{\beta+p_i-2}{\beta+p_i-1}$ and $l_i(x) = \frac{1}{|x_i|^{\alpha(\beta+p_i-1)}}$ therefore by using lemma,

we get(12).

Corollary : 3.1. For $u \in C_o{}'(E)$ it proves,

$$\sum_{i=1}^{n} \int \left|\frac{\partial^{\alpha} u}{\partial x_i^{\alpha}}\right|^{\beta+p_i-1} d_{\alpha}x \geq n^{\beta+3-2\alpha} \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \int \frac{|u|^{\beta+1}}{|x_i|^{\beta+3-2\alpha}} d_{\alpha}x \tag{15}$$

Proof : In (12) with $p_i = 2$ (i=1 to n), consider the basic inequality

$$n\left[\sum_{i=1}^{n} \frac{1}{a_i}\right]^{-1} \leq \frac{1}{n}\left[\sum_{i=1}^{n} a_i\right] \text{ for } a_i \geq 0, i = 1,2..n$$

Now, substitute $a_i = |x_i|^{\beta+3-2\alpha}$

$$\sum_{i=1}^{n} \int_E \left|\frac{\partial^{\alpha} u}{\partial x_i^{\alpha}}\right|^{\beta+1} d_{\alpha}x = \sum_{i=1}^{n} \int_E \left|\frac{\partial^{\alpha} u}{\partial x_i^{\alpha}}\right|^{\beta+p_i-1} d_{\alpha}x$$

$$\geq \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \sum_{i=1}^{n} \int_E \left|\frac{\partial^{\alpha} u}{\partial x_i^{\alpha}}\right|^{\beta+1} \frac{1}{|x_i|^{\beta+3-2\alpha}} d_{\alpha}x$$

$$\geq \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \sum_{i=1}^{n} \int_E \left|\frac{\partial^{\alpha} u}{\partial x_i^{\alpha}}\right|^{\beta+1} \frac{n^{\beta+3-2\alpha}}{|x_i|^{\beta+3-2\alpha}} d_{\alpha}x$$

$$\geq n^{\beta+3-2\alpha} \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \sum_{i=1}^{n} \int_E \left|\frac{\partial^{\alpha} u}{\partial x_i^{\alpha}}\right|^{\beta+1} \frac{1}{|x_i|^{\beta+3-2\alpha}} d_{\alpha}x$$

IV. Picone's Identity of Nonlinear Analogue

This chapter will commence with a special set of elliptic equations that frequently appear in the heterogeneous catalyst dynamics in chemistry. Let will explain the linear relationship between u and v that emerges from the picone identity. Singular elliptic equations are discussed in further detail in [XVI] and its reference [IX]

Consider the elliptic equation's singular nonlinearity.

$$\begin{aligned} -\sum_{i=1}^{n} \frac{\partial^{2\alpha} u}{\partial x_i^{2\alpha}} &= f(v) \text{ in } \Omega \\ -\sum_{i=1}^{n} \frac{\partial^{2\alpha} v}{\partial x_i^{2\alpha}} &= \frac{(f(v))^2}{u^{\beta+p_i-1}} \text{ in } \Omega \end{aligned} \tag{16}$$

Theorem: 4.1. If (u, v) is a weak solution to (16), then f must be satisfied

$$f^{\alpha}(y) \leq (\beta + p_i - 1)\left[f(y)^{\frac{\beta+p_i-3}{\beta+p_i-2}}\right]$$

Then $u = c_1 v$, here c_1 is a constant.

Proof : Assuming (u, v) is a weaker solution to (16) that satisfies the criteria. Next, consider any φ_1 and φ_2 in $W_0^{1,p}(\Omega)$.

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right| \left|\frac{\partial^\alpha \varphi_1}{\partial x_i^\alpha}\right| d_\alpha x = \int_\Omega f(v)\varphi_1 d_\alpha x \tag{17}$$

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right| \left|\frac{\partial^\alpha \varphi_2}{\partial x_i^\alpha}\right| d_\alpha x = \int_\Omega \frac{(f(v))^2}{u^{\beta+p_i-2}}\varphi_2 d_\alpha x \tag{18}$$

Substituting u for $\varphi_1 = \mathrm{u}$ and $\varphi_2 = \frac{u^{\beta+p_i-2}}{f(v)}$ in (17) and (18). By using (17) we obtain

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} d_\alpha x = \int_\Omega f(v) u d_\alpha x$$

Substituting $\varphi_2 = \left(\frac{u^{\beta+p_i-1}}{v^{\beta+p_i-1}}\right)$ in (17), we get

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha u}{\partial x_i^\alpha} \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{\partial^\alpha}{\partial x_i^\alpha} \frac{u^{\beta+p_i-1}}{f(v)}\right) d_\alpha x$$
$$= \sum_{i=1}^{n} \int_\Omega \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u^{\beta+p_i-1}}{f(v)}\right) \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha u}{\partial x_i^\alpha} d_\alpha x$$
$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha u}{\partial x_i^\alpha} \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{\partial^\alpha}{\partial x_i^\alpha} \frac{u^{\beta+p_i-1}}{f(v)}\right) d_\alpha x = \int_\Omega f(v) u d_\alpha x$$

Then, we get

$$\sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|_i^p d_\alpha x = \int_\Omega u f(v) d_\alpha x$$
$$= \sum_{i=1}^{n} \int_\Omega \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u^{\beta+p_i-1}}{f(v)}\right) \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha v}{\partial x_i^\alpha} d_\alpha x$$

Hence we have

$$\int_\Omega \Psi_1(u,v) d_\alpha x = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-1} - \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u^{\beta+p_i-1}}{f(v)}\right) \left|\frac{\partial^\alpha u}{\partial x_i^\alpha}\right|^{\beta+p_i-3} \frac{\partial^\alpha v}{\partial x_i^\alpha} d_\alpha x = 0$$

By virtue of $\Psi_1(u, v)$ being positive, we know that $\Psi_1(u, v) = 0$ and $\sum_{i=1}^{n} \frac{\partial^\alpha}{\partial x_i^\alpha}\left(\frac{u}{v}\right) = 0$ as a result.

Theorem: 4.2. For $C_0 > 0$ and $\beta + p_i - 1 > 1$, then the inequality

$$C_0 f(v) \leq -\sum_{i=1}^{n} \frac{\partial^{2\alpha} v}{\partial x_i^{2\alpha}} \tag{19}$$

has no positive solution in $\mathrm{W}_{\mathrm{loc}}^{1,p}(R^N)$.

V. Conclusion

Following the paper, to obtain quantitative results we have largely concentrated on obtaining anisotropic picone identities for half linear conformable elliptic equations by using the p- laplacian operator and hardy type inequality. In spite of many studies based on nonlinearity, we have also made additional discussion on nonlinear analogue. Varied applications that are of interest can be applied for further higher-order studies. These results extend and complement some of the existing literature.

Conflict of Interest:

There was no relevant conflict of interest regarding this paper.

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