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THE TIME-FRACTIONAL PERTURBED NONLINEAR SCHRÖDINGER EQUATION WITH BETA DERIVATIVE

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Abstract

In this article, we extract the diverse solitary wave solutions to the time-fractional perturbed nonlinear Schrödinger equation describing the dynamics of optical solitons travelling through nonlinear optical fibers. The nonlinear fractional differential equation is transformed into a nonlinear differential equation using a traveling wave transformation relating to the beta derivative. After that, the resulting equation is explained using the extended Riccati equation method. Abundant soliton and soliton-type solutions are extracted, comprising trigonometric and hyperbolic functions. The nature of the solutions varies qualitatively depending on distinct parameters. Additionally, graphical representations of the constructed solutions exhibit various physical forms, including kink, bell-shaped, periodic, anti-coupon etc. Moreover, the achieved solutions play a significant role in interpreting wave propagation studies and are essential for validating numerical and experimental findings in the fields of nonlinear optics, quantum mechanics, engineering, etc.

Keywords: Beta Derivative, Extended Riccati Equation method, Optical Solitons, Time-fractional Perturbed Nonlinear Schrödinger Equation, Traveling Wave Transformation.

I. Introduction

In recent years, fractional calculus has gained widespread attention for its extensive applications in various fields, such as engineering, chemistry, medicine, physics, biology, control theory, etc. The memory and genetic characteristics of fractional differential operators have led to the development of useful methods for exploring a range of phenomena and processes through fractional-order equations. Researchers have extensively investigated traveling wave solutions to both integer and non-integer order nonlinear partial differential equations (NLPDEs), exploring their relevance in applied mathematics [XIV, X, IX, II]. Consequently, a thorough literature review reveals a reliable impact on constructing exact solutions for nonlinear fractional differential equations (NLFDEs) with physical significance. The numerous applications of NLFDEs, including viscoelasticity, heat conduction, electromagnetic waves, diffusion equations, electrode-electrolyte polarization, and biogenetics, have captured global interest in this area [XVII]. To obtain exact soliton solutions for fractional order NLPDEs, some efficient methods have been proposed, such as the sub-equation method [VI], (G'/G)-expansion method [XII], exp-function method [VII], modified auxiliary equation method [XIV], extended Sinh-Gordon equation expansion method [VIII], modified trial equation method [XVIII], extended tanh-function method [XXXVIII], Laplace optimized decomposition method [V], etc.

Laskin initially introduced the fractional Schrödinger equation, a fundamental concept in quantum mechanics [XVI]. The fractional nonlinear Schrödinger (NLS) equation is the most frequently used nonlinear model in nonlinear science, engineering, and technology due to its wide-ranging applications [XXIII, XIX, XI]. Optical soliton solutions of fractional NLS equations are frequently used in various branches of nonlinear science and engineering, such as optical fibers, optoelectronics, quantum electronics, photonics, Nanofibers, etc. [XXI, XV, XXVI, XIII]. Various numerical methods have been suggested in the literature to obtain approximate solutions for the fractional Schrödinger equations. The perturbed nonlinear Schrödinger equation (PNLSE) with fractional temporal evolution is expressed as [XXVII]:

$$i\left(\frac{\partial^{\beta} p}{\partial t^{\beta}}\right) + p_{xx} + \varrho_0 p|p|^2 + i(\varrho_1 p_{xxx} + \varrho_2 |p|^2 p_x + \varrho_3 (|p|^2)_x\, p) = 0, t > 0, 0 < \beta \leq 1. \quad (1)$$

Here in equation (1), the first term represents the time-fractional derivative in the sense of the beta derivative, where ϱ_1, ϱ_2 and ϱ_3 represent third-order dispersion, nonlinear dispersion, and version of nonlinear dispersion respectively. The time-fractional PNLSE describes the dynamics of optical solitons travelling via nonlinear optical fibers, as well as has applications in plasma physics, fluid dynamics, optics, and so on [XXV, I]. Time fractional PNLSE was discussed by Younis et al. [XXVII] to obtain dark, combined bright-dark, singular, and combined dark-singular optical dromions with the help of the extended Fan sub-equation method along with modified Riemann-Liouville derivatives. To examine the perturbed time-fractional nonlinear Schrödinger equation, analytical and semi-analytical wave solutions, the conformable fractional derivatives modified Khater technique, and the Adomian decomposition method was used by Wang et al [XXV]. Owyed et al. [XX] achieved trigonometric, hyperbolic, and rational function solutions exploring our governing model with the aid of generalized $\exp(-w\xi)$ and (G'/G^2)-expansion processes. Recently, various optical dromions for the time fractional PNLSE were established by Rizvi et al. [XXII] with the application of an extended modified auxiliary equation mapping method.

This research project is motivated by a keen interest in soliton solutions in the context of fractional-temporal perturbed nonlinear Schrödinger equation (PNLSE). The aim is to extensively investigate the unique properties and behaviours of solitons in the fractional-temporal framework and contribute to the knowledge of nonlinear optics and mathematical physics. The extended Riccati method [XIII] is used for this investigation.
The rest of the article is organized as follows: Section 2 contains the definition of beta derivatives. Section 3 gives a review of the method. Section 4 provides the mathematical analysis for our equation. Section 5 represents the graphical interpretations of the solutions. Section 6 devotes the conclusions based on the results.

II. The beta derivative

The beta-derivative is defined as [IV]:

$$ {}_0^{\mathrm{Z}}D_x^{\beta}\{f(x)\} = \lim_{\epsilon\to 0}\frac{f\left(x+\epsilon\left(x+\frac{1}{\Gamma(\beta)}\right)\right)^{1-\beta}-f(x)}{\epsilon},\ 0<\beta\le 1. \tag{2}$$

Some properties of the beta derivatives are:

i) Assuming that m and n are real numbers, $g\neq 0$ and f are two functions β-differentiable and $\beta\in(0,1]$, we have

$$ {}_0^{\mathrm{Z}}D_x^{\beta}\{mf(x)+ng(x)\} = m\,{}_0^{\mathrm{Z}}D_x^{\beta}f(x) + n\,{}_0^{\mathrm{Z}}D_x^{\beta}g(x)$$

ii) ${}_0^{\mathrm{Z}}D_x^{\beta}\{c\}=0$, for c any given constant.

iii) ${}_0^{\mathrm{Z}}D_x^{\beta}\{f(x)\cdot g(x)\} = g(x)\,{}_0^{\mathrm{Z}}D_x^{\beta}\{f(x)\} + f(x)\,{}_0^{\mathrm{Z}}D_x^{\beta}\{g(x)\}$.

iv) ${}_0^{\mathrm{Z}}D_x^{\beta}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,{}_0^{\mathrm{Z}}D_x^{\beta}\{f(x)\}-f(x)\,{}_0^{\mathrm{Z}}D_x^{\beta}\{g(x)\}}{g^2(x)}$.

v) ${}_0^{\mathrm{Z}}D_x^{\beta}\left(\frac{f(\zeta))}{g(x)}\right) = T\frac{df(\zeta)}{d\zeta}$, with $\zeta=\frac{T}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta}$, where T is a constant.

The proof of the above relations is given in [III].

Relation between classical and beta derivative

Considering $\epsilon=\left(x+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}h$, and $h\to 0$, when $\epsilon\to 0$, therefore we have

${}_0^{Z}D_x^{\beta}\{f(x)\} = \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \frac{df(x)}{dx}$. This is the relation between classical and beta derivative.

III. Outline of the scheme

Let us suppose a fractional order nonlinear partial differential equation by

$$\Theta\left(\mathrm{u}, \mathrm{M}_t^{\beta}\mathrm{u}, \mathrm{M}_x^{\beta}\mathrm{u}, \mathrm{M}_{tt}^{2\beta}\mathrm{u}, \mathrm{M}_{xx}^{2\beta}\mathrm{u}, \mathrm{M}_t^{\beta}\mathrm{u}\mathrm{M}_x^{\beta}\mathrm{u}, \ldots\right) = 0,\ 0 < \beta \leq 1, \tag{3}$$

where Θ is a polynomial in its arguments. The subsequent transformation

$$u = u(x,t) = U(\zeta),\ \zeta = \zeta(x,t), \tag{4}$$

turns Eq. (3) into a differential equation due to ζ as follows:

$$\Lambda(U, U', U'', U''', \ldots \ldots \ldots) = 0. \tag{5}$$

Integrating Eq. (5) if permits and set integral constant zero as we seek for soliton solutions.

Suppose the solution formula of Eq. (3) stands as

$$U(\zeta) = \frac{a_0 + \sum_{i=1}^{n}(a_i\phi^i(\zeta) + b_i\phi^{-i}(\zeta))}{c_0 + \sum_{i=1}^{n}(c_i\phi^i(\zeta) + d_i\phi^{-i}(\zeta))}, \tag{6}$$

where, $i = 1,2,3,\ldots,n$; the involved free constants Eq. (6) will be found later, imposing homogenous technique provides the value of n and $\phi = \phi(\zeta)$ satisfies the Riccati differential equation,

$$\phi'(\zeta) = \mu + \lambda\phi(\zeta) + \nu\phi^2(\zeta). \tag{7}$$

The solutions of Eq. (7) are available in [30].

By means of Eqs. (4) and (5) into Eq. (3) results in a polynomial in ϕ. The values of the unknown parameters in equations (4) and (5) can be determined by solving the algebraic equation formed by setting each coefficient to zero. Wave solutions for Equation 4 are derived using the parameter values and solutions of Eq. (5).

IV. Mathematical analysis

We assume the following changes in the wave variable:

$$p(x,t) = P(\zeta) \text{ and } \zeta = kx + \frac{u}{\beta}\left(t + \frac{1}{\Gamma\beta}\right)^{\beta}, \tag{8}$$

where k and u are constants. Now using Eq. (2) in Eq. (1), we get the subsequent result,

$$iuP_{\zeta} + k^2P_{\zeta\zeta} + \varrho_0 P|P|^2 + i(\varrho_1 k^3 P_{\zeta\zeta\zeta} + \varrho_2 k|P|^2P_{\zeta} + \varrho_3 k(|P|^2{}_{\zeta}P)) = 0, \tag{9}$$

where P is a complex function. We consider a change of wave variable as follows:

$$P(\zeta) = e^{i\zeta}R(\zeta), \tag{10}$$

where $R(\zeta)$ is a real function.

From Eq. (3) we can separate the real and imaginary parts as follows:

$$(k\varrho_2 - \varrho_0)R^3 + (u + k^2 - k^3\varrho_1)R + (3k^3\varrho_1 - k^2)R'' = 0. \quad (11)$$

$$(k(\varrho_2 + 2\varrho_3)R'R^2 + (u - 3k^3\varrho_1 + 2k^2))R' + \varrho_1 k^3 R''' = 0. \quad (12)$$

After integration of Eq. (12), we get,

$$\frac{k(\varrho_2+2\varrho_3)R^3}{3} + (u - 3k^3\varrho_1 + 2k^2)R + \varrho_1 k^3 R'' = 0. \quad (13)$$

From Eq. (11) and Eq. (13), we obtain the following relations,

$$\frac{k\varrho_2-\varrho_0}{\frac{1}{3}k(\varrho_2+2\varrho_3)} = \frac{(u+k^2-k^3\varrho_1)}{(u-3k^3\varrho_1+2k^2)} = \frac{(3k^3\varrho_1 - k^2)}{\varrho_1 k^3}. \quad (14)$$

From Eq. (14), we obtain,

$$k = \frac{\varrho_2+2\varrho_3-3\varrho_0\varrho_1}{6\varrho_1\varrho_3},\ u = 2(2k\varrho_1 - 1)k^2. \quad (15)$$

Finally, Eq. (13) and Eq. (15) can be written as

$$m_1R^3(\zeta) + m_2R(\zeta) + m_3R''(\zeta) = 0, \quad (16)$$

where, $m_1 = \frac{1}{3}k(\varrho_2 + 2\varrho_3)$, $m_2 = (u - 3k^3\varrho_1 + 2k^2)$, $m_3 = \varrho_1 k^3$.

Now using the homogenous balance principle, between R'' and R^3 in equation (10), we obtain $M = 1$. Therefore, the shape of the solution (6) reduces to the subsequent form

$$R(\zeta) = \frac{a_0+a_1\phi(\zeta)+\frac{b_1}{\phi(\zeta)}}{c_0+c_1\phi(\zeta)+\frac{d_1}{\phi(\zeta)}}. \quad (17)$$

Setting (17) and using equation (7), Eq. (16) reduces to a polynomial in $\phi(\zeta)$. Collecting the coefficients of this polynomial, set them to zero, and solve by Maple, as a result, we obtain the following values of the parameters:

$$a_0 = -\frac{c_0\left(\frac{\pm\lambda a_1\sqrt{-m_2m_1}}{m_2}-2\nu c_0\right)m_2}{\pm 2\nu c_0\sqrt{-m_2m_1}+\lambda a_1 m_1},\ b_1 = 0,\ c_1 = \pm\frac{a_1\sqrt{-m_2m_1}}{m_2},\ d_1 = 0,$$

where c_0, a_1 are arbitrary.

Inserting the values of the parameters into the solution (17) and then utilizing equation (10), we attain

$$P(\zeta) = \frac{m_2\{a_1c_0(2\nu\phi(\zeta)-\lambda)\sqrt{-m_2m_1}+\phi(\zeta)\lambda a_1^2 m_1+2\nu m_2c_0^2\}}{(2\sqrt{-m_2m_1}\nu c_0+\lambda a_1 m_1)(\sqrt{-m_2m_1}a_1\phi(\zeta)+c_0m_2)}e^{i\zeta}. \quad (18)$$

When $\lambda^2 - 4\mu\nu > 0$ and $\mu\nu \neq 0$ or $\lambda\nu \neq 0$.

$$P_1(\zeta) = \frac{m_2\left((2c_0\nu\theta+\lambda a_1m_1)\omega\, a_1\tanh((\omega/2)\zeta)+4a_1c_0\lambda\nu\theta+\lambda^2a_1^2m_1-4\nu^2m_2c_0^2\right)}{(2c_0\nu\theta+\lambda a_1m_1)(\theta\omega a_1\tanh((\omega/2)\zeta)+\lambda a_1\theta-2c_0m_2\nu)}e^{i\zeta}, \quad (19)$$

$$P_2(\zeta) = \frac{m_2\big(\omega a_1(2c_0\nu\theta+\lambda a_1 m_1)\coth((\omega/2)\zeta)+4c_0\nu a_1\lambda\theta-4c_0^2 m_2\nu^2+\lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta+\lambda a_1 m_1)(\coth((\omega/2)\zeta)\theta\omega a_1+\theta\lambda a_1-2c_0 m_2\nu)} e^{i\zeta}, \quad (20)$$

$$P_3(\zeta) = \frac{m_2\big((\tanh(\omega\zeta)+I\,\mathrm{sech}(\omega\zeta))(2\theta c_0\nu+\lambda a_1 m_1)\omega a_1+4\theta\lambda\nu a_1 c_0-4c_0^2 m_2\nu^2+\lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta+\lambda a_1 m_1)\{(\tanh(\omega\zeta)+I\,\mathrm{sech}(\omega\zeta))\theta\omega a_1+\theta\lambda a_1-2c_0 m_2\nu\}} e^{i\zeta}, \quad (21)$$

$$P_4(\zeta) = \frac{m_2\big((\tanh(\omega\zeta)-I\,\mathrm{sech}(\omega\zeta))(2\theta c_0\nu+\lambda a_1 m_1)\omega a_1-4\theta\lambda\nu c_0 a_1+4\nu^2 m_2 c_0^2-\lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta+\lambda a_1 m_1)\{(\tanh(\omega\zeta)-I\,\mathrm{sech}(\omega\zeta))\theta a_1\omega-\theta\lambda a_1+2c_0 m_2\nu\}} e^{i\zeta}, \quad (22)$$

$$P_5(\zeta) = \frac{m_2\big((\coth(\omega\zeta)+\mathrm{csch}(\omega\zeta))(2c_0\nu\theta+\lambda a_1 m_1)\omega a_1+4c_0\nu a_1\lambda\theta-4\nu^2 m_2 c_0^2+\lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta+\lambda a_1 m_1)\{(\coth(\omega\zeta)+\mathrm{csch}(\omega\zeta))\theta a_1\omega+\theta\lambda a_1-2c_0 m_2\nu\}} e^{i\zeta}, \quad (23)$$

$$P_6(\zeta) = \frac{m_2\big((\coth(\omega\zeta)-\mathrm{csch}(\omega\zeta))(2\theta c_0\nu+\lambda a_1 m_1)a_1\omega+4c_0\nu a_1\lambda\theta-4\nu^2 m_2 c_0^2+\lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta+\lambda a_1 m_1)\{(\coth(\omega\zeta)-\mathrm{csch}(\omega\zeta))\theta a_1\omega+\theta\lambda a_1-2c_0 m_2\nu\}} e^{i\zeta}, \quad (24)$$

$$P_7(\zeta) = \frac{m_2\begin{pmatrix}(\tanh((\omega/4)\zeta)+\coth((\omega/4)\zeta))(2c_0\nu\theta+\lambda a_1 m_1)\omega a_1+8c_0\nu a_1\lambda\theta \\ -8\nu^2 m_2 c_0^2+2\lambda^2 a_1^2 m_1\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)\{(\tanh((\omega/4)\zeta)+\coth((\omega/4)\zeta))\theta a_1\omega+2\theta\lambda a_1-4c_0 m_2\nu\}} e^{i\zeta}, \quad (25)$$

$$P_8(\zeta) = \frac{m_2\begin{pmatrix}2a_1c_0\theta\{-2\lambda(A\sinh(\omega\zeta)+B)+(\omega C-A\omega\cosh\omega)\}+4\nu^2 m_2 c_0^2(A\sinh(\omega\zeta)+B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sinh(\omega\zeta)+B)+(\omega C-A\omega\cosh(\omega\zeta))\}\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)\{2\nu c_0 m_2(A\sinh(\omega\zeta)+B)+a_1\theta(-\lambda(A\sinh(\omega\zeta)+B)+(\omega C-A\omega\cosh(\omega\zeta)))\}} e^{i\zeta}, \quad (26)$$

$$P_9(\zeta) = \frac{m_2\begin{pmatrix}2a_1c_0\theta\{-2\lambda(A\sinh(\omega\zeta)+B)+(-\omega C-A\omega\cosh\omega)\}+4\nu^2 m_2 c_0^2(A\sinh(\omega\zeta)+B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sinh(\omega\zeta)+B)+(-\omega C-A\omega\cosh(\omega\zeta))\}\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)\{2\nu c_0 m_2(A\sinh(\omega\zeta)+B)+a_1\theta(-\lambda(A\sinh(\omega\zeta)+B)+(-\omega C-A\omega\cosh(\omega\zeta)))\}} e^{i\zeta}, \quad (27)$$

$$P_{10}(\zeta) = \frac{m_2\begin{pmatrix}2a_1c_0\theta\{-2\lambda(A\sinh(\omega\zeta)+B)-(\omega C+A\omega\cosh\omega)\}+4\nu^2 m_2 c_0^2(A\sinh(\omega\zeta)+B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sinh(\omega\zeta)+B)-(\omega C+A\omega\cosh(\omega\zeta))\}\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)\{2\nu c_0 m_2(A\sinh(\omega\zeta)+B)+a_1\theta(-\lambda(A\sinh(\omega\zeta)+B)-(\omega C+A\omega\cosh(\omega\zeta)))\}} e^{i\zeta}, \quad (28)$$

$$P_{11}(\zeta) = \frac{m_2\begin{pmatrix}2a_1c_0\theta\{-2\lambda(A\sinh(\omega\zeta)+B)-(-\omega C+A\omega\cosh\omega)\}+4\nu^2 m_2 c_0^2(A\sinh(\omega\zeta)+B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sinh(\omega\zeta)+B)-(-\omega C+A\omega\cosh(\omega\zeta))\}\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)\{2\nu c_0 m_2(A\sinh(\omega\zeta)+B)+a_1\theta(-\lambda(A\sinh(\omega\zeta)+B)-(-\omega C+A\omega\cosh(\omega\zeta)))\}} e^{i\zeta}, \quad (29)$$

when A and B are two non-zero real constants and satisfy $B^2 - A^2 > 0$ and $C = \sqrt{(A^2 + B^2)}$.

$$P_{12}(\zeta) = \frac{m_2\begin{pmatrix}-\sinh((\omega/2)\zeta)\omega c_0(a_1\lambda\theta-2\nu m_2 c_0)+ \\ (-2\lambda\nu m_2 c_0^2+a_1((\theta(4\mu\nu+\lambda^2)c_0+2\mu\lambda a_1 m_1))\cosh((\omega/2)\zeta)\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)(\sinh((\omega/2)\zeta)\omega c_0 m_2+\cosh((\omega/2)\zeta)(2\mu a_1\theta-\lambda m_2 c_0))} e^{i\zeta}, \quad (30)$$

$$P_{13}(\zeta) = \frac{m_2\begin{pmatrix}-\cosh((\omega/2)\zeta)\omega c_0(a_1\lambda\theta-2\nu m_2 c_0)+ \\ (-2\lambda\nu m_2 c_0^2+a_1((\theta(4\mu\nu+\lambda^2)c_0+2\mu\lambda a_1 m_1))\sinh((\omega/2)\zeta)\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)(\cosh((\omega/2)\zeta)\omega c_0 m_2+\sinh((\omega/2)\zeta)(2\mu a_1\theta-\lambda m_2 c_0))} e^{i\zeta}, \quad (31)$$

$$P_{14}(\zeta) = -\frac{m_2\begin{pmatrix}(-4a_1\theta(4\mu\nu+\lambda^2)c_0-8\lambda(\mu a_1^2 m_1-\nu m_2 c_0^2))\cosh(\omega\zeta) \\ +4c_0(I+\sinh(\omega\zeta))(\lambda a_1\theta-2\nu c_0 m_2)\omega\end{pmatrix}}{4(2c_0\nu\theta+\lambda a_1 m_1)((2\mu a_1\theta-\lambda m_2 c_0)\cosh(\omega\zeta)+c_0(I+\sinh(\omega\zeta))\omega m_2)} e^{i\zeta}, \quad (32)$$

$$P_{15}(\zeta) = -\frac{m_2\begin{pmatrix}(-4a_1\theta(4\mu\nu+\lambda^2)c_0+8\lambda(\mu a_1^2 m_1-\nu m_2 c_0^2))\cosh(\omega\zeta) \\ +4c_0(I-\sinh(\omega\zeta))(\lambda a_1\theta-2\nu c_0 m_2)\omega\end{pmatrix}}{4(2c_0\nu\theta+\lambda a_1 m_1)((-2\mu a_1\theta+\lambda m_2 c_0)\cosh(\omega\zeta)+c_0(I-\sinh(\omega\zeta))\omega m_2)} e^{i\zeta}, \quad (33)$$

$$P_{16}(\zeta) = -\frac{m_2\begin{pmatrix}a_1c_0\theta\{4\nu\mu\sinh(\omega\zeta)-\lambda(-\lambda\sinh(\omega\zeta)+\omega\cosh(\omega\zeta)+\omega))+ \\ 2((\mu\sinh(\omega\zeta)\lambda a_1^2 m_1)+\nu m_2 c_0^2(-\lambda\sinh(\omega\zeta)+\omega\cosh(\omega\zeta)+\omega))\}\end{pmatrix}}{(2c_0\nu\theta+\lambda a_1 m_1)((2\mu a_1\theta\sinh(\omega\zeta))+c_0 m_2(-\lambda\sinh(\omega\zeta)+\omega\cosh(\omega\zeta)+\omega))} e^{i\zeta}, \quad (34)$$

$$P_{17}(\zeta) = \frac{m_2\begin{pmatrix} a_1 c_0 \theta\{4\nu\mu \sinh(\omega\zeta) - \lambda(-\lambda \sinh(\omega\zeta) + \omega \cosh(\omega\zeta) - \omega)) + \\ 2((\mu \sinh(\omega\zeta)\lambda a_1^2 m_1) + \nu m_2 c_0^2(-\lambda \sinh(\omega\zeta) + \omega \cosh(\omega\zeta) - \omega))\} \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)((2\mu a_1 \theta \sinh(\omega\zeta)) + c_0 m_2(-\lambda \sinh(\omega\zeta) + \omega \cosh(\omega\zeta) - \omega))} e^{i\zeta}, \quad (35)$$

$$P_{18}(\zeta) = \frac{m_2\begin{pmatrix} (2\cosh((\omega/4)\zeta)^2 - 1)c_0(\theta\lambda a_1 - 2\nu m_2 c_0)\omega - \\ 2(-2\lambda\nu m_2 c_0^2 + a_1(c_0\theta(4\mu\nu + \lambda^2) + 2\mu\lambda a_1 m_1)) \\ \sinh((\omega/4)\zeta)\cosh((\omega/4)\zeta) \end{pmatrix}}{\begin{matrix}(2c_0\nu\theta + \lambda a_1 m_1)(m_2\,(2\cosh((\omega/4)\zeta)^2 - 1)c_0\omega + 4\sinh((\omega/4)\zeta) \\ (a_1\mu\theta - (\lambda m_2 c_0/2))\cosh((\omega/4)\zeta))\end{matrix}} e^{i\zeta}, \quad (36)$$

where, $\omega = \sqrt{\lambda^2 - 4\mu\nu}$, $\theta = \sqrt{-m_2 m_1}$.

When $\lambda^2 - 4\mu\nu < 0$ and $\lambda\mu \neq 0$ or $\mu\nu \neq 0$,

$$P_{19}(\zeta) = \frac{m_2\big(\omega_1 a_1(2c_0\nu\theta + \lambda a_1 m_1)\tan(\omega_1\zeta/2) - 4c_0\nu a_1\lambda\theta + 4\nu^2 m_2 c_0^2 + \lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)(\tan(\omega_1\zeta/2)\theta a_1\,\omega_1 - \theta\lambda a_1 + 2c_0 m_2\nu)} e^{i\zeta}, \quad (37)$$

$$P_{20}(\zeta) = \frac{m_2\big(\omega_1 a_1(2c_0\nu\theta + \lambda a_1 m_1)\cot(\omega_1\zeta/2) + 4c_0\nu a_1\lambda\theta - 4\nu^2 m_2 c_0^2 + \lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)(\cot(\omega_1\zeta/2)\theta a_1\,\omega_1 + \theta\lambda a_1 - 2c_0 m_2\nu)} e^{i\zeta}, \quad (38)$$

$$P_{21}(\zeta) = \frac{m_2\big((\tan(\omega_1\zeta) - \sec(\omega_1\zeta))(2c_0\nu\theta + \lambda a_1 m_1)a_1\omega_1 - 4c_0\nu a_1\lambda\theta + 4\nu^2 m_2 c_0^2 - \lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)((\tan(\omega_1\zeta) - \sec(\omega_1\zeta))\theta a_1\omega_1 - \theta\lambda a_1 + 2c_0 m_2\nu)} e^{i\zeta}, \quad (39)$$

$$P_{21}(\zeta) = \frac{m_2\big((\tan(\omega_1\zeta) + \sec(\omega_1\zeta))(2c_0\nu\theta + \lambda a_1 m_1)a_1\omega_1 - 4c_0\nu a_1\lambda\theta + 4\nu^2 m_2 c_0^2 - \lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)((\tan(\omega_1\zeta) + \sec(\omega_1\zeta))\theta a_1\omega_1 - \theta\lambda a_1 + 2c_0 m_2\nu)} e^{i\zeta}, \quad (40)$$

$$P_{23}(\zeta) = \frac{m_2\big((\cot(\omega_1\zeta) + \csc(\omega_1\zeta))(2c_0\nu\theta + \lambda a_1 m_1)a_1\omega_1 + 4c_0\nu a_1\lambda\theta - 4\nu^2 m_2 c_0^2 + \lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)(\theta a_1\omega_1\,(\cot(\omega_1\zeta) + \csc(\omega_1\zeta)) + \theta\lambda a_1 - 2c_0 m_2\nu)} e^{i\zeta}, \quad (41)$$

$$P_{23}(\zeta) = \frac{m_2\big((\cot(\omega_1\zeta) - \csc(\omega_1\zeta))(2c_0\nu\theta + \lambda a_1 m_1)a_1\omega_1 + 4c_0\nu a_1\lambda\theta - 4\nu^2 m_2 c_0^2 + \lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)(\theta a_1\omega_1\,(\cot(\omega_1\zeta) - \csc(\omega_1\zeta)) + \theta\lambda a_1 - 2c_0 m_2\nu)} e^{i\zeta}, \quad (42)$$

$$P_{25}(\zeta) = \frac{m_2\big((\cot(\omega_1\zeta/4) - \tan(\omega_1\zeta/4))(2c_0\nu\theta + \lambda a_1 m_1)a_1\omega_1 + 8c_0\nu a_1\lambda\theta - 8\nu^2 m_2 c_0^2 + 2\lambda^2 a_1^2 m_1\big)}{(2c_0\nu\theta + \lambda a_1 m_1)((\cot(\omega_1\zeta/4) - \tan(\omega_1\zeta/4))\theta a_1\omega_1 + 2\theta\lambda a_1 - 4c_0 m_2\nu)} e^{i\zeta}, \quad (43)$$

$$P_{26}(\zeta) = \frac{m_2\begin{pmatrix} 2a_1c_0\theta\{-2\lambda(A\sin(\omega_1\zeta) + B) + (C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\} \\ +4\nu^2 m_2 c_0^2(A\sin(\omega_1\zeta) + B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sin(\omega_1\zeta) + B) + (C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\} \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\begin{Bmatrix} 2\nu c_0 m_2(A\sin(\omega_1\zeta) + B) \\ +a_1\theta\big((-\lambda(A\sin(\omega_1\zeta) + B) + (C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\big)\big) \end{Bmatrix}} e^{i\zeta}, \quad (44)$$

$$P_{27}(\zeta) = \frac{m_2\begin{pmatrix} 2a_1c_0\theta\{-2\lambda(A\sin(\omega_1\zeta) + B) + (-C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\} + 4\nu^2 m_2 c_0^2(A\sin(\omega_1\zeta) + B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sin(\omega_1\zeta) + B) + (-C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\} \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\begin{Bmatrix} 2\nu c_0 m_2(A\sin(\omega_1\zeta) + B) \\ +a_1\theta\big((-\lambda(A\sin(\omega_1\zeta) + B) + (-C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\big)\big) \end{Bmatrix}} e^{i\zeta}, \quad (45)$$

$$P_{28}(\zeta) = \frac{m_2\begin{pmatrix} 2a_1c_0\theta\{-2\lambda(A\sin(\omega_1\zeta) + B) - (C_1\omega_1 + A\omega_1\cos(\omega_1\zeta))\} + 4\nu^2 m_2 c_0^2(A\sin(\omega_1\zeta) + B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sin(\omega_1\zeta) + B) - (C_1\omega_1 + A\omega_1\cos(\omega_1\zeta))\} \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\begin{Bmatrix} 2\nu c_0 m_2(A\sin(\omega_1\zeta) + B) \\ +a_1\theta\big((-\lambda(A\sin(\omega_1\zeta) + B) + (C_1\omega_1 + A\omega_1\cos(\omega_1\zeta))\big)\big) \end{Bmatrix}} e^{i\zeta}, \quad (46)$$

$$P_{29}(\zeta) = \frac{m_2\begin{pmatrix} 2a_1c_0\theta\{-2\lambda(A\sin(\omega_1\zeta) + B) - (-C_1\omega_1 + A\omega_1\cos(\omega_1\zeta))\} + 4\nu^2 m_2 c_0^2(A\sin(\omega_1\zeta) + B) \\ +\lambda a_1^2 m_1\{-\lambda(A\sin(\omega_1\zeta) + B) - (-C_1\omega_1 + A\omega_1\cos(\omega_1\zeta))\} \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\begin{Bmatrix} 2\nu c_0 m_2(A\sin(\omega_1\zeta) + B) + \\ a_1\theta\big((-\lambda(A\sin(\omega_1\zeta) + B) - (-C_1\omega_1 - A\omega_1\cos(\omega_1\zeta))\big)\big) \end{Bmatrix}} e^{i\zeta}, \quad (47)$$

where in the solutions (45), (46), and (47), A and B are non-zero real constants that satisfy $A^2 - B^2 > 0$, $C_1 = \sqrt{(A^2 - B^2)}$.

$$P_{30}(\zeta) = \frac{m_2\begin{pmatrix} \sin(\omega_1\zeta/2)\omega_1 c_0(a_1\lambda\theta - 2\nu m_2 c_0) + \\ \cos(\omega_1\zeta/2)\,(-2\lambda\nu m_2 c_0^2 + (c_0\theta(4\mu\nu+\lambda^2) + 2\mu\lambda a_1 m_1)a_1) \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)(-\sin(\omega_1\zeta/2)\omega_1 c_0 m_2 + \cos(\omega_1\zeta/2)(2\mu a_1\theta - \lambda m_2 c_0))} e^{i\zeta}, \tag{48}$$

$$P_{31}(\zeta) = \frac{m_2\begin{pmatrix} -\cos(\omega_1\zeta/2)\omega_1 c_0(a_1\lambda\theta - 2\nu m_2 c_0) + \\ \sin(\omega_1\zeta/2)\,(-2\lambda\nu m_2 c_0^2 + (c_0\theta(4\mu\nu+\lambda^2) + 2\mu\lambda a_1 m_1)a_1) \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)(\cos(\omega_1\zeta/2)\omega_1 c_0 m_2 + \sin(\omega_1\zeta/2)(2\mu a_1\theta - \lambda m_2 c_0))} e^{i\zeta}, \tag{49}$$

$$P_{32}(\zeta) = \frac{m_2\begin{pmatrix} (a_1 c_0\theta(4\mu\nu+\lambda^2) + 2\lambda(\mu a_1^2 m_1 - \nu m_2 c_0^2))\cos(\omega_1\zeta) + \\ (1+\sin\omega_1\zeta)c_0\omega_1\,(\lambda a_1\theta - 2\nu c_0 m_2) \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)((2\mu a_1\theta - \lambda m_2 c_0)\cos\omega_1\zeta - (1+\sin\omega_1\zeta)c_0 m_2\omega_1)} e^{i\zeta}, \tag{50}$$

$$P_{33}(\zeta) = \frac{m_2\begin{pmatrix} (a_1 c_0\theta(4\mu\nu+\lambda^2) + 2\lambda(\mu a_1^2 m_1 - \nu m_2 c_0^2))\cos(\omega_1\zeta) + \\ (-1+\sin\omega_1\zeta)c_0\omega_1\,(\lambda a_1\theta - 2\nu c_0 m_2) \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)((2\mu a_1\theta - \lambda m_2 c_0)\cos\omega_1\zeta - (-1+\sin\omega_1\zeta)c_0 m_2\omega_1)} e^{i\zeta}, \tag{51}$$

$$P_{34}(\zeta) = \frac{m_2\begin{pmatrix} (c_0 a_1\theta(4\mu\nu+\lambda^2) + 2\lambda(\mu a_1^2 m_1 - \nu m_2 c_0^2))\sin(\omega_1\zeta) - \\ (1+\cos(\omega_1\zeta)c_0\omega_1\,(\lambda a_1\theta - 2\nu c_0 m_2) \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\{(2\mu a_1\theta - \lambda m_2 c_0)\sin(\omega_1\zeta) + \omega_1\, c_0 m_2(1+\cos(\omega_1\zeta))\}} e^{i\zeta}, \tag{52}$$

$$P_{35}(\zeta) = \frac{m_2\begin{pmatrix} (c_0 a_1\theta(4\mu\nu+\lambda^2) + 2\lambda(\mu a_1^2 m_1 - \nu m_2 c_0^2))\sin(\omega_1\zeta) - \\ (-1+\cos(\omega_1\zeta)c_0\omega_1\,(\lambda a_1\theta - 2\nu c_0 m_2) \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\{(2\mu a_1\theta - \lambda m_2 c_0)\sin(\omega_1\zeta) + \omega_1\, c_0 m_2(-1+\cos(\omega_1\zeta))\}} e^{i\zeta}, \tag{53}$$

$$P_{36}(\zeta) = \frac{\begin{matrix} (2(-(cos(\omega_1\zeta/4)^2 c_0(\theta\lambda a_1 - 2\nu m_2 c_0)\omega) + (c_0 a_1\theta(4\mu\nu+\lambda^2) + 2(\mu\lambda a_1^2 m_1 - \lambda\nu m_2 c_0^2)) \\ \sin(\omega_1\zeta/4)\cosh(\omega_1\zeta/4) + \omega_1 c_0(\lambda a_1\theta - 2\nu c_0 m_2))m_2) \end{matrix}}{\begin{matrix} ((2c_0\nu\theta + \lambda a_1 m_1)(\,2(\cosh(\omega_1\zeta/4)^2 m_2 c_0\omega) + 2\sin(\omega_1\zeta/4)(a_1\mu\theta - \lambda m_2 c_0) \\ \cos(\omega_1\zeta/4) - \omega_1 m_2 c_0)) \end{matrix}} e^{i\zeta}, \tag{54}$$

where, $\omega_1 = \sqrt{-\lambda^2 + 4\mu\nu}\,, \theta = \sqrt{-m_2 m_1}$.

When $\mu = 0$ and $\lambda\nu \neq 0$,

$$P_{37}(\zeta) = \frac{(\,m_2(c_0 a_1\nu(-2\lambda f_1 - \lambda)\theta + 2\nu^2 m_2 c_0^2(f_1 + \cosh(\lambda\zeta) - \sinh(\lambda\zeta)) - f_1\lambda^2 a_1^2 m_1)}{(2c_0\nu\theta + \lambda a_1 m_1)(-\theta\lambda a_1 f_1 + c_0 m_2\nu(f_1 + \cosh(\lambda\zeta) - \sinh(\lambda\zeta)))} e^{i\zeta}, \tag{55}$$

$$P_{38}(\zeta) = \frac{m_2\begin{pmatrix} c_0 a_1\theta(-(2\lambda\,(\cosh(\lambda\zeta) + \sinh(\lambda\zeta)) - \lambda(f_1 + \cosh(\lambda\zeta) + \sinh(\lambda\zeta)))) \\ +2\nu^2 m_2 c_0^2 2(f_1 + \cosh(\lambda\zeta) + \sinh(\lambda\zeta)) - \lambda^2\,(\cosh(\lambda\zeta) + \sinh(\lambda\zeta))a_1^2 m_1 \end{pmatrix}}{(2c_0\nu\theta + \lambda a_1 m_1)\{\,-\theta\lambda a_1\,(\cosh(\lambda\zeta) + \sinh(\lambda\zeta)) + c_0 m_2\nu(l_1 + \cosh(\lambda\zeta) + \sinh(\lambda\zeta))\}} e^{i\zeta}, \tag{56}$$

where l_1 is an arbitrary constant.

When $\nu \neq 0, \mu = \lambda = 0$,

$$P_{39}(\zeta) = \frac{m_2(a_1 c_0(-2\nu - \lambda(\nu\zeta + l_1))\theta + 2(\nu\zeta + l_1)\nu m_2 c_0^2 - \lambda a_1^2 m_1)}{(2c_0\nu\theta + \lambda a_1 m_1)(-\,\theta a_1 + c_0 m_2(\nu\zeta + l_1))} e^{i\zeta}, \tag{57}$$

where l_1 is an arbitrary constant.

V. Results and discussion

This section explores the explicit representations of solutions to the time fractional PNLSE equation achieved through the extended Riccati technique. The exact traveling wave solutions are depicted in the mentioned figures by assigning the different parameter values. Figure 1 depicts kink solitons, while Figures 2 and 3 portray singular kink-type solitons. Figures 4 and 5 represent peakon and anti-bell type solitons, respectively. Figure 6 displays a singular periodic shape, whereas Figures 7 and 8 illustrate periodic shape solitons. Lastly, Figure 9 showcases an anti-bell solitary wave.

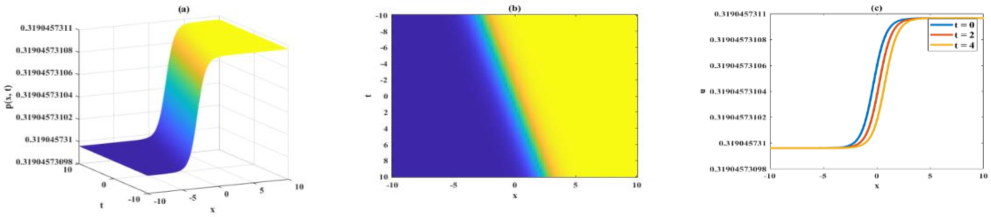

Fig.1: (a) represents a 3D plot of the solution (1) for the values, $a_1 = c_0 = c_1 = \beta = 1$, $\lambda = 5$, $\mu = \nu = \varrho_2 = \varrho_3 = 2$, $\varrho_0 = \varrho_1 = 0.75$ where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

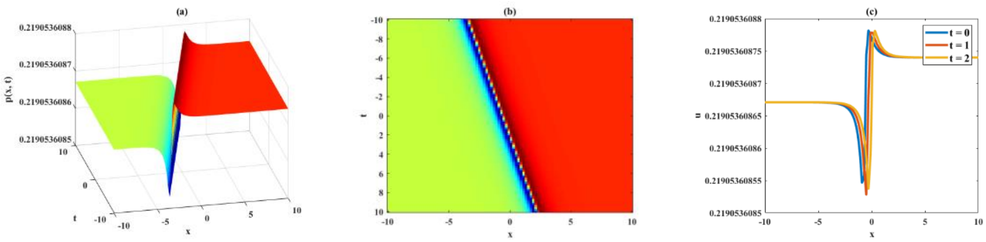

Fig.2: (a) represents a 3D plot of the solution (2) for the values $a_1 = c_0 = c_1 = \beta = 1$, $\lambda = 5$, $\mu = \nu = 2$, $\varrho_0 = \varrho_1 = 0.75$, $\varrho_2 = 2$, $\varrho_3 = 4$ where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

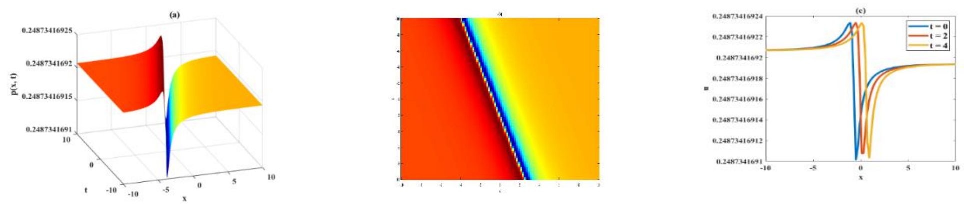

Fig. 3: (a) represents a 3D plot of the solution (5) for the values $a_1 = c_1 = \sigma = 1$, $c_0 = 1.5$, $\lambda = 5$, $\mu = 3$, $\nu = 2$, $\varrho_0 = \varrho_2 = 0.5$, $\varrho_1 = 0.75$, $\varrho_3 = 2.5$, where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

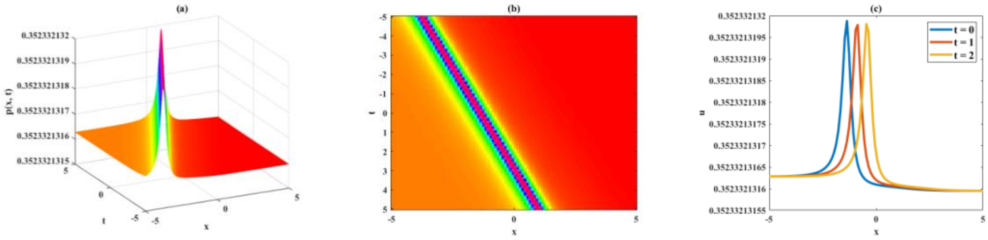

Fig.4:(a) represents a 3D plot of the solution (7) for the values $A = \varrho_0 = \varrho_1 = 0.5$, $a_1 = \mu = 1.5$, $B = c_0 = \beta = 1$, $c_1 = 1.25$, $1.5, \lambda = 4$, $\nu = 1.5$, $\varrho_2 = 0.35$, $\varrho_3 = 2$, where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

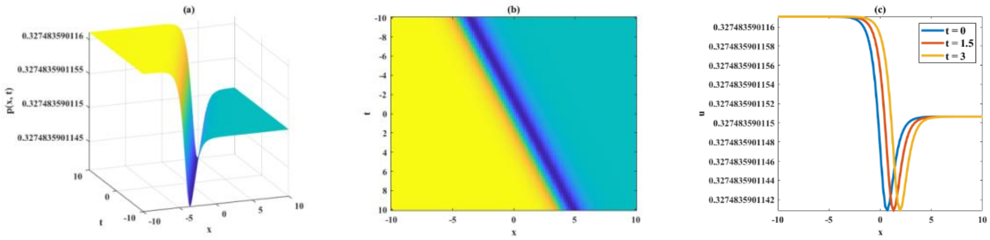

Fig.5:(a) represents a 3D plot of the solution (10) for the values $a_1 = \mu = \nu = \varrho_3 = 2$, $c_0 = c_1 = \beta = \varrho_2 = 1$, $\lambda = 5$, $\varrho_0 = 0.75$, $\varrho_1 = 0.55$ where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

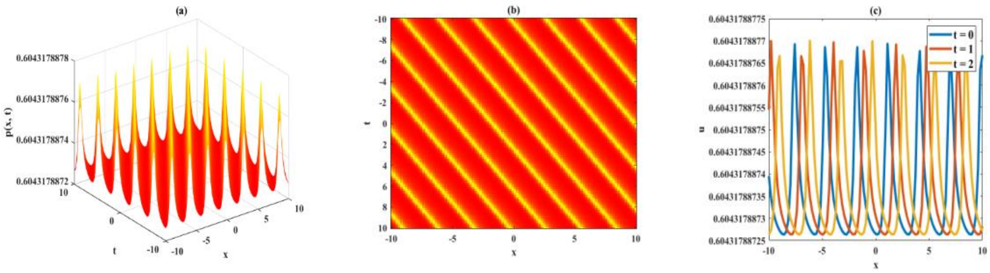

Fig.6: (a) represents a 3D plot of the solution (15) for the values $a_1 = 0.95$, $c_0 = \beta = \mu = 1$, $c_1 = 0.5, \lambda = \varrho_3 = 2$, $\nu = 1.5$, $\varrho_0 = \varrho_1 = 0.25$, $\varrho_2 = 0.75$ where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

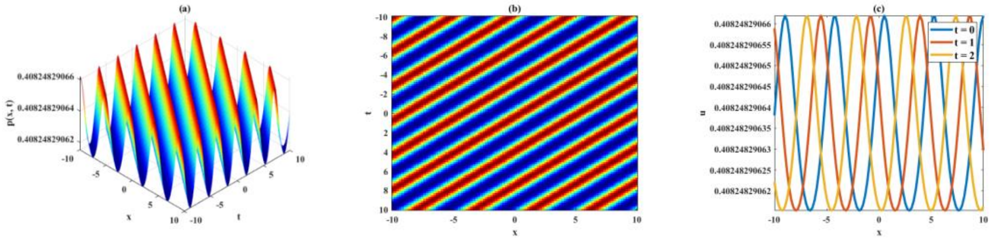

Fig.7: (a) represents a 3D plot of the solution (16) for the values $A = 1.25$, $B = c_0 = c_1 = \beta = \varrho_0 = 1$, $a_1 = \lambda = \varrho_3 = 0.5$, $\mu = \nu = \varrho_2 = 2$, $\varrho_1 = 1.5$ where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

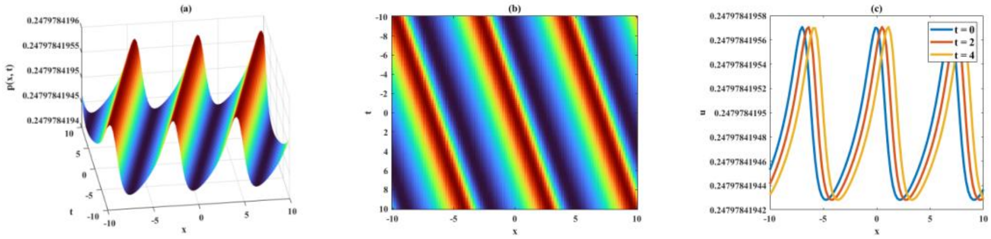

Fig.8: (a) represents a 3D plot of the solution (16) for the values $a_1 = \lambda = 0.5$, $c_0 = c_1 = \beta = \mu = \nu = 1$, $\varrho_0 = \varrho_1 = 0.75$, $\varrho_2 = 2$, $\varrho_3 = 3$ where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

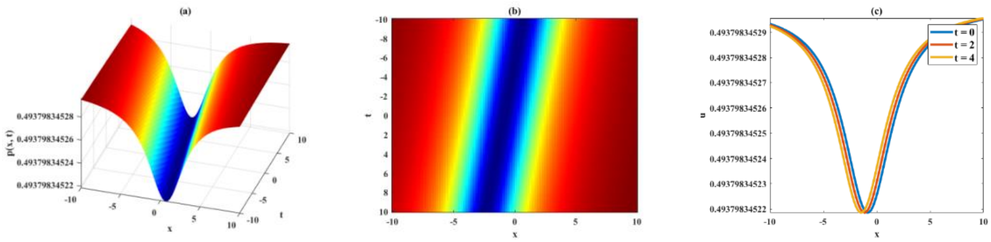

Fig. 9: (a) represents a 3D plot of the solution (16) for the values $a_1 = c_0 = c_1 = \beta = \nu = \varrho_3 = 1$, $l_1 = 0.5, \lambda = \mu = 0$, $\varrho_0 = \varrho_1 = 0.75$, $\varrho_2 = 3$, where $x \in [-10,10]$ and $t \in [-10,10]$; (b) visualizes the density plot for $x \in [-10,10]$ and $t \in [-10,10]$; (c) displays multiple 2D figure for $x \in [-10,10]$.

VI. Conclusion

This study successfully uncovered various solitary wave solutions to the time-fractional perturbed nonlinear Schrödinger equation describing the behavior of solitons in nonlinear optical fibers. The comprehensive analysis of solutions has been facilitated by transforming a nonlinear fractional differential equation into a nonlinear differential equation through the utilization of traveling wave transformation and the beta derivative. The application of the extended Riccati equation method elucidated the nature of these solutions, revealing solitons and soliton-type solutions characterized by trigonometric and hyperbolic functions. The obtained solutions comprehend numerous arbitrary parameters that might be useful to analyze the underlying characteristics of complex physical phenomena. By comparing the model (1) studied by several authors with other methods [XXVII, XX, XXII], we deduce that our method provides new solutions for the same model studied. The nature of solutions varies qualitatively depending on individual parameters. Graphical representations exhibit a rich structure of physical forms, including kink, bell-shaped, periodic, and anti-cuspon, among others. These physical solutions play an important role in the interpretation of wave propagation studies and are necessary to verify numerical and experimental results in the fields of nonlinear optics, quantum mechanics, engineering, etc. Moreover, in view of mathematical analysis, we see that the used method is efficient integrability for constructing exact soliton solutions.

Conflict of Interest:

The author declares that there was no conflict of interest regarding this paper.

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