JOURNAL OF MECHANICS OF CONTINUA AND MATHEMATICAL SCIENCES
www.journalimcms.org

ISSN (Online) : 2454 -7190 , Special Issue, No.-11, May (2024) pp 230-239 ISSN (Print) 0973-

ERROR ANALYSIS OF THE SOLUTIONS OF (1+1)-DIMENSIONAL & (2+1)-DIMENSIONAL HEAT-LIKE EQUATIONS USING HE'S POLYNOMIAL

Mankirat Kaur[1], Rantej Sharma[2], Kashish Wadhawan[3], Abhinav Dhiman[4]

[1,3,4] Department of Computer Science and Engineering (Artificial Intelligence), Chitkara University Institute of Engineering & Technology, Chitkara University, Rajpura, Punjab, India.

[2] Department of Civil Engineering, Chandigarh University, Mohali, India.

Emails: [1]mankirat3869.beai23@chitkara.edu.in, [2]rantej_thapars@yahoo.com, [3]kashish3857.beai23@chitkara.edu.in, [4]abhinav3762.beai23@chitkara.edu.in

Corresponding Author: **Rantej Sharma**

https://doi.org/10.26782/jmcms.spl.11/2024.05.00016

(Received: March 22, 2024; Revised: May 3, 2024; Accepted: May 19, 2024)

Abstract

In this paper, we are examining He's polynomial method for solving (1+1)-dimensional and (2+1)-dimensional heat-like equations that arise in various diffusion processes. The absolute error is calculated from the exact solution and numerical solution by taking different iterations of the He's polynomial. This method is also called the homotopy perturbation method (HPM). The nonlinear terms can be easily handled by the use of He's polynomials. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids round-off errors. Some examples are given to show the efficiency and accuracy of the He's polynomial used to solve Heat-like equations.

Keywords: Boundary Conditions, Error Analysis, He's Polynomial, Heat Equations, Nonlinear Terms, Homotopy Perturbation Method.

I. Introduction

The heat-like models arise in various physical circumstances and are an important section considered in applied sciences. These techniques have been used for solving various complex problems including characteristic, modified variational iteration moreover Adomian's decomposition methods, as outlined in references [I–III]. These techniques often face significant challenges. He devised and structured HPM by integrating the conventional HPM approach [V-X]. This methodology has been tried on a wide class of functional equations and it has demonstrated compatibility with the versatile nature of the problems at hand, as evidenced by references [IV–XVIII]. In this methodology, the solution is represented by an infinite series, typically yielding a solution with minimal error refer to [I-XVIII] and the cited references. This method is error-free from round-off errors and worth mentioning that

HPM is applied without any discretization, limited presumption, or modification. This method is employed for handling nonlinear terms within the problem, avoiding partitioning by finite differences or spline techniques at intersections. However, it necessitates rigorous calculations and often leads to ill-conditioned resultant equations, complicating the solution process. In contrast to the method of separation of variables, which necessitates starting and termination conditions, the HPM offers an analytical solution without such requirements. Solving nonlinear problems without employing Adomian's polynomials is an advantage of the HPM over the decomposition method. It is worth noting that He's polynomials were introduced by [I, II] through the decomposition of the nonlinear term. Their compatibility with Adomian's polynomials was established, leading to the conclusion that He's polynomials are simpler to compute and do not involve the complexities associated with calculating Adomian's polynomials. It is noteworthy that He's polynomials can be derived from HPM, a fact underscored by ongoing research in this field. Encouraged and motivated by this insight, we employ He's polynomials to address heat and wave-like equations. Introduced an HPM approach adds up to supplementary terms and consequently leads towards complexities, whereas the approach is easily executable. Therefore, it removes the unnecessary calculation that comes up in [XIX]. Numerous examples are provided to validate the reliability and effectiveness of the algorithm.

II. Homotopy Perturbation Method (HPM) and He's Polynomials

The concept of the HPM can be elucidated through a general equation provided as follows

$$L(u)=0, \tag{1}$$

Here *L* represents any integral or differential operator while a convex homotopy *H(u, p)* is defined as follows

$$H\,(u,p)\;=\;(1-p)F(u)\;+\;pL(u), \tag{2}$$

F(u) denotes a functional operator with established solutions denoted by *v0*

$$H(u,p)=0, \tag{3}$$

we have

$$H(u,0)\;=\;F(u),\quad H(u,1)\;=\;L(u). \tag{4}$$

Consequently, it can be observed that *H(u, p)* follows an implicitly defined trajectory from an initial point *H(v0, 0)* to a solution function denoted as *H(f, 1).* As the trivial problem *F(u) = 0* undergoes continuous deformation, the parameter monotonically increases from zero to unity, thereby transforming the original problem *L(u) = 0*. In the HPM, the homotopy parameter $p\in(0,1]$ serves as an expanding parameter, which is utilized to achieve a solution, as detailed in references [III-X].

$$\sum_{i=0}^{\infty} p^i\,u_i = u_0 + pu_1 + \;p^2u_2\; + \;p^3u_3 + \cdots, \tag{5}$$

As p approaches 1, equation (5) aligns with (2) and transitions into an approximate solution structured as follows

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_{i.} \tag{6}$$

In most cases, the convergence rate of series (6) is acknowledged to be contingent upon *L(u)*, as extensively discussed in references [III-X]. Considering the assumption of a unique solution for (2.6), comparisons of similar powers of p yield solutions of different orders. In summary, based on references [I,II], He's HPM approaches the solution *u(x)* of the homotopy equation through a series expansion in terms of *p* as presented below:

$$u(x) = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots, \tag{7}$$

Additionally, the consideration of the nonlinear term $N(u)$ is incorporated as follows:

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + pH_i + p^2 H_2 + \cdots, \tag{8}$$

The He's polynomials H_n'sas detailed in references [12,13], can be computed utilizing the following formula:

$$H_n(u_{0,\cdots,}u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i u_i\right)\right) \quad , \qquad n = 1,2,3 \ldots \tag{9}$$

III. Numerical Applications

In this section, we use He's polynomials which are computed from HPM for solving heat-like equations.

Example 3.1 Analysis of Heat-like equations in (1+1)-dimensional starting (initial) and boundary conditions

$$u_t = \frac{1}{2}x^2 u_{xx}, \quad 0 < x < 1, t > 0,$$

with constraints

$$u(1,t) = e^t,$$

Starting conditions

$$u(x,0) = x^2,$$

Now applying the method of complex homotopy,

$$u_0 + pu_1 + p^2u_2 + \cdots = y^2 + \frac{1}{2}\int_0^1 (x^2(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots)dt$$

Comparing the coefficient of similar powers of p,

$$p^0: \quad u_0(x,t) = x^2,$$

$$p^1: \quad u_1(x,t) = x^2 t,$$

$$p^2: \quad u_2(x,t) = x^2\frac{t^2}{2!},$$

$$p^3 : u_3(x,t) = x^2 \frac{t^3}{3!},$$

$$p^4 : u_4(x,t) = x^2 \frac{t^4}{4!},$$

where $p^{(i)}s$ are He's polynomials equations. The solution is given in the next equation :

$$u(x,t) = x^2(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots)$$

and the short form of the equation:

$$u(x,t) = x^2 e^t$$

Table 1 : Error Analysis of Exact Solution and Numerical Solution (up to 4th iteration) at x = 1

t	Exact Solution (u)	Approx Solution (u*)	|u-u*|
0.1	1.105170918075648	1.105170833333333	8.4742e-08
0.2	1.221402758 160170	1.2214000000 00000	2.7582e-06
0.3	1.349858807 576003	1.3498375000 00000	2.1308e-05
0.4	1.491824697 641270	1.4917333333 33333	9.1364e-05
0.5	1.648721270 700128	1.6484375000 00000	2.8377e-04
0.6	1.822118800 390509	1.8214000000 00000	7.1880e-04
0.7	2.013752707 470477	2.0121708333 33333	1.5819e-03
0.8	2.225540928492468	2.2224000000 00000	3.1409e-03
0.9	2.459603111156950	2.4538375000 00000	5.7656e-03
1.0	2.718281828459045	2.7083333333 33333	9.9485e-03

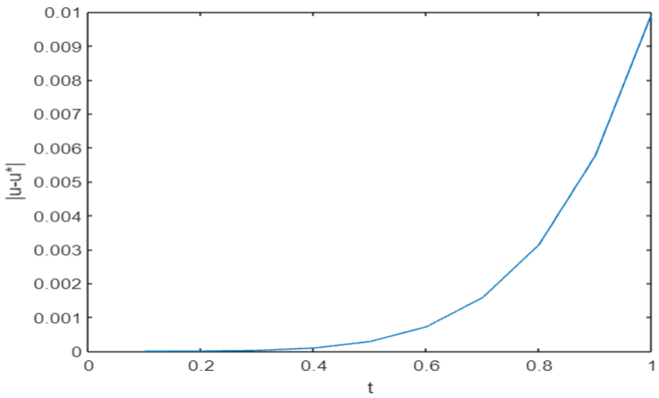

Fig. 1. Error Analysis of the numerical solutions (up to 4th iteration)

Table 2: Error Analysis of Exact Solution and Numerical Solution (up to 8th iteration) at x = 1

t	Exact Solution (u)	Approx Solution (u*)	|u-u*|
0.1	1.105170918075648	1.105170918075645	3.1086e-15
0.2	1.221402758160170	1.221402758158730	1.4397e-12
0.3	1.349858807576003	1.349858807520089	5.5914e-11
0.4	1.491824697641270	1.491824696888889	7.5238e-10
0.5	1.648721270700128	1.648721265035962	5.6642e-09
0.6	1.822118800390509	1.822118770857143	2.9533e-08
0.7	2.013752707470477	2.013752587956597	1.1951e-07
0.8	2.225540928492468	2.225540526730159	4.0176e-07
0.9	2.459603111156950	2.459601938948661	1.1722e-06
1.0	2.718281828459045	2.718278769841270	3.0586e-06

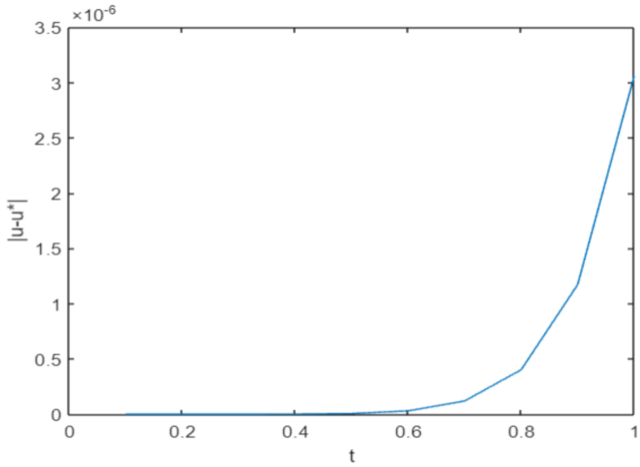

Fig. 2. Error Analysis of the Numerical solutions (up to 8th iteration)

Example3.2 Analysis of Heat-like equations in (2+1)-dimensional starting (initial) and boundary conditions

$$u_t = 2(y^2 u_{xx} + x^2 u_{yy}), 0 < x, y < 1, t > 0$$

with constraints

$$ux(0, y, t) = 0, \quad ux(1, y, t) = 2sinht,$$

$$uy(x, 0, t) = 0, \quad uy(x, 1, t) = 2cosht,$$

Starting conditions

$$u(x, y, 0) = y^2$$

Now applying the convex homotopy method,

$$u_0 + pu_1 + p^2 u_2 + \cdots$$
$$= y^2 + \frac{1}{2}\int_0^1 (y^2(\frac{\partial^2 u_0}{\partial x^2} + p\frac{\partial^2 u_1}{\partial x^2} + p^2\frac{\partial^2 u_2}{\partial x^2} + \cdots) + (x^2(\frac{\partial^2 u_0}{\partial y^2} + p\frac{\partial^2 u_1}{\partial y^2}$$
$$+ p^2\frac{\partial^2 u_2}{\partial y^2} + \cdots))ds$$

Compare the coefficient of similar powers of p:

$$p^0 : u_0(x, y, t) = y^2,$$

$$p^1 : u_1(x, y, t) = x^2 t,$$

$$p^2 : u_2(x, y, t) = y^2 \frac{t^2}{2!},$$

$$p^3 : u_3(x, y, t) = x^2 \frac{t^3}{3!},$$

$$p^{(4)}: u_4(x, y, t) = y^2 \frac{t^4}{4!},$$

where $p^{(i)}$ s are He's polynomials equations. The solution is given below:

$$u(x, y, t) = x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) + y^2\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)$$

and the short form of the equation:

$$u = x^2 \sinh t + y^2 \cosh t$$

Table 3: Error Analysis of Exact Solution and Numerical Solution (up to 4th iteration) at x = 1 & y = 2

t	Exact Solution (u)	Approx Solution (u*)	|u-u*|
0.1	4.120183422243058	4.120183333333332	8.8910e-08
0.2	4.281603025017398	4.281600000000000	3.0250e-06
0.3	4.485874349962584	4.485850000000000	2.4350e-05
0.4	4.735041813156635	4.734933333333334	1.0848e-04
0.5	5.031599166319270	5.031250000000000	3.4917e-04
0.6	5.378514455117312	5.377599999999999	9.1446e-04
0.7	5.779259724363305	5.777183333333332	2.0764e-03
0.8	6.237845767407002	6.233600000000000	4.2458e-03
0.9	6.758862267503273	6.750850000000001	8.0123e-03
1.0	7.347523732904776	7.333333333333334	1.4190e-02

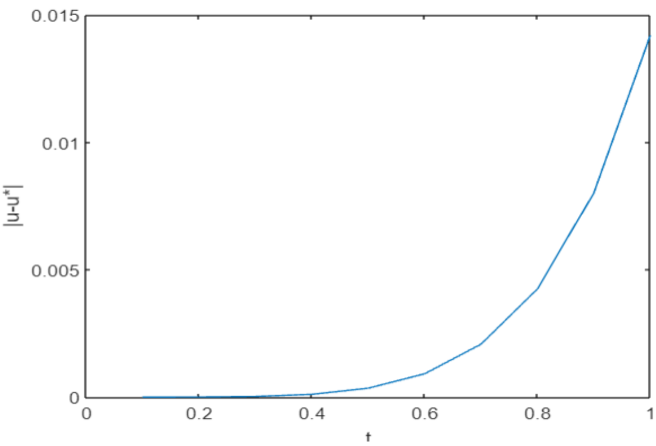

Fig. 3. Error Analysis of the Numerical solutions (upto 4th iteration)

Table 4: Error Analysis of Exact Solution and Numerical Solution (upto 8th iteration) at x = 1 & y = 2

t	Exact Solution (u)	Approx Solution (u*)	|u-u*|
0.1	4.120183422243058	4.120183422243054	4.4409e-15
0.2	4.281603025017398	4.281603025015873	1.5241e-12
0.3	4.485874349962584	4.485874349901787	6.0798e-11
0.4	4.735041813156635	4.735041812317461	8.3917e-10
0.5	5.031599166319270	5.031599159846230	6.4730e-09
0.6	5.378514455117312	5.378514420571427	3.4546e-08
0.7	5.779259724363305	5.779259581409721	1.4295e-07
0.8	6.237845767407002	6.237845276444444	4.9096e-07
0.9	6.758862267503273	6.758860805258930	1.4622e-06
1.0	7.347523732904776	7.347519841269842	3.8916e-06

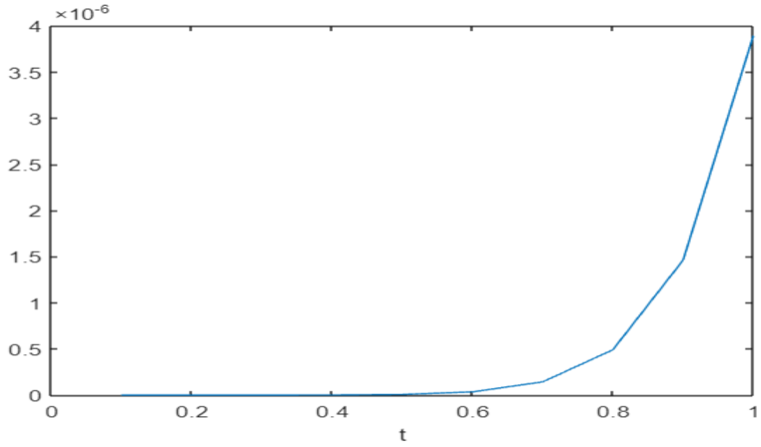

Fig. 4. Error Analysis of the Numerical solutions (up to the 8th iteration)

IV. Conclusion

As far as the above examples are considered, it has been observed that by taking more iterations of He’s Polynomial the absolute error is decreased. This paper employs He’s polynomials, acquired by the HPM, to label heat-like equations. Our approach demonstrates a direct application of the method without relying on linearization or revolution. This paper's assumption proposes that the submitted plan of action proves highly efficient and logical in acquiring systematic solutions for a range of margin issues. Particularly, the method yields solutions that coincide quickly with physical scenarios, enhancing realism. Additionally, He’s polynomials present compatibility with Adomian’s Polynomials while offering simpler calculation and greater understandability.

Conflict of Interest

The author declares that they have no conflict of interest regarding this paper.

References

I. Ghorbani. : ‘Beyond Adomian polynomials: He polynomials’. *Chaos, Solitons & Fractals*. vol. 39(3), pp. 1486–1492, 2009. 10.1016/j.chaos.2007.06.034

II. Ghorbani and J. Saberi-Nadjafi. : ‘He’s homotopy perturbation method for calculating adomian polynomials’. *International Journal of Nonlinear Sciences and Numerical Simulation*. vol. 8(2), pp. 229–232, 2007. 10.1515/IJNSNS.2007.8.2.229

III. J.-H. He. : ‘A coupling method of a homotopy technique and a perturbation technique for nonlinear problems’. *International Journal of Non-Linear Mechanics*. vol. 35(1), pp. 37–43, 2000. 10.1016/S0020-7462(98)00085-7

IV. J.-H. He. : ‘An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering’. *International Journal of Modern Physics B*. vol. 22(21), pp. 3487–3578, 2008. 10.1142/S0217979208048668

V. J.-H. He. : ‘Comparison of homotopy perturbation method and homotopy analysis method’. *Applied Mathematics and Computation*. vol. 156(2), pp. 527–539, 2004. 10.1016/j.amc.2003.08.008

VI. J.-H. He. : ‘Homotopy perturbation method for bifurcation of nonlinear problems’. *International Journal of Nonlinear Sciences and Numerical Simulation*. vol. 6(2), pp. 207–208, 2005. 10.1515/IJNSNS.2005.6.2.207

VII. J.-H. He. : ‘Homotopy perturbation method for solving boundary value problems’. *Physics Letters A*. vol. 350(1-2), pp. 87–88, 2006. 10.1016/j.physleta.2005.10.005

VIII. J.-H. He. : ‘Recent development of the homotopy perturbation method’, Topological Methods in Nonlinear Analysis’. vol. 31(2), pp. 205–209, 2008.

IX. J.-H. He. : ‘Some asymptotic methods for strongly nonlinear equations’. *International Journal of Modern Physics B*’. vol. 20(10) pp. 1141–1199, 2006. 10.1142/S0217979206033796

X. J.-H. He. : ‘The homotopy perturbation method for nonlinear oscillators with discontinuities’. *Applied Mathematics and Computation*. vol. 151(1), pp. 287–292, 2004. 10.1016/S0096-3003(03)00341-2

XI. L. Xu. : 'He's homotopy perturbation method for a boundary layer equation in unbounded domain'. *Computers & Mathematics with Applications*. vol. 54(7-8), pp. 1067–1070, 2007. 10.1016/j.camwa.2006.12.052

XII. M. A. Noor and S. T. Mohyud-Din. : 'Homotopy perturbation method for solving nonlinear higher order boundary value problems'. *International Journal of Nonlinear Sciences and Numerical Simulation*. vol. 9(4), pp. 395–408, 2008. 10.1515/IJNSNS.2008.9.4.395

XIII. M. A. Noor and S. T. Mohyud-Din. : 'Modified variational iteration method for heat and wave-like equations'. *Acta Applicandae Mathematicae*. vol. 104(3), pp. 257–269, 2008. 10.1007/s10440-008-9255-x

XIV. M. A. Noor and S. T. Mohyud-Din. : 'Modified variation alliteration method for heat and wave-like equations'. *Acta Applicandae Mathematicae*. vol. 104(3), pp. 257–269, 2008. 10.1007/s10440-008-9255-x

XV. M. Wazwaz and A. Gorguis. : 'Exact solutions for heat-like and wave-like equations with variable coefficients'. *Applied Mathematics and Computation*. vol. 149(1), pp. 15–29, 2004. 10.1016/S0096-3003(02)00946-3

XVI. R. Wilcox. : 'Closed-form solution of the differential equation (∂^2/∂x∂y+ax(∂/∂x)+by(∂/∂y)+cxy+∂/∂t)P(x,y,t) = 0 by normal-ordering exponential operators'. *Journal of Mathematical Physics*. vol. 11(4), pp. 1235–1237, 1970. 10.1063/1.1665252

XVII. S. T. Mohyud-Din, M. A. Noor, and K. I. Noor. : 'Traveling wave solutions of seventh-order generalized KdV equations using He's polynomials'. *International Journal of Nonlinear Sciences and Numerical Simulation*. vol. 10(2), pp. 227–233, 2009. 10.1515/IJNSNS.2009.10.2.227

XVIII. S. T. Mohyud-Din and M. A. Noor. : 'Homotopy perturbation method for solving fourth-order boundary value problems'. *Mathematical Problems in Engineering*'. vol. 2007, Article ID 98602, 15 pages, 2007. 10.1155/2007/98602

XIX. S. T. Mohyud-Din, M. A. Noor, and K. I. Noor. : 'Homotopy perturbation method for unsteady flow of gas through a porous medium'. *International Journal of Modern Physics B. In press*.

XX. S. T. Mohyud-Din, M. A. Noor, and K. I. Noor. : 'On the coupling of polynomials with correction functional'. *International Journal of Modern Physicss B. In press*.