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APPLICATIONS OF CHEBYSHEV WAVELET OF THE SECOND KIND FOR SOLVING LOGISTIC DIFFERENTIAL EQUATIONS

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Abstract

This research paper focuses on the comparison study of wavelet solutions for solving logistic differential equations. For this purpose, we are utilizing Chebyshev wavelets of the second kind and Haar wavelets. Various numerical tests have been conducted to demonstrate the ease of use, precision, and effectiveness of the solutions provided by various wavelet techniques. The implications of these results are discussed within the broader context of mathematical and scientific research.

Keywords: Wavelets, Chebyshev wavelets of the second kind, Haar wavelets, Operational metrics of integrations, Logistic differential equations, Numerical examples.

I. Introduction

The main purpose of this research is to establish the comparison studies of different wavelet solutions for solving logistic differential equations of the form:

$$Du(\tau) = \delta u(\tau)\big(1 - u(\tau)\big), \ \ \tau > 0, \tag{1}$$

with the initial condition $u(0) = \rho_0$.

The logistic differential equation is instrumental in modeling population growth with environmental constraints. This equation's versatility extends to economic modeling, tumor growth studies, and analyzing innovation diffusion. In essence, the logistic differential equation serves as a crucial mathematical framework, offering insights into the intricate interplay between growth and limitations in various natural and societal systems, fostering a deeper understanding of complex phenomena and informing decision-making processes across diverse scientific domains. The authors explored solutions in a coupled system of time-fractional differential equations in [I]. To achieve a numerical solution of multiterm variable-order fractional differential equations in [II], Chebyshev polynomials of the fourth kind were employed. Chebyshev wavelet-based method has been presented for solving linear and nonlinear differential equations in [III]. In this research paper, an analysis of time-varying functional differential equations has been introduced in [IV]. Fractional calculus, denoting derivatives and integrals with fractions, inspired the study in [V]. A discrete logistic equation was described with a Volterra convolution type in [VI]. In [VII, XXVI], an innovative operational matrix algorithm based on Haar wavelets is described for tackling linear and nonlinear fractional order differential equations. For the solution of linear and non-linear reaction-diffusion equations, a thorough review has been presented in [VIII]. The numerical solution of ordinary differential equations having fractional order derivatives utilizing the Legendre wavelet method has been introduced in [IX]. For the solution of linear and non-linear delay differential equations, a Haar wavelet and Hermite wavelet collocation method have been discussed in [X, XI, XXII]. To solve fractional differential equations, a computational method based on the shifted Chebyshev wavelet has been suggested in [XII]. The authors introduced a wavelet collocation method for the solution of a neutral delay differential equation using various wavelets in [XIII]. General non-existence results for the solution of the logistic differential equation were proposed in [XIV]. Operational matrices of integration for Chebyshev and Legendre wavelets have been formed and applied to multi-order fractional differential equations in [XV]. In [XVI], a numerical strategy for analyzing non-linear stochastic differential equations resulting from fractional Brownian motion is introduced. The authors apply Legendre wavelets to address the fractional Logistic differential equation (FLDE) in [XVII]. The authors introduced the Legendre wavelet method for the solution of delay differential equations in [XVIII]. The numerical solution of Riccati, second and third-order nonlinear ordinary differential equations were presented in logistic function in [XIX]. The authors establish the theory of existence for sequential fractional differential equations with Caputo fractional derivative in [XX]. A numerical technique has been proposed to solve a system of linear differential equations using a second kind of Chebyshev wavelet in [XXI]. Operational matrix of derivative derived for Chebyshev wavelets applied to ODEs with analytic solutions in [XXIII]. The authors described the use of the Hermite wavelet method for solving non-linear heat diffusion equations in [XXIV]. The numerical solution of various fractional equations has been addressed in [XXV]. Theoretical and numerical simulations of generalized logistic differential equations on real-world problems have been described in [XXVII]. The concept of local fractional integral transforms and their applications has been studied in [XXVIII]. A proficient approach, utilizing the Chebyshev wavelet variant in conjunction with the Picard technique, is formulated for the resolution of fractional nonlinear differential equations in [XXIX]. The authors presented the study to investigate the presence and singular nature of the solution for the fractional logistic differential equation in [XXX].

II. Wavelets and their properties

In this section, we will discuss the theory of the different wavelets such as Chebyshev wavelets of the second kind and Haar wavelets.

II.i. Chebyshev wavelets of the second kind and its properties

Wavelets-based numerical algorithms have been utilized extensively in the previous few decades to address a wide range of scientific, engineering, and technological issues. Wavelets constitute a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\varphi_{a,b}(t) = |a|^{-1/2}\varphi\left(\frac{t-b}{a}\right), \quad a,b \in R, \quad a \neq 0$$

The second kind of wavelet $\varphi_{n,m} = \varphi(k,n,m,t)$ have four arguments; k is any positive integer, $n = 1,2,3,4,\ldots,2^{k-1}$, m is the degree of second-kind of Chebyshev polynomials and t is normalized time. It is defined on the interval [0,1) as follows:

$$\varphi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\widetilde{U}\left(2^k t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}} \\ 0\,, & otherwise \end{cases}$$

where

$$\widetilde{U}_m(t) = \sqrt{\frac{2}{\pi}}U_m(t), \tag{1}$$

where $m = 0,1,2,3,\ldots,M-1$ and M is fixed integer. In relation given by equation (1) is for orthonormality. Here $U_m(t)$ are the second kind of Chebyshev polynomials of degree m which are orthogonal concerning the weight function $\omega(t) = \sqrt{1-t^2}$ on the interval [-1,1], and satisfy the satisfy the following recursive formula:

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1,2,3,4,\ldots ....$$

It is observed that the weight function for second-kind Chebyshev wavelets must be translated and dilated as $\omega_n(t) = \omega\left(2^k t - 2n + 1\right)$. Here we find out the integral of second kind Chebyshev wavelets functions with $k = 1,\ M = 6$. For this, the six basis functions in [0,1] are as follows:

$$\varphi_{1,0}(t) = \frac{2}{\sqrt{\pi}},$$

$$\varphi_{1,1}(t) = \frac{2}{\sqrt{\pi}}(4t-2)\,,$$

$$\varphi_{1,2}(t) = \frac{2}{\sqrt{\pi}}(16t^2-16t+3)\,,$$

$$\varphi_{1,3}(t) = \frac{2}{\sqrt{\pi}}(64t^3-96t^2+40t-4)\,,$$

$$\varphi_{1,4}(t) = \frac{2}{\sqrt{\pi}}(256t^4-512t^3+336t^2-80t+5)\,,$$

$$\varphi_{1,5}(t) = \frac{2}{\sqrt{\pi}}(1024t^5-2560t^4+2304t^3-896t^2+140t-6)$$

II.ii. Haar wavelets and their properties

Haar functions constitute a set of orthogonal waveforms characterized by rectangular shapes with varying amplitudes. The Haar wavelet, on the other hand, consists of a series of square-shaped functions that have been rescaled, collectively forming a basis or family of wavelets. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha\ \gamma]$ as:

$$h_i(x) = \begin{cases} 1, & \alpha \le x < \beta \\ -1, & \beta \le x < \gamma \\ 0, & elsewhere\,, \end{cases} \tag{3}$$

where $\alpha = \frac{k}{m}, \beta = \frac{k+0.5}{m},\ \gamma = \frac{k+1}{m}, m = 2^j$ and $j = 0,1,2,3,\dots,J$. The level of the resolution is denoted by J. The integer $k = 0,1,2,\dots,m-1$ is the translation parameter. The index i is calculated as $i = m+k+1$. The minimal and maximal values i are $i = 2$ and $i = 2^{j+1}$ respectively.

The collocation points are calculated as

$$x_l = \frac{(l-0.5)}{2M},\ \ l = 1,2,3,\dots\dots,2M. \tag{4}$$

The operational matrix is represented by P, having order $2M \times 2M$, and is calculated as below:

$$P_{1,i}(x) = \int_0^{x_l} h_i(x)dx \tag{5}$$

$$P_{n+1,i}(x) = \int_0^x P_{n,i}(x)dx, \qquad n = 1,2,3,\dots. \tag{6}$$

From (5), we obtain:

$$P_{i,1}(x) = \begin{cases} x-\alpha, & \alpha \le x < \beta \\ \gamma - x, & \beta \le x < \gamma \\ 0\,, & elsewhere. \end{cases}$$

III. Proposed methodology for solving Logistic differential equations

From (1), we have

$$Du(\tau) = \delta u(\tau) - \delta u^2(\tau),$$

dividing by $u^2(\tau)$, we obtain

$$\frac{1}{u^2(\tau)} Du(\tau) = \delta \frac{1}{u(\tau)} - \delta. \tag{7}$$

Substituting

$$\frac{1}{u(\tau)} = w(\tau)$$

This implies

$$-\frac{1}{u^2(\tau)} Du(\tau) = Dw(\tau).$$

From (7), we obtain

$$-Dw(\tau) = \delta . w(\tau) - \delta$$

Or

$$\frac{dw}{d\tau} + \delta . w(\tau) = \delta \tag{8}$$

Assume that

$$\frac{dw}{d\tau} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(\tau) \tag{9}$$

Integrating (9) concerning τ, we obtain

$$w(\tau) = w(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^t \varphi_{n,m}(\tau) d\tau \tag{10}$$

Applying initial conditions, we obtain

$$w(\tau) = \rho_0 + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^t \varphi_{n,m}(\tau) d\tau \tag{11}$$

Substituting the values from (9), (10), and (11) in (8), we obtain

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \left\{ \varphi_{n,m}(\tau) + R. \int_0^t \varphi_{n,m}(\tau) d\tau \right\} = f(\tau) \tag{12}$$

where

$$f(\tau) = E(\tau) - R. q_0 - C. \{q_{00} + \tau. q_0\}$$

After discretizing (12), we get a system of linear equations, which can be solved with any classical scheme. After solving the system of equations, we obtain the wavelet coefficients. Substituting the values of wavelet coefficients into (9), we obtain the required solution.

IV. Results and analysis

In this section, we provide numerical experiments aimed at demonstrating the accuracy and efficiency of the proposed scheme. Here, we take different values of δ, K, N_0 along with initial conditions, and solve the Logistic Differential equation with the help of the proposed method.

Case I: For $\delta = 0.0760,\ K = 267.5301,\ N_0 = 0.5$
The logistic model is reduced to

$$Du(\tau) = \delta u(\tau)\left(1 - \frac{u(\tau)}{K}\right),\ \ \tau > 0, \tag{13}$$

with the initial condition $u(0) = 0.5$.

Dividing by $u^2(\tau)$, we obtain

$$\frac{1}{u^2(\tau)} Du(\tau) = \delta \frac{1}{u(\tau)} - \delta. \tag{14}$$

3Substituting

$$\frac{1}{u(\tau)} = w(\tau)$$

This implies

$$-\frac{1}{u^2(\tau)} Du(\tau) = Dw(\tau).$$

From (7), we obtain

$$-Dw(\tau) = \delta. w(\tau) - \delta$$

This implies

$$\frac{dw}{d\tau} + \delta. w(\tau) = \delta \tag{15}$$

Assume that

$$w'(\tau) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \varphi_{n,m}(\tau) \tag{16}$$

Integrating (16) concerning t, we obtain

$$w(\tau) = w(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^t \varphi_{n,m}(\tau) d\tau \tag{17}$$

Applying initial conditions, we obtain

$$w(\tau) = \rho_0 + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^t \varphi_{n,m}(\tau) d\tau \tag{18}$$

Substituting the values from (16), (17) and (18) in (15), we obtain

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \left\{\varphi_{n,m}(t) + \delta. \int_0^t \varphi_{n,m}(t) dt\right\} = \delta \tag{19}$$

After discretizing (19), we get a system of linear equations, which can be solved with any classical scheme. After solving the system of equations, we obtain the wavelet coefficients. Substituting the values of wavelet coefficients into (16), we obtain the required solution.

Table 1: Comparison of solutions of Example 1

τ	Exact Solutions	Chebyshev wavelets solutions	Haar wavelets solutions
0.0625	0.5023761791	0.5023761791	0.5023705402
0.1875	0.5071624039	0.5071624039	0.5071567475
0.3125	0.5119941404	0.5119941404	0.5119884668
0.4375	0.5168718199	0.5168718199	0.5168661291
0.5625	0.5217958773	0.5217958773	0.5217901696
0.6875	0.5267667519	0.5267667519	0.5267610276
0.8125	0.5317848872	0.5317848872	0.5317791463
0.9375	0.5368507305	0.5368507305	0.5368449734

Table 2: Comparison of absolute errors of Example 1

τ	Absolute Errors (Chebyshev Wavelets)	Absolute Errors (Haar Wavelets)
0.0625	0	5.6389e-006
0.1875	0	5.6564e-006
0.3125	1.1102e-016	5.6736e-006
0.4375	0	5.6907e-006
0.5625	1.1102e-016	5.7076e-006
0.6875	1.1102e-016	5.7243e-006
0.8125	1.1102e-016	5.7408e-006
0.9375	1.1102e-016	5.7571e-006

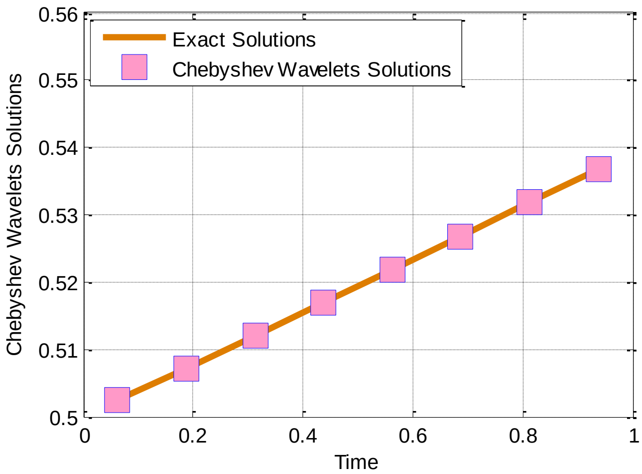

Fig. 1. Comparison of Exact and Chebyshev Wavelets Solutions

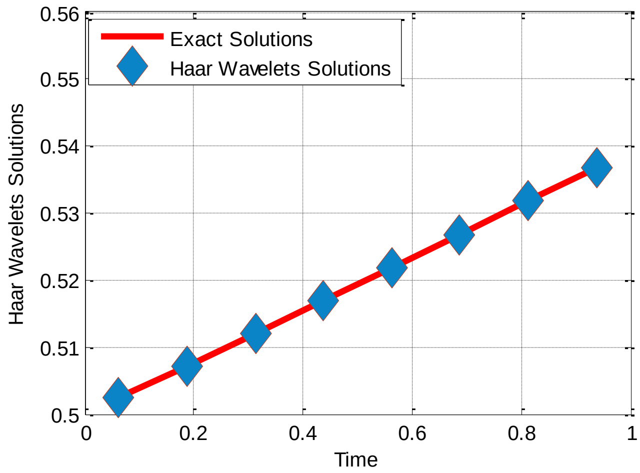

Figure 2: Comparison of Exact and Haar Wavelets Solutions

Table 1 and Table 2 show the comparison of numerical solutions obtained by Chebyshev wavelets of the second kind and the Haar wavelets with the exact solutions for Case – I. Figure 1 and Figure 2 show the comparison of absolute errors obtained by Chebyshev wavelets of the second kind and the Haar wavelets with exact solutions.

Case II: For $\delta = 1, \ K = 1, \ \alpha = 1, \ N_0 = 0.2$

The logistic model reduces to

$$Du(\tau) = u(\tau)\big(1 - u(\tau)\big), \ \ \tau > 0, \tag{20}$$

with the initial condition $u(0) = 0.2$.

Table 3: Comparison of solutions of Example 2

τ	Exact Solutions	Chebyshev wavelets solutions	Haar wavelets solutions
0.0625	0.2101877033	0.2101877033	0.2098765432
0.1875	0.2316897623	0.2316897623	0.2313851080
0.3125	0.2546821715	0.2546821715	0.2543882359
0.4375	0.2791273503	0.2791273502	0.2788485615
0.5625	0.3049588164	0.3049588164	0.3046996935
0.6875	0.3320795876	0.3320795876	0.3318445932
0.8125	0.3603617576	0.3603617575	0.3601551344
0.9375	0.3896474557	0.3896474556	0.3894730562

Table 4: Comparison of absolute errors of Example 2

τ	Absolute Errors (Chebyshev Wavelets)	Absolute Errors (Haar Wavelets)
0.0625	2.7184e-011	3.1116e-004
0.1875	2.0665e-011	3.0465e-004
0.3125	2.4250e-011	2.9394e-004
0.4375	2.4524e-011	2.7879e-004
0.5625	2.6927e-011	2.5912e-004
0.6875	2.6493e-011	2.3499e-004
0.8125	3.2023e-011	2.0662e-004
0.9375	8.4879e-012	1.7440e-004

Table 3 and Table 4 show the comparison of numerical solutions obtained by Chebyshev wavelets of the second kind and the Haar wavelets with the exact solutions for Case – II. Figure 3 and Figure 4 show the comparison of absolute errors obtained by Chebyshev wavelets of the second kind and the Haar wavelets with exact solutions.

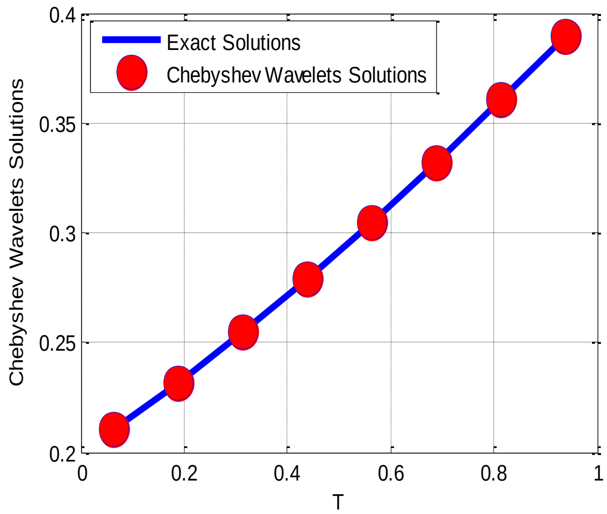

Fig. 3: Comparison of Exact and Chebyshev Wavelets Solutions

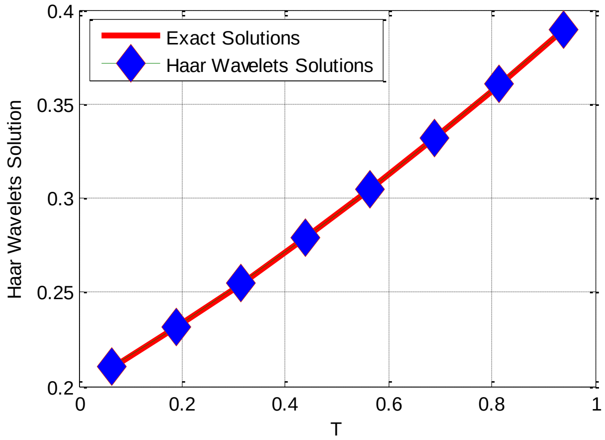

Fig. 4: Comparison of Exact and Haar Wavelets Solutions

V. Conclusion

Chebyshev wavelet is a powerful numerical tool for solving logistic differential equations arising in many applications of sciences and engineering. In this paper, we have utilized operational matrices of integration for solving such equations. Also, from the literature review, we concluded that the technique based on operational matrices is very fast, simple, and accurate rather than the method based on operational matrices of differentiation. Additionally, it is feasible to extend this method to matrices of higher orders, such as $100^{x}100$. By doing so, the number of collocation points increases, resulting in a solution that approaches the exact solution even more closely. Also, the comparison study shows that the proposed technique is well-established and efficient for solving such types of models. For future scope, such a technique will be applicable for two- and three-dimensional differential equations.

Conflict of interests

There is no conflict of interest in this study.

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