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NUMERICAL AND ANALYTICAL SOLUTION OF (1+1) DIMENSIONAL TELEGRAPH EQUATIONS USING LAPLACE VARIATIONAL ITERATION TECHNIQUE

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Abstract

This study adopts a novel approach that integrates the Laplace transform and the variational iteration method to tackle the (1+1)-Dimensional Telegraph equations, representing the current or voltage flow in electrical circuits. The methodology commences with transforming the telegraph equation into a modified format using Laplace transformation. Following this, the variational iteration method is utilized to derive both numerical and approximate analytical solutions. The paper incorporates practical examples to demonstrate the effectiveness of the proposed approach, supplemented with graphical illustrations depicting the outcomes achieved through the suggested techniques.

Keywords: Telegraph Equations, Laplace Transform, Variational iteration method.

I. Introduction

Numerous challenges involve problems related to initial values and boundary values featuring (1+1)-dimensional partial differential equations. Various numerical and analytical methods are available for addressing (1+1) dimensional problems encountered in scientific and engineering applications.

Several numerical techniques have gained importance in the area of research, including but not limited to the wavelet method, A domain decomposition method, finite volume method, and so on. These methodologies are known for their effectiveness in handling various computing problems and uses. The Laplace transform is a vital tool in the domain of computing and applied mathematics, particularly when working with differential and integral equations. Researchers have shifted their attention towards a hybrid methodology that combines variational iterative techniques with the Laplace transform. Due to its ability to generate numerical outcomes characterized by significant convergence and a high level of precision, this approach has garnered considerable interest. Variational iteration techniques, explored in [I, X, XIV, XXIII], have been instrumental in solving differential equations.

In [IV], a novel hybrid approach is introduced, seamlessly integrating algorithms of variational iteration combined with the Laplace transform. The use of variational iterative methods to the solution of the time-fractional diffusing equation in a penetrable environment is investigated [XIX]. The limitations inherent in the variational iteration method are scrutinized, and how the Laplace transform method overcomes these challenges are elucidated [VI, XV]. In [VII], the Laplace variational approach is implemented, tailored specifically to accommodate adjusted fractional derivatives. that include a non-singular kernel. Furthermore, the exploration extends to [XXI, XX], where the Laplace variational iteration method is devised for the solution of nonlinear PDE. In addition, [VIII, XVIII] proposes an efficient numerical technique grounded in wavelets, showcasing its efficacy in solving three-dimensional PDE.

Examine the hyperbolic telegraph equation of second-order linear nature in a (1+1)-dimensional space, presented as follows:

$$\frac{\partial^2 \omega}{\partial t^2}(x,t) + 2\rho\frac{\partial \omega}{\partial t}(x,t) + \sigma.\omega(x,t) = \xi\nabla^2\omega(x,t) + J(x,t), \quad 0 \le t \le T \tag{1}$$

the initial conditions

$$\omega(x,0) = f(x), \frac{\partial \omega}{\partial t}(x,0) = g(x)$$

The significance of hyperbolic PDE becomes apparent due to its crucial function in establishing fundamental equations in the fields of mechanics.

These equations prove invaluable for comprehending various phenomena across Various fields within the domain of applied sciences, including mechanical, electrical engineering, industry, aerospace, chemistry, and biology. The literature indicates many efforts directed toward addressing the challenges posed by one, 2D telegraph equations. "Numerous computational methods were developed to compute approximate solutions. In [III], the Taylor matrix approach is presented for resolving linear hyperbolic equations in two dimensions. Meanwhile, the dual reciprocity boundary integral method, as described in [V], tackles the resolution of 1D second-order hyperbolic telegraph equations. Unconditionally stable finite difference schemes, detailed in [XIII], offer a solution for one-dimensional linear hyperbolic equations. In [IX], a technique utilizing Haar wavelets is introduced for the resolution of two-dimensional telegraph equations. The utilization of interpolating scaling functions is documented in [XVI] for determining solutions to telegraph equations. Various methods for solving the differential equations have been discussed in [XI, XII]. For 1D hyperbolic telegraph equations, a cubic B-spline collocation method is introduced in [XVII]. Moreover, an adapted approach using B-spline differential quadrature method is formulated to address the resolution of 2D telegraph equations [XXIII]. Moreover, the Chebyshev tau technique is employed to address the telegraph equation [II].

II. Definition and properties of Laplace Transform

Consider $\phi(t)$ as a function defined over all positive values of t.

The L.T. of the function $\phi(t)$, indicated as $L\{\phi(t)\}$, is defined in the following manner:

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)dt = \bar{\phi}(s),$$

Assuming the integral exists, 's' serves as a parameter that may take on either real or complex values. Consequently,

$$L\{\phi(t)\} = \bar{\phi}(s),$$

that is

$$\phi(t) = L^{-1}\{\bar{\phi}(s)\}.$$

The expression $L^{-1}\{\bar{\phi}(s)\}$, refers to the inverse Laplace transform of $\bar{\phi}(s)$.

(a) Consider three functions of time $\phi(t),\ \psi(t),\ \chi(t)$, defined for all t-values greater than zero. In this context,

$$L\{c.\phi(t) + d.\psi(t) + e.\chi(t)\} = c.L\{\phi(t)\} + d.L\{\psi(t)\} + e.L\{\chi(t)\}$$

$c, d,$ and e are arbitrary constants.

(b) The L.T. is highly effective when dealing with the nth-order differentiation of a function. Examine a function $\phi(t)$ that is defined for all t-values greater than zero. In this context, L.T. of the nth derivative of $\phi(t)$ is expressed as follows:

$$L\left[\frac{d^n(\phi(t))}{dt^n}\right] = s^n\bar{\phi}(s) - s^{n-1}\phi(0) - s^{n-2}\phi'(0) - s^{n-3}\phi''^{(0)} - \cdots - s\phi^{(n-2)}(0) - \phi^{(n-1)}(0)$$

where $\bar{\phi}(s) = L\{\phi(t)\}$.

(c) This also extends to the inverse Laplace transform. Consider any three functions of time, $\phi(t)$, $\psi(t)$, and $\chi(t)$, defined $\forall\ \ t > 0$. Additionally, let $\bar{\phi}(s)$, $\bar{\psi}(s)$, and $\bar{\chi}(s)$ represent their respective functions in the Laplace transform domain. In this context,

$\bar{\phi}(s) = L\{\phi(t)\}, \bar{\psi}(s) = L\{y(t)\}$ and $\bar{\chi}(s) = L\{\chi(t)\}$. Then

$$L^{-1}\{c.\bar{\phi}(s) + d.\bar{\psi}(s) + e.\bar{\chi}(s)\} = c.L^{-1}\{\bar{\phi}(s)\} + d.L\{\bar{\psi}(s)\} + e.L\{\bar{\chi}(s)\}$$
$$= c.\phi(t) + d.\psi(t) + e.\chi(t)$$

where $c, d,$ and e are arbitrary constants.

(d) Consider two functions, $\phi(t)$ *and* $\psi(t)$, along with their respective Laplace transformations denoted as $\bar{\phi}(s)$ *and* $\bar{\psi}(s)$. In this context, the inverse L. T. of the product of $\bar{\phi}(s)$,$\bar{\psi}(s)$ is expressed as:

$$L^{-1}\{\bar{\phi}(s).\bar{\psi}(s)\} = \int_0^t \phi(u) \,.\psi\,(t-u)du$$

III. Variational iterative method (VIM)

One particularly useful approach for estimating solutions to a wide range of issues found in a variety of scientific and technical applications is the variational iterative method. This approach is notable for its quick convergence rate, which results in solutions in a series that converges infinitely.

When working with the non-linear terms found in PDE, the effectiveness of the variational iteration approach is very apparent. To illustrate, let's explore the following set of differential equations:

$$\mathrm{P}\omega(\mathrm{x},\mathrm{t}) + \mathrm{Q}\omega(\mathrm{x},\mathrm{t}) = \mathrm{R}(\mathrm{x},\mathrm{t}) \tag{2}$$

with specified initial conditions

$$\omega(\mathrm{x}, 0) = \mathrm{G}(\mathrm{x}) \ \text{ and } \ \omega'(\mathrm{x}, 0) = \mathrm{H}(\mathrm{x}) \tag{3}$$

In the context of the equation, P stands for either a first or second-order linear operator, while Q represents a nonlinear operator, while R denotes the non-homogeneous term. Apply the VIM to derive a corrective functional as follows:

$$\omega_{k+1} = \omega_k + \int_0^t \lambda[\mathbf{P}\omega_k(\mathrm{x},\mathrm{s}) + \mathbf{Q}\tilde{\omega}_k(\mathrm{x},\mathrm{s}) - \mathbf{R}(\mathrm{x},\mathrm{s})]\mathrm{ds} \tag{4}$$

Here, λ represents Lagrange's multiplier, while **k** signifies the k^{th} approximations. The function ω_k is recognized as the constrained function, adhering to the condition $\delta\tilde{\omega}_k = 0$. Subsequently, the successive approx. ω_{k+1} of the solution, ω is derived using λ and the initial approximation w_0. The resulting solution is expressed as:

In this context, λ functions as Lagrange's multiplier, with k denoting kth approximations. The function $\tilde{\omega}_k$ is acknowledged as the constrained function, adhering to the condition $\delta\tilde{\omega}_k = 0$. Subsequently, the next approximation, ω_{k+1},for the solution w is computed utilizing λ and the initial approximation w_0. The resulting solution is articulated as follows:

$$\omega = \lim_{k\to\infty} \omega_k$$

IV. Proposed Method to Solve (1+1) dimensional Telegraph Equations

A hybrid strategy that combines the L. T. method with a modified variational iteration method is presented. This integrated approach is used to solve (1+1)-dimensional Telegraphic partial differential equations, which are often encountered in several fields of engineering. The objective is to achieve the precise or analytical solution for the (1+1)-dimensional telegraphic equation, presented in a convergent series structure with components that may be readily computed.

Moreover, this hybrid approach showcases its adaptability in other fields and also provides a more efficient computing procedure, hence improving the ease of accessing the solution.

Let P is a second-order operator, denoted by $\frac{\partial^2}{\partial t^2}$. In this context, Equation (2) transforms into:

$$\frac{\partial^2}{\partial t^2}\omega(x,t) + \mathbf{Q}w(x,t) = \mathbf{R}(x,t) \tag{5}$$

Upon employing L.T. at equation (5), we have:

$$L\left\{\frac{\partial^2}{\partial t^2}\omega(x,t)\right\} + L\{\mathbf{Q}\omega(x,t)\} = L\{\mathbf{R}(x,t)\} \tag{6}$$

$$s^2L\{\omega(x,t)\} - sG(x) - H(x) = L\{\mathbf{R}(x,t)\} - L\{\mathbf{Q}\omega(x,t)\} \tag{7}$$

Upon performing the inverse Laplace transformation on both ends of equation (7):

$$\omega(x,t) = \mathbf{R}^1(x,t) - L^{-1}\left[\frac{1}{s^2}L\{\mathbf{Q}\omega(x,t)\}\right] \tag{8}$$

Here, $\mathbf{R}^1$ represents the component arising from the source term and the given initial conditions. To address this, we leverage the correctional function within the variational iteration method.

$$\omega_{k+1}(x,t) = \mathbf{R}^1(x,t) - L^{-1}\left[\frac{1}{s^2}L\{\mathbf{Q}\omega_k(x,t)\}\right] \tag{9}$$

Equation (9) signifies the updated and modified correction functional derived from the L.T. of variation method. The solution to the aforementioned equation is expressed as follows:

$$\omega(x,t) = \lim_{k\to\infty}\omega_k(x,t) \tag{10}$$

V. Numerical Applications

To evaluate the efficiency and accuracy of the suggested hybrid method, we provide a set of illustrative examples. This evaluation aims to showcase the actual implementation and effectiveness of the proposed strategy in various situations.

Example 1: Take into consideration the subsequent (1+1)-dimensional telegraphic equation

$$\frac{\partial^2\omega(x,t)}{\partial t^2} + 2\frac{\partial\omega(x,t)}{\partial t} + \omega(x,t) = \nabla^2\omega(x,t) \tag{10}$$

with the initial conditions

$$\omega(x,0) = \sinh x \text{ and } \omega_t(x,0) = -\sinh x$$

With Laplace transformation on both sides,

$$L\left\{\frac{\partial^2\omega(x,t)}{\partial t^2} + 2\frac{\partial\omega(x,t)}{\partial t} + \omega(x,t)\right\} = L\{\nabla^2\omega(x,t)\} \tag{11}$$

This implies

$$s^2L\{\omega(x,t)\} - s\omega(x,0) - \omega_t(x,0) + 2sL\{\omega(x,t)\} - 2\omega(x,0) + L\{\omega(x,t)\} = L\{\nabla^2\omega(x,t)\}$$

i.e.

$$s^2L\{\omega(x,t)\} - s(\sinh x) + \sinh x + 2sL\{\omega(x,t)\} - 2\sinh x + L\{\omega(x,t)\} = L\{\nabla^2\omega(x,t)\}$$

Utilizing the given initial conditions, we derive the following:

$$(s+1)^2L\{\omega(x,t)\} = (s+1)(\sinh x) + L\{\nabla^2\omega(x,t)\}$$

Divide by $(s+1)^2$,

$$L\{\omega(x,t)\} = \frac{(\sinh x)}{(s+1)} + \frac{1}{(s+1)^2}L\{\nabla^2\omega(x,t)\} \qquad (12)$$

By utilizing inverse L.T. at equation, (12), we get the resultant expression:

$$\omega = e^{-t}(\sinh x) + L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2\omega(x,t)\}\right] \qquad (13)$$

By using the variational iteration approach to equation (13), so:

$$\omega_{m+1} = e^{-t}(\sinh x) + L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2\omega_m\}\right] \qquad (14)$$

From (14), we obtain

$$\omega_0 = \sinh x,$$

$$\omega_1 = \sinh x\,(1 - t\,e^{-t}),$$

$$\omega_2 = \sinh x\left(1 - t\,e^{-t} - \frac{t^3e^{-t}}{3!} - \frac{t^5e^{-t}}{5!}\right),$$

$$\omega_3 = \sinh x\left(1 - t\,e^{-t} - \frac{t^3e^{-t}}{3!} - \frac{t^5e^{-t}}{5!} - \frac{t^7e^{-t}}{7!} - \frac{t^9e^{-t}}{9!}\right),$$

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and so on.

The solution is as follows

$$\omega = \lim_{m\to\infty}\omega_m$$

Upon simplification, the expression transforms into:

$$\omega = \sinh x\left(1 - t\,e^{-t} - \frac{t^3e^{-t}}{3!} - \frac{t^5e^{-t}}{5!} - \frac{t^7e^{-t}}{7!} - \frac{t^9e^{-t}}{9!} \ldots\ldots\ldots\ldots\right),$$

i.e

$$\omega = \text{sinhx}\left(1 - e^{-t}\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} \dots \dots \dots \dots\right)\right)$$

This implies

$$\omega = \text{sinhx}\left(1 - e^{-t}\left(\frac{e^t - e^{-t}}{2}\right)\right)$$

Hence

$$\omega = \text{sinhx}\,(1 - e^{-t}\text{sinht}) \tag{15}$$

Table 1: Comparison between Exact and Approximate solutions at x = 1

t	Exact Solution	Approximate Solution	Absolute Error
0	1.1752011936438	1.1752011936438	0
0.1	1.06868727596697	1.06868727596697	0
0.2	0.981481055934126	0.981481055934127	9.99201E-16
0.3	0.910082641733554	0.910082641733593	3.89688E-14
0.4	0.851626564306546	0.851626564307375	8.29004E-13
0.5	0.803766776012747	0.803766776021481	8.73401E-12
0.6	0.764582495500813	0.764582495559568	5.8755E-11
0.7	0.732501120008457	0.732501120298455	2.89998E-10
0.8	0.706235111291824	0.706235112432846	1.14102E-09
0.9	0.684730322194901	0.68473032597077	3.77587E-09
1	0.667123690022795	0.667123700923393	1.09006E-08

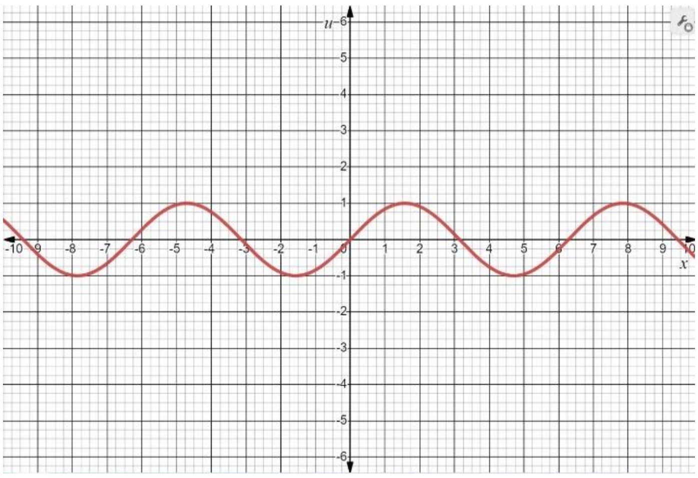

Fig. 1.a. Physical representation of solution at t = 0 sec

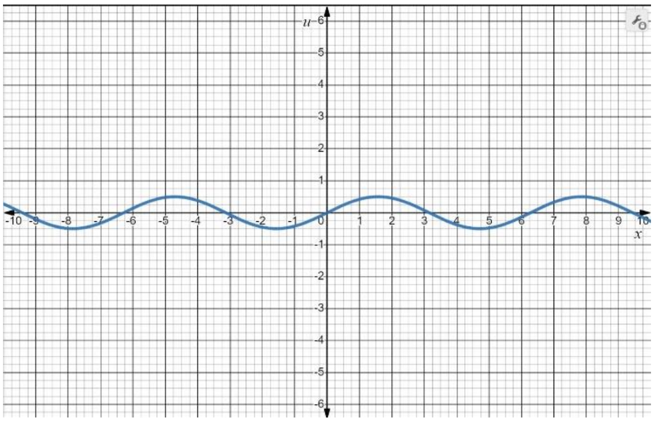

Fig. 1.b. Physical representation of solution at t = 5 sec

Figures 2a and 2b represent the actual characteristics or performance of the solution $\omega = \sinh x\,(1 - e^{-t}\sinh t)$ at t = 0 sec and at t = 5 sec respectively.

Example 2: Consider the (1+1) dimensional telegraph equation

$$\frac{\partial^2\omega(x,t)}{\partial t^2} + 2\frac{\partial\omega(x,t)}{\partial t} + \omega(x,t) = \nabla^2\omega(x,t) \tag{16}$$

With initial conditions

$$\omega(x,0) = e^x \text{ and } \omega_t(x,0) = -e^x$$

Utilizing the L.T. on equation (16), the resulting expression is:

$$L\left\{\frac{\partial^2\omega(x,t)}{\partial t^2} + 2\frac{\partial\omega(x,,t)}{\partial t} + \omega(x,t)\right\} = L\{\nabla^2\omega(x,t)\}$$

This implies

$$s^2L\{\omega(x,t)\} - s\omega(x,0) - \omega_t(x,0) + 2sL\{\omega(x,t)\} - 2\omega(x,0) + L\{\omega(x,t)\} = L\{\nabla^2\omega(x,t)\}$$

This becomes

$$s^2L\{\omega(x,t)\} - s(e^x) + 2\,e^x + 2sL\{\omega(x,t)\} - 2\,e^x + L\{\omega(x,t)\} = L\{\nabla^2\omega(x,t)\}$$

Using the initial conditions,

$$(s+1)^2L\{\omega(x,t)\} = (s+1)(e^x) + L\{\nabla^2\omega(x,t)\}$$

Divide by $(s+1)^2$,

$$L\{\omega(x,t)\} = \frac{(e^x)}{(s+1)} + \frac{1}{(s+1)^2}L\{\nabla^2\omega(x,t)\} \tag{17}$$

Upon performing the inverse Laplace transformation of equation (17), resulting expression is obtained as follows:

$$\omega = e^{-t}(e^{x}) + L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2\omega(x,t)\}\right] \tag{18}$$

Utilizing the variational iteration approach on equation (18), we attain the following equation:

$$\omega_{m+1} = e^{x-t} + L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2\omega_m\}\right] \tag{19}$$

Deriving from equation (19), we have

$$\omega_0 = e^{x},$$

$$\omega_1 = e^{x}(1 - t\,e^{-t}),$$

$$\omega_2 = e^{x}\left(1 - t\,e^{-t} - \frac{t^3e^{-t}}{3!} - \frac{t^5e^{-t}}{5!}\right),$$

$$\omega_3 = e^{x}\left(1 - t\,e^{-t} - \frac{t^3e^{-t}}{3!} - \frac{t^5e^{-t}}{5!} - \frac{t^7e^{-t}}{7!} - \frac{t^9e^{-t}}{9!}\right),$$

and so on. The solution is

$$\omega = \lim_{m\to\infty}\omega_m$$

Following the simplification process, the expression transforms into:

$$\omega = e^{x}\left(1 - t\,e^{-t} - \frac{t^3e^{-t}}{3!} - \frac{t^5e^{-t}}{5!} - \frac{t^7e^{-t}}{7!} - \frac{t^9e^{-t}}{9!} \dots\dots\dots\dots\right)$$

This becomes

$$\omega = e^{x}\left(1 - e^{-t}\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} \dots\dots\dots\dots\right)\right)$$

This implies

$$\omega = e^{x}\left(1 - e^{-t}\left(\frac{e^{t} - e^{-t}}{2}\right)\right)$$

Therefore

$$\omega = e^{x}\left(\frac{1 + e^{-2t}}{2}\right)$$

Table 2 : Comparison between Exact and approximate solutions at x = 1

t	Exact Solution	Approximate Solution	Absolute Error
0	2.71828182845905	2.71828182845905	0
0.1	2.47191137847576	2.47191137847576	0
0.2	2.27020031442478	2.27020031442478	0
0.3	2.10505326305016	2.10505326305025	8.9706E-14
0.4	1.96984229330961	1.96984229331152	1.91003E-12
0.5	1.85914091422952	1.85914091424972	2.01998E-11
0.6	1.76850629076851	1.76850629090442	1.3591E-10
0.7	1.69430093724734	1.69430093791812	6.7078E-10
0.8	1.63354673227654	1.63354673491576	2.63922E-09
0.9	1.58380539628813	1.58380540502185	8.73372E-09
1	1.54308063481524	1.54308066002871	2.52135E-08

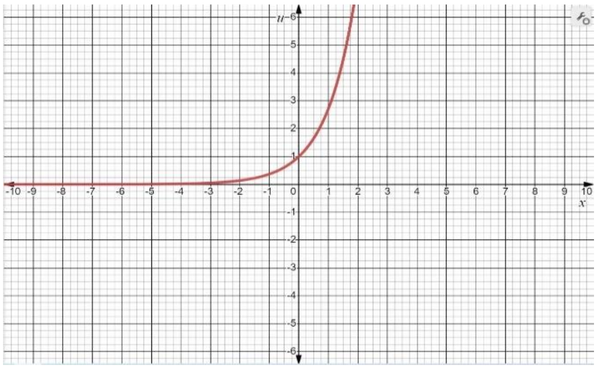

Fig. 2.a. Physical representation of solution at t = 0 sec

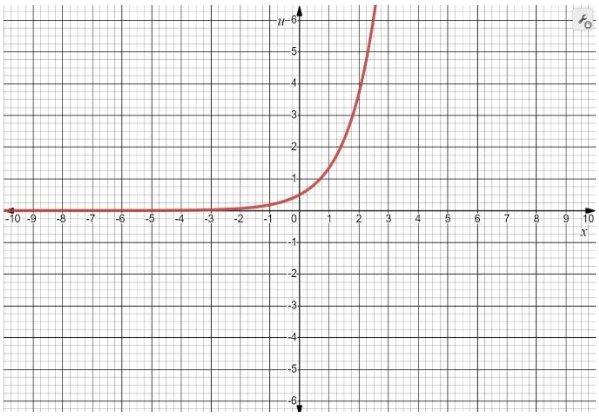

Fig. 2.b. Physical representation of solution at t = 5 sec

Figures 2a and 2b represent the physical behavior of the solution $\omega = e^{x}\left(\frac{1+e^{-2t}}{2}\right)$ at t = 0 sec and at t = 5 sec respectively.

V. Conclusions

The outcomes suggest that the novel hybrid approach, incorporating the Laplace transform method and the modified variational iteration method, serves as a highly accurate and efficient semi-analytical approach in addressing (1+1) dimensional telegraph differential equations. As a prospective avenue, this method holds promise for extending its application to tackle (1+1)-dimensional nonlinear partial differential equations in future investigations.

Conflict of Interest:

The author declares that they have no conflict of interest.

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