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AN EFFICIENT TECHNIQUE FOR SOLVING ONE-DIMENSIONAL HEAT EQUATIONS ARISING IN THE DIFFUSION PROCESS

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Abstract

This study utilizes an innovative approach by combining the Laplace transform with the variational iteration method to address one-dimensional heat equations encountered in diffusion phenomena. Initially, the heat equation is transformed into a modified form using the Laplace transformation. Subsequently, the variational iteration method is employed to obtain both numerical and approximate analytical solutions. In addition to graphical representations of the outcomes obtained using the suggested, the study includes practical instances to demonstrate the efficacy of the suggested approach.

Keywords: Laplace Transform, heat equations, Variational iteration method, Partial differential equations.

I. Introduction

The study and analysis of one-dimensional heat equations play a pivotal role in understanding diffusion processes across various scientific domains. Heat conduction, a fundamental concept in physics and engineering, is often modeled by one-dimensional heat equations. Researchers and scientists have continuously sought efficient methods to solve these equations, aiming to gain insights into the temperature distribution and heat transfer phenomena. In this context, a noteworthy development emerges in the form of a hybrid method designed specifically for addressing one-dimensional heat equations. This creative method provides a thorough and precise response by fusing the advantages of many mathematical methodologies. Temperature changes in one-dimensional heat equations often take place in relation to time and space, and the hybrid technique provides a special combination of mathematical tools to deal with the complexities involved in such processes of diffusion. The hybrid method often incorporates established methodologies like the Laplace transform and variational iteration techniques. By merging these approaches, researchers aim to capitalize on the advantages each method brings to the table, resulting in a more robust and efficient solution strategy. This hybrid technique not only enhances the method's application but also extends the applicability of method to a wider range of situations. The following is an explanation of the one-dimensional heat equation:

$$\frac{\partial u(x,t)}{\partial t} = c^2\nabla^2 u(x,t) \tag{1}$$

Laplace transform, which converts differential equations into simple equations and convolutions into multiplications, is a crucial analytical technique for solving differential equations. A well-known numerical technique for solving differential equations is the variational iterative approach. If there is an exact solution to the differential equations, variational iterations can be used to find it.

There are several new mathematical techniques available for evaluating two-dimensional heat equations. Two-dimensional heat equations have been solved using the finite difference approach [V]. In [I], the Chebyshev series solution for the two-dimensional heat equations has been is presented. In [VII], a methodical fusion of the Finite Difference Method and Collocation method has been devised to tackle the challenges posed by two-dimensional heat equations. This collaborative approach showcases a strategic integration of numerical techniques, demonstrating its applicability and efficiency in the realm of solving complex heat equation scenarios. In [XIII], the Radial basis function method has been illustrated to solve two-dimensional heat equations. Meanwhile, the variational iteration technique has been introduced in [IV, VI, XIV], for solving -nonlinear equations. Various methods for solving the one-dimensional heat equations have been discussed in [II, VIII, XI].

The implications of this hybrid method extend beyond the realm of theoretical exploration. Its practical significance is particularly pronounced in fields such as materials science, environmental engineering, and industrial processes, where a precise understanding of heat distribution is crucial. By providing a versatile tool for solving one-dimensional heat equations, this hybrid method contributes significantly to advancing our ability to model and predict heat-related phenomena, offering valuable insights for both academic research and practical applications. Some other methods and applications related to differential equations have been discussed in [III, IX, X XII].

II. New Hybrid method for solving one-dimensional heat equations

This part introduces a hybrid approach that combines the L. T. method with the modified variational iteration method. The goal is to address one-dimensional heat P.D.E. that emerges in diffusion processes. The one-dimensional heat equation's exact or analytical solution has been derived as a convergent series, featuring easily computable components.

Consider the following one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = c^2\nabla^2 u(x,t) \tag{2}$$

with initial conditions

$$u(x,0) = f(x)$$

Applying L. T. of (2), we obtain

$$L\left\{\frac{\partial u(x,t)}{\partial t}\right\} = c^2 L\{\nabla^2 u(x,t)\} \tag{3}$$

This implies

$$pL\{u(x,t)\} - u(x,0) = c^2 L\{\nabla^2 u(x,t)\} \tag{4}$$

Applying initial conditions, we obtain

$$pL\{u(x,t)\} = f(x) + c^2 L\{\nabla^2 u(x,t)\} \tag{5}$$

Divide both sides by p, so,

$$L\{u(x,t)\} = \frac{f(x)}{p} + \frac{c^2}{p} L\{\nabla^2 u(x,t)\} \tag{6}$$

By applying inverse L.T. of equation (6), we have,

$$u = f(x) + L^{-1}\left[\frac{c^2}{p} L\{\nabla^2 u(x,t)\}\right] \tag{7}$$

By employing the iteration method, we derive the following from equation (7)

$$u_{m+1} = f(x) + L^{-1}\left[\frac{c^2}{p} L\{\nabla^2 u_m\}\right] \tag{8}$$

III. Numerical Experiments:

Here, some examples of heat equations are given and solved by using a new hybrid method.

Example 1: Consider the following one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \nabla^2 u(x,t) \tag{9}$$

with initial conditions

$$u(x,0) = \sin x$$

Applying L.T. of (9), we have

$$L\left\{\frac{\partial u(x,t)}{\partial t}\right\} = L\{\nabla^2 u(x,t)\} \tag{10}$$

This implies

$$pL\{u(x,t)\} - u(x,0) = L\{\nabla^2 u(x,t)\}$$

Applying initial conditions,

$$pL\{u(x,t)\} = \sin x + L\{\nabla^2 u(x,t)\}$$

Divide both sides by p,

$$L\{u(x,t)\} = \frac{\sin x}{p} + \frac{1}{p} L\{\nabla^2 u(x,t)\} \quad (11)$$

By applying inverse L.T. to equation (6), we have,

$$u = \sin x + L^{-1}\left[\frac{1}{p} L\{\nabla^2 u(x,t)\}\right] \quad (12)$$

By employing the iteration method, we derive the following from equation (12)

$$u_{m+1} = \sin x + L^{-1}\left[\frac{1}{p} L\{\nabla^2 u_m\}\right] \quad (13)$$

From (13), we obtain

$$u_0 = \sin x$$
$$u_1 = \sin x\,(1 - t)$$
$$u_2 = \sin x\left(1 - t + \frac{(t)^2}{2!}\right)$$
$$u_3 = \sin x\left(1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!}\right)\ldots$$
$$u_m = \sin x\left(1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!} + \cdots + \frac{(-1)^m (t)^m}{m!}\right)$$

The solution is

$$u = \lim_{n\to\infty} u_m$$

After simplification, we obtain

$$u = \sin x\left(1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!}\ldots\right)$$
$$u = \sin x\ e^{-t} \quad (14)$$

Table 1: Comparison between Exact and Approximate solutions at x = 1

t	Exact Value	Approximate Value (upto 6th iteration)	Absolute Error
0	0.841470984807897	0.841470984807897	0
0.1	0.761394433245753	0.761394433262243	1.649E-11
0.2	0.68893817308504	0.688938175169847	2.08481E-09
0.3	0.623377037720679	0.62337707290955	3.51889E-08
0.4	0.564054869274084	0.564055129726112	2.60452E-07
0.5	0.510377951544573	0.510379178697828	1.22715E-06
0.6	0.461809067898073	0.461813413175118	4.34528E-06
0.7	0.417862124622486	0.417874758690123	1.26341E-05
0.8	0.37809728593843	0.378129086337263	3.18004E-05
0.9	0.342116571974933	0.342188267624795	7.16956E-05
1	0.309559875653112	0.309708070797351	1.48195E-04

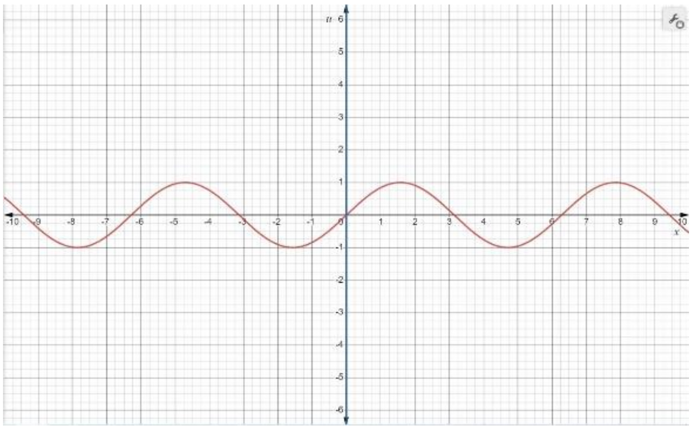

Fig. 1.a. Physical behaviour of solution at t = 0

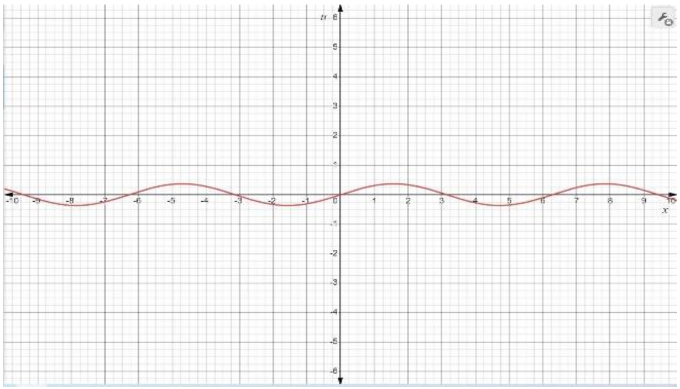

Fig. 1.b. Physical behaviour of solution at t = 1 sec

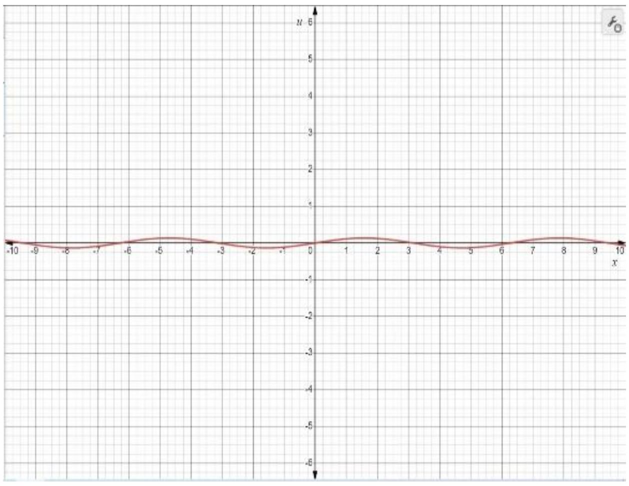

Fig. 1.c. Physical behaviour of solution at t = 2 sec

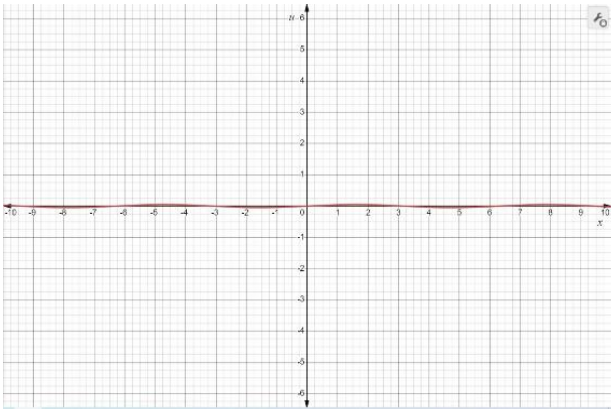

Fig. 1.d. Physical behaviour of solution at t = 3 sec

The above figures represent the physical behavior of the solution $u = \sin x \ e^{-t}$ at t = 0 sec, t = 1 sec, t = 2 sec and t = 3 sec respectively.

Example 2: Consider a two-dimensional Heat equation

$$\frac{\partial u(x,t)}{\partial t} = \nabla^2 u(x,t) \tag{15}$$

where

$$u(x,0) = e^x$$

By applying L. T. on both sides of (15),

$$L\left\{\frac{\partial u(x,t)}{\partial t}\right\} = L\{\nabla^2 u(x,t)\}$$

This implies

$$pL\{u(x,t)\} - u(x,0) = L\{\nabla^2 u(x,t)\}$$

Applying initial conditions,

$$pL\{u(x,t)\} = e^x + L\{\nabla^2 u(x,t)\}$$

Divide by p,

$$L\{u(x,t)\} = \frac{e^x}{p} + \frac{1}{p} L\{\nabla^2 u(x,t)\} \tag{16}$$

By applying the inverse L.T. to equation (16), we have,

$$u = e^x + L^{-1}\left[\frac{1}{p} L\{\nabla^2 u(x,t)\}\right] \tag{17}$$

By employing the iteration method, we derive the following from equation (17)

$$u_{m+1} = e^x + L^{-1}\left[\frac{1}{p} L\{\nabla^2 u_m\}\right] \tag{18}$$

From (18), we obtain

$$u_0 = e^x$$

$$u_1 = e^x(1+t)$$

$$u_2 = e^x\left(1+t+\frac{(t)^2}{2!}\right)$$

$$u_3 = e^x\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}\right)\ldots$$

$$u_m = e^x\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots+\frac{(t)^m}{m!}\right)$$

The solution is obtained as

$$u = \lim_{n\to\infty} u_m$$

$$u = e^x\left(1+t+\frac{(t)^2}{2!}+\frac{(t)^3}{3!}+\cdots\right)$$

$$u = e^x(e^t)$$

$$u = e^{x+t}$$

Table 2 : Comparison between Exact and Approximate solutions at x = 1

t	Exact Value	Approximate Value (upto 6th iteration)	Absolute Error
0	2.71828182845905	2.71828182845905	0
0.1	3.00416602394643	3.00416602389182	5.46097E-11
0.2	3.32011692273655	3.32011691565647	7.08008E-09
0.3	3.66929666761924	3.66929654508996	1.22529E-07
0.4	4.05519996684467	4.05519903695929	9.29885E-07
0.5	4.48168907033806	4.48168457798435	4.49235E-06
0.6	4.95303242439511	4.95301611364263	1.63108E-05
0.7	5.4739473917272	5.4738987632557	4.86285E-05
0.8	6.04964746441295	6.04952195335757	1.25511E-04
0.9	6.68589444227927	6.68560426934487	2.90173E-04
1	7.38905609893065	7.38844102540882	6.15074E-04

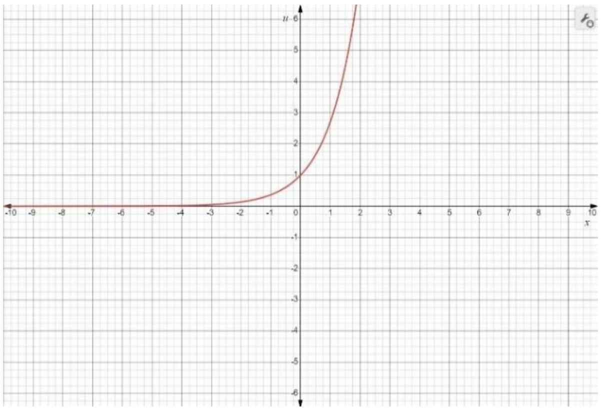

Fig. 2.a. Physical behavior of solution at t = 0

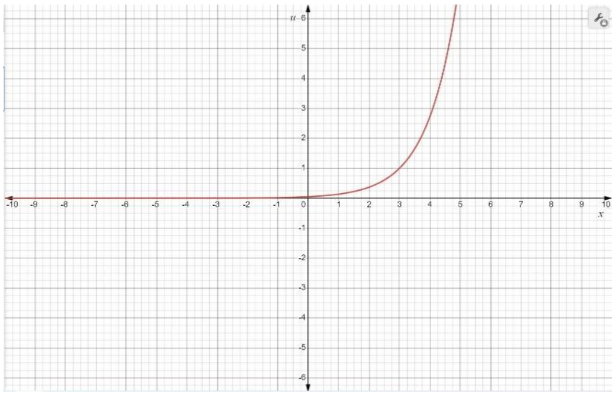

Fig. 2.b. Physical behavior of solution at t = 3

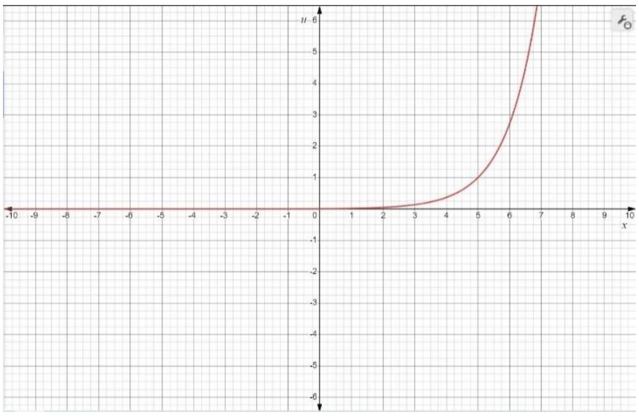

Fig. 2.c. Physical behavior of solution at t = 5

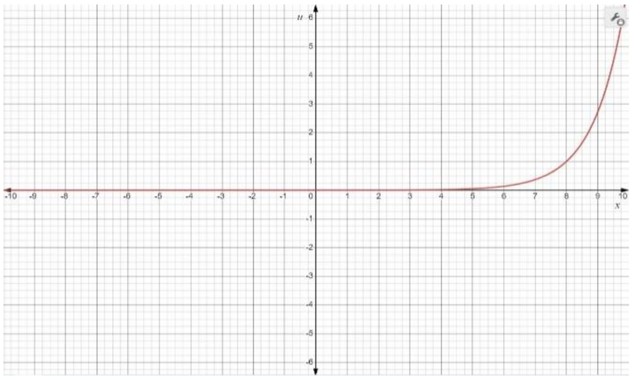

Fig. 2.d. Physical behavior of solution at t = 8

The above figures represent the physical behavior of the solution $u = e^{x+t}$ at t = 0 sec, t = 3 sec, t = 5 sec and t = 8 sec respectively.

V. Conclusion

Based on the outcomes of the resolved numerical examples, it is evident that the recently introduced semi-analytical technique, formed by combining the Laplace transform and the modified variational iterative method, proves to be a highly effective mathematical approach for resolving one-dimensional heat equations with ease. The demonstrated efficiency of this method suggests its potential applicability to address two-dimensional and three-dimensional linear as well as non-linear heat equations in subsequent research endeavors.

Conflict of Interest:

The authors declare that they have no conflict of interest.

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