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SOLVING 2D AND 3D TELEGRAPH EQUATIONS WITH ELZAKI TRANSFORM AND HOMOTOPY PERTURBATION METHOD

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Abstract

This study investigates the solution of complex mathematical problems of two-dimensional and three-dimensional telegraph equations. To solve these equations, we use a comprehensive approach that combines the Elzaki transform and the homotopy perturbation method (HPM) and provides a systematic and efficient means of obtaining exact solutions to these problems. Our methodology is rigorously tested in both 2 and 3 dimensions, demonstrating its effectiveness.

Keywords: Telegraph equation, Homotopy Perturbation method, Elzaki transform, numerical problems.

I. Introduction

Exploring mathematical models is crucial for comprehending and predicting real-world phenomena, and within this realm, the application of 2-dimensional (2D) and 3-dimensional (3D) telegraph equations hold significant importance. Heaviside initially formulated the telegraph equation in 1876, focusing on the dynamics of charging and discharging a finite length of cable in the presence of induction [I]. These equations play a vital role in various scientific domains, including physics, engineering, and telecommunications, by effectively describing the characteristics of waves, signals, and electrical currents in spatial dimensions. Practically, the relevance of 2D and 3D telegraph equations is evident in their application to analyze signal transmission through intricate mediums like electrical conductors and communication channels. Their scope extends across diverse fields, encompassing electromagnetic wave propagation in waveguides and the modeling of signal transmission in advanced communication networks. A profound understanding of these equations is essential for optimizing signal processing, designing efficient communication systems, and gaining valuable insights into wave behaviors. Despite the established utility of 2D and 3D telegraph equations, solving them remains a formidable task due to their inherent complexity. This study aims to address this challenge by introducing an innovative methodology that incorporates the Elzaki transform along with the homotopy perturbation method (HPM). Through this inventive approach, we seek to offer a systematic and efficient means of obtaining analytical solutions for 2D and 3D telegraph equations, contributing to the progression of mathematical modeling and its applications across various scientific disciplines.

2D Telegraph equation [XIV]:

$$\frac{\partial^2\varphi}{\partial t^2} + 2\alpha\frac{\partial\varphi}{\partial t} + \beta^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + f(x, y, t), \quad t > 0 \tag{1}$$

defined in the region $\Omega \times \partial\Omega$ where $\Omega = [a < x < b, c < y < d] \times [t > 0]$, with boundary $\partial\Omega, \alpha > 0, \beta \geq 0$ are real numbers and a, b, c, d are arbitrary constants.

The initial conditions are given by

$$\varphi(x, y, 0) = \xi(x, y), \qquad \varphi_t(x, y, 0) = \vartheta(x, y),$$

3D Telegraph equation [XIV]:

$$\frac{\partial^2\varphi}{\partial t^2} + 2\alpha\frac{\partial\varphi}{\partial t} + \beta^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} + f(x, y, z, t), \quad t > 0 \tag{2}$$

Defined in the region $\Omega \times \partial\Omega$where $\Omega = [a_1 < x < b_1, a_2 < y < b_2, a_3 < z < b_3] \times [t > 0]$, with boundary $\partial\Omega, \alpha > 0, \beta \geq 0$ are real numbers and $a_1, b_1, a_2, b_2, a_3, b_3$ are arbitrary constants.

The initial conditions are given by

$$\varphi(x, y, z, 0) = \xi(x, y, z), \qquad \varphi_t(x, y, z, 0) = \vartheta(x, y, z),$$

Multi-dimensional telegraph equations have been studied using a variety of techniques [I], [II], [VI], [VII], [XIII], [XIV] the findings in these studies expedite the investigation of this problem. [III-V] authors have provided a summary of the homotopy perturbation technique and explored its uses for tackling various types of differential equations. A novel 'Elzaki transform' is described in [IX-XI], along with its applications in solving partial differential equations. Its correlation with the Laplace transform is also discussed. In [VIII] a hybrid method of Elzaki transform and homotopy perturbation method has been discussed, whereas [XII], provided a convergence and error estimation for the suggested method.

This research paper is arranged as: Section II provides an introduction to the Elzaki integral transform along with an exploration of its properties. In section III, the combined form of the homotopy perturbation method and the Elzaki integral transform (ETHPM) has been discussed. The convergence analysis of the proposed technique has been outlined in Section IV. In section V, the results and discussion regarding the validation and efficacy of the proposed method have been presented. A series of numerical exercises investigated the solution of two- and three-dimensional telegraph equations, demonstrating the simplicity and accuracy of the proposed method. Finally, in section VI, the conclusion of this study has been discussed.

This study investigates the semi-analytical solution of well-known (2+1)-D and (3+1)-D telegraph equations. For this purpose, we have used the combination of Elzaki transform and the homotopy perturbation method. To illustrate the simplicity and accuracy of the proposed technique, some test examples have been performed. The results provide accurate solutions and improve our understanding of temporal changes in these dimensions.

II. Elzaki transform and its properties [IX]

The Elzaki transform is defined for the functions of exponential order. Consider the set S, as defined in [IX].

$$S = \left\{ h(t) : \exists\, M, k_1, k_2 > 0, |h(t)| < M\, e^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

For this set S, $M < \infty$, k_1, k_2 can be either finite or infinite.
Assume any function h(t), the Elzaki transform of h$(t)$is termed as [IX]:

$$E_L\{h(t)\} = T(\mathrm{v}) = \mathrm{v} \int_0^\infty h(t).\, e^{-\frac{t}{\mathrm{v}}} d\mathrm{t}\,, \qquad t > 0, k_1 \le \mathrm{v} \le k_2$$

where$k_1$and k_2 can take on either finite values or infinite values.

III. Combined Form of Elzaki Transform and Homotopy Perturbation Method

Here, we present the procedure employed for ETHPM (see [VIII]). The equation for the problem is given below:

$$D\{u(x, y, z, t)\} + R\{u(x, y, z, t)\} + N\{u(x, y, z, t)\} = g(x, y, z, t) \qquad (3)$$

With IC $u(x, y, z, 0) = h(x, y, z), \quad u_t(x, y, z, 0) = f(x, y, z)$

Here $D = \frac{\partial^2}{\partial t^2}$,$R = \frac{\partial}{\partial t}$, N is non linear differential operator, $g(x, y, z, t)$ is source term. Applying the Elzaki transform (we denote it by E_L) on equation (3),

$$E_L[D\{u(x, y, z, t)\}] = E_L[R\{u(x, y, z, t)\}] + E_L[N\{u(x, y, z, t)\}] + E_L[g(x, y, z, t)] \quad (4)$$

This implies

$$E_L[u(x, y, z, t)] = v^2 E_L[g(x, y, z, t)] + v^2 h(x, y, z) + v^3 f(x, y, z) + \mathrm{v}^2 E_L[R\{u(x, y, z, t)\} + N\{u(x, y, z, t)\}]$$

Applying the inverse Elzaki transform to the above equation

$$u(x, y, z, t) = \delta(x, y, z, t) - E_L^{-1}\left[\mathrm{v}^2 E_L\left\{R\{u(x, y, z, t)\} + N\{u(x, y, z, t)\}\right\}\right] \qquad (5)$$

Here $\delta(x, y, z, t)$ represents the term, which arises from initial conditions and source term. Now apply the homotopy perturbation method, the linear term can be written as

$$u(x, y, z, t) = \sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \qquad (6)$$

The non-linear term will be decomposed as follows:

$$N[u(x,y,z,t)] = \sum_{n=0}^{\infty} p^n H_n(u) \quad (7)$$

Where $H_n(\eta)$ is the He's polynomial,

$$H_n(u_0,u_1,u_2,\dots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2,3,\dots \quad (8)$$

Put eq. (6) and eq. (7)in eq.(5), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = \delta(x,y,z,t) - pE_L{}^{-1}[\mathrm{v}^2 E_L\{R\{\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\} + \sum_{n=0}^{\infty} p^n H_n(u)\}]$$

This is the combined form of Elzaki transform and HPM. Now make the Comparison of the coefficients of the same power of p on both sides of the above equation, then

$$p^0: u(x,y,z,t) = \delta(x,y,z,t),$$

$$p^1: u_1(x,y,z,t) = -E_L{}^{-1}\{\mathrm{v}^2 E_L[Ru_0(x,y,z,t) + H_0(u)]\},$$

$$p^2: u_2(x,y,z,t) = -E_L{}^{-1}\{\mathrm{v}^2 E_L[Ru_1(x,y,z,t) + H_1(u)]\},$$

And proceed the same way. So the required solution is

$$\Phi(x,y,z,t) = \lim_{p\to 1} u_n(x,y,z,t)$$

$$= u_0(x,y,z,t) + u_1(x,y,z,t) + u_2(x,y,z,t) + \cdots \quad (9)$$

IV. Convergence Analysis of the proposed technique [XII]

Here we discuss the theorem, which will help us to understand the convergence of the recommended technique (see [XII]).

Theorem: Consider the Banach space $C[0,\mathcal{T}]$ $\forall$ continuous functions on $C[0,\mathcal{T}]$ with supremum norm. In the theorem we take $\Phi(x,y,z,t), \Phi_n(x,y,z,t) \in C[0,\mathcal{T}]$. See [XII]

$$\sum_{n=0}^{\infty} \Phi_n \to \Phi, if\ \exists\ y \in (0,1)$$

such that

$$\|\varphi_n\| \le y\|\varphi_{n-1}\|, \quad \forall n \in N$$

Proof: For the sake of convergence of sequence $\{s_n\}$of the partial sums of series (9), we first prove that $\{\hat{S}_n\}$ is a Cauchy sequence in $(C[0,\mathcal{T}], \|.\|)$.

$$\|\hat{S}_{n+1} - \hat{S}_n\| = \|\varphi_{n+1}\| \le y\|\varphi_n\| \le y^2\|\varphi_{n-1}\| \le \cdots \le y^{n+1}\|\varphi_0\|$$

Therefore

$$\|\hat{S}_n - \hat{S}_m\| = \left\|\sum_{i=m+1}^{n} \varphi_i\right\| \le \sum_{i=m+1}^{n} \|\varphi_i\| \le y^{m+1}\left(\sum_{i=0}^{n-m-1} y^i\right)\|\varphi_0\|$$

$$= y^{m+1}\frac{1-y^{n-m}}{1-y}\|\varphi_0\|, \quad n,m \in N$$

Since $0 < y < 1$,therefore $\|\hat{S}_n - \hat{S}_m\| \leq \frac{y^{m+1}}{1-y}\|\varphi_0\|$. Also φ_0 is bounded, so $\{\hat{S}_n\}$ is a Cauchy sequence in $C[0,\mathcal{T}]$, hence $\sum_{n=0}^{\infty}\Phi_n$ is convergence.

V. Results and Discussions

In this section, we have presented and discussed the findings of the work. We previously introduced the standard 2D and 3D telegraph equations. Here we employ different values of α, β, and f in the telegraph equation, along with various initial conditions, and solve these numerical problems with the proposed Elzaki transform homotopy perturbation method.

Example 1. Let $\alpha = 2, \beta = 1, and\ f = x^2 + y^2 + t - 2$ in equation (1)

$$\varphi_{tt} + 2\varphi_t + u = \varphi_{xx} + \varphi_{yy} + (x^2 + y^2 + t - 2) \tag{10}$$

Initial conditions $\varphi(x, y, 0) = x^2 + y^2$, $\quad \varphi_t(x, y, 0) = 1$,

The exact solution is $\varphi(x, y, t) = x^2 + y^2 + t$

Solution: First of all, the Elzaki transform has been applied to the equation(10), then utilize the differential properties of the Elzaki transform

$$E_L(\varphi_{tt}) = E_L\left(\varphi_{xx} + \varphi_{yy} - 2\varphi_t - \varphi + (x^2 + y^2 + t - 2)\right)$$

$$\frac{T(v)}{v^2} - \varphi(x, y, 0) - \varphi_t(x, y, 0) = E_L\left(\varphi_{xx} + \varphi_{yy} - 2\varphi_t - \varphi + (x^2 + y^2 + t - 2)\right)$$

Use initial conditions, then inverse Elzaki transform has been applied we get,

$$\varphi(x, y, t) = (x^2 + y^2 + t) + E_L{}^{-1}v^2E_L\{\varphi_{xx} + \varphi_{yy} - 2\varphi_t - \varphi + (x^2 + y^2 + t - 2)\}$$

Now apply the homotopy perturbation method to both sides of the above equation

$$\sum_{n=0}^{\infty} p^n\varphi_n(x, y, t) = (x^2 + y^2 + t) + pE_L{}^{-1}\left\{v^2E_L\left(\sum_{i=0}^{\infty} p^nH_n(\varphi)\right)\right\}$$

Compare the identical powers of p

$$p^0 = \varphi_0 = (x^2 + y^2 + t)$$

$$p^1 = \varphi_1 = E_L{}^{-1}v^2E_L\{(\varphi_0)_{xx} + (\varphi_0)_{xx} - 2(\varphi_0)_t - \varphi_0 + (x^2 + y^2 + t - 2)\} = 0$$

$$p^2 = \varphi_2 = E_L{}^{-1}v^2E_L\{(\varphi_1)_{xx} + (\varphi_1)_{yy} - 2(\varphi_1)_t - \varphi_1\} = 0$$

Also $\varphi_3 = 0$ and so on. Now the solution is:

$$\varphi(x, y, t) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

This implies

$$\varphi(x, y, t) = (x^2 + y^2 + t)$$

which is the exact solution.

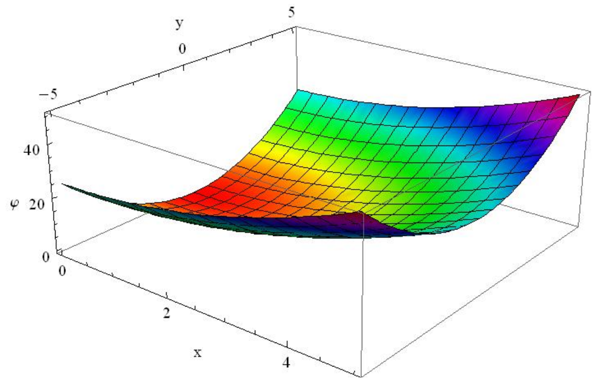

Fig. 1.

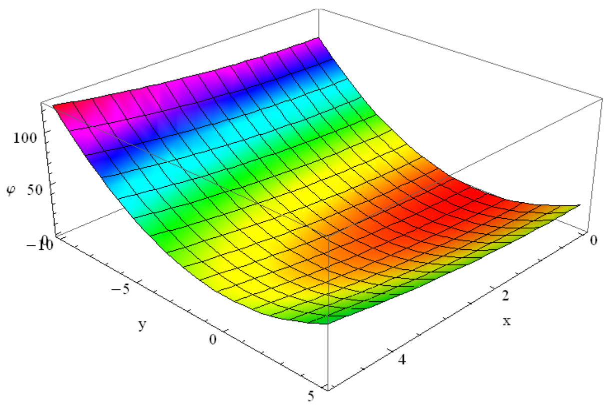

Fig. 2.

Figure 1 and Figure 2 display how the solution of Example 1 behaves at different values of x and y at t = 0.5. From this numerical experiment, it is clear that the proposed method can tackle such complex problems. The exact solution also validates the method.

Example 2:put$\alpha = 0, \beta = -2\pi^2, and\ f = 0$ in equation (1)

$$\varphi_{tt} - 2\pi^2\varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz} \tag{11}$$

Initial conditions $\varphi(x, y, 0) = \sin \pi x \sin \pi y \sin \pi z$, $\varphi_t(x, y, 0) = 0$,

Exact solution is $\varphi(x, y, t) = \sin \pi x \sin \pi y \sin \pi z \cos \pi t$

Solution: Apply Elzaki transform on both sides of the equation(11), and use the differential properties of the Elzaki transform

$$E_L(\varphi_{tt}) = E_L\left(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} + 2\pi^2\varphi\right)$$

This implies

$$\frac{T(v)}{v^2} - \varphi(x, y, 0) - \varphi_t(x, y, 0) = E_L\left(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} + 2\pi^2\varphi\right)$$

After rearranging and using initial conditions, then applying the inverse Elzaki transform we get,

$$\varphi(x, y, t) = \sin \pi x \sin \pi y \sin \pi z + E_L{}^{-1}\left(v^2 E_L\{\varphi_{xx} + \varphi_{yy} + \varphi_{zz} + 2\pi^2\varphi\}\right)$$

Now use homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n \varphi_n(x, y, z, t) = \sin \pi x \sin \pi y \sin \pi z + pE_L{}^{-1}\left\{v^2 E_L\left(\sum_{i=0}^{\infty} p^n H_n(\varphi)\right)\right\}$$

Now comparing the like powers of p

$$p^0 = \varphi_0 = \sin \pi x \sin \pi y \sin \pi z$$

$$p^1 = \varphi_1 = E_L{}^{-1} v^2 E_L\{(\varphi_0)_{xx} + (\varphi_0)_{yy} + (\varphi_0)_{zz} - 2\pi^2\varphi_0\}$$
$$= -\pi^2 \frac{t^2}{2!} \sin \pi x \sin \pi y \sin \pi z$$

$$p^2 = \varphi_2 = E_L{}^{-1} v^2 E_L\{(\varphi_1)_{xx} + (\varphi_1)_{yy} + (\varphi_2)_{zz} - 2\pi^2\varphi_1\}$$
$$= \pi^4 \frac{t^4}{4!} \sin \pi x \sin \pi y \sin \pi z$$

$$p^3 = \varphi_3 = E_L{}^{-1} v^2 E_L\{(\varphi_2)_{xx} + (\varphi_2)_{yy} + (\phi_2)_{zz} - 2\pi^2\varphi_2\}$$
$$= -\pi^6 \frac{t^6}{6!} \sin \pi x \sin \pi y \sin \pi z$$

The solution is :

$$\varphi = \sin \pi x \sin \pi y \sin \pi z - \pi^2 \frac{t^2}{2!} \sin \pi x \sin \pi y \sin \pi z + \pi^4 \frac{t^4}{4!} \sin \pi x \sin \pi y \sin \pi z - \pi^6 \frac{t^6}{6!} \sin \pi x \sin \pi y \sin \pi z + \cdots$$

Or

$$\varphi = \sin \pi x \sin \pi y \sin \pi y \cos \pi t$$

which is exact solution.

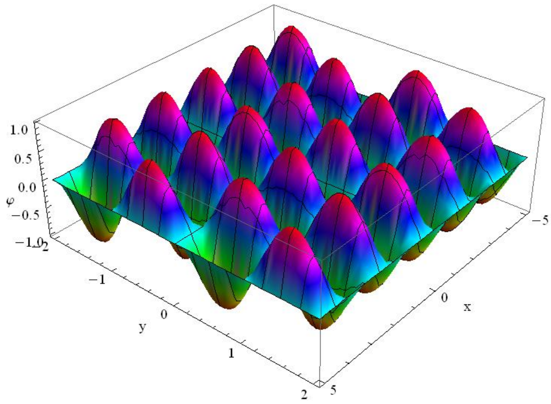

Fig. 3.

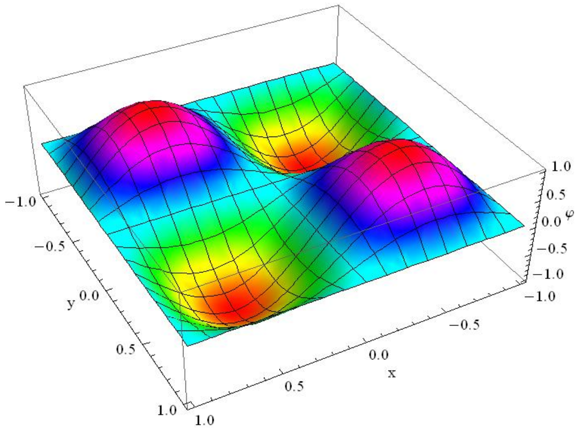

Fig. 4.

Figure 3 and Figure 4 display how the solution of Example 2 behaves at different values of x and y at $t = 1$ and $z = 0.5$. The exact solution of the numerical experiment confirms the efficacy of the proposed method.

Example 3: Put $\alpha = 0, \beta = \sqrt{2}, and\ f = 0$ in equation (1)

$$\varphi_{tt} + 2\varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz} \tag{12}$$

With initial conditions $\varphi(x, y, 0) = e^{x+y+z}, \varphi_t(x, y, 0) = e^{x+y+z}$.

The exact solution is $\varphi(x, y, t) = e^{x+y+z+t}$

Solution: Apply Elzaki transform on both sides of the equation (12), and use the differential properties of the Elzaki transform

$$E_L(\varphi_{tt}) = E_L\left(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - 2\varphi\right)$$

It implies

$$\frac{T(v)}{v^2} - \varphi(x,y,0) - v\varphi_t(x,y,0) = E_L\left(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - 2\varphi\right)$$

After rearranging and using initial conditions, then applying the inverse Elzaki transform we get,

$$\varphi(x,y,t) = e^{x+y+z} + E^{-1}\left\{v^2 E_L\{\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - 2\varphi\}\right\}$$

Using the homotopy perturbation method, we obtain

$$\sum_{n=0}^{\infty} p^n \varphi_n(x,y,t) = e^{x+y+z} + pE_L{}^{-1}\left\{v^2 E_L\left(\sum_{n=0}^{\infty} p^n H_n(\varphi)\right)\right\}$$

Comparing the like powers of p, we obtain

$$p^0 = \varphi_0 = e^{x+y+z}(1+t),$$

$$p^1 = \varphi_1 = E_L{}^{-1}v^2 E_L\{(\varphi_0)_{xx} + (\varphi_0)_{yy} + (\varphi_0)_{zz} - 2\varphi_0\} = e^{x+y+z}\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right),$$

$$p^2 = \varphi_2 = E_L{}^{-1}v^2 E_L\{(\varphi_1)_{xx} + (\varphi_1)_{yy} + (\varphi_2)_{zz} - 2\pi^2\varphi_1\} = e^{x+y+z}\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right),$$

$$p^3 = \varphi_3 = E_L{}^{-1}v^2 E_L\{(\varphi_2)_{xx} + (\varphi_2)_{yy} + \varphi_{zz} - 2\pi^2\varphi_2\} = e^{x+y+z}\left(\frac{t^6}{6!} + \frac{t^7}{7!}\right),$$

and so on

Now $\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \cdots$

It implies

$$\varphi = e^{x+y+z}(1+t) + e^{x+y+z}\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right) + e^{x+y+z}\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right) + e^{x+y+z}\left(\frac{t^6}{6!} + \frac{t^7}{7!}\right) + \cdots$$

or

$$\varphi = e^{x+y+z+t}$$

which is the exact solution.

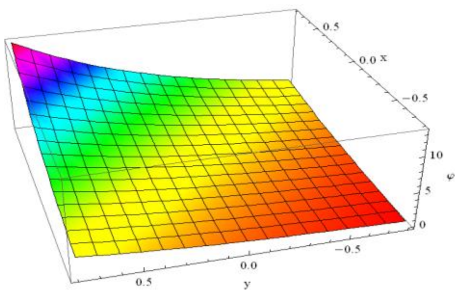

Fig. 5.

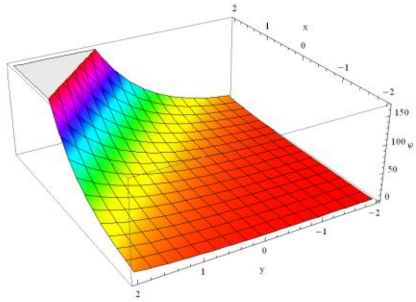

Fig. 6.

Figure 5 and Figure 6 display how the solution of Example 3 behaves at different values of x and y at $t = 0.5$, $z = 0.5$, and $z = 2$ respectively. From the numerical experiments, it is clear that the proposed method can reduce the computational size and is found to be very effective.

VI. Conclusion

In this paper, we have combined two different techniques, namely the Elzaki Transform and the homotopy perturbation method, to tackle 2D and 3D telegraph equations. The main purpose of this paper is to demonstrate the effectiveness of this hybrid method through illustrations. This hybrid technique successfully provides the exact solution of the equation. From the above result and graphical representations, we conclude that this is a nice refinement of the existing techniques. In the future, this technique will be applicable for solving higher-dimensional systems of linear and nonlinear PDES

Conflict of Interest

There is no Conflict of Interest regarding this paper.

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