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# SOLVING NONLINEAR COUPLED FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY ZZ TRANSFORM AND ADOMIAN POLYNOMIALS

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## Abstract

*By combining the ZZ transform with Adomian polynomials, the semi-analytical solutions to nonlinear Caputo partial fractional differential equations have been derived in this work. The Caputo sense has been applied to the fractional derivative. Using the proposed method, several fractional partial differential equations have been resolved. When compared to other similar procedures, it has been shown that applying the ZZ transform and breaking down the nonlinear components using Adomian polynomials is quite convenient.*

**Keywords:** ZZ Transform, Sumudu Transform, Adomian Polynomials, Caputo's system of Fractional Partial Differential Equations (FPDE).

## I. Introduction

There has been a trend in the use of fractional order derivatives to obtain a thorough grasp of differential equations of integer order. The enduring nonlinearity in these equations makes efficient solutions difficult to achieve, regardless of the sequence in which derivatives are obtained.

In the applied sciences, systems of fractional nonlinear partial differential equations frequently arise [VI, XII, XIV]. It is also evident that such a differential equation exists in nonlinear dynamics [II, III]. Analytical solutions to these problems are quite challenging. Therefore, one must take into account numerical or other methodologies. Semi-analytical techniques have become widely used in recent decades to solve both linear and nonlinear dynamical systems. These are a handful of them, Homotopy Perturbation Method (HPM) [XI, XXI], Variational Iteration Method (VIM) [XX], and New Iteration Method [IV]. In addition, nonlinearity is being addressed using other techniques such as the Laplace homotopy perturbation method [XI], and the Laplace Adomian decomposition method [XVI]. Furthermore, the Shehu transform and Sumudu transforms—also used in conjunction with these decomposition techniques and referred to as STADM [XVII], Sumudu ADM[VII, VIII], and Sumudu NIM [XIX] is employed in place of the Laplace transform.

Zafar has developed a novel transformation known as the ZZ transform [XXII], which he mixes with HPM [X, XV] to solve coupled fractional differential equations that are nonlinear. To solve a particular system of fractional partial differential equations, we are now combining the ZZ transform with the Adomian decomposition approach in this work.

This paper introduces a novel approach to solving nonlinear systems of fractional partial differential equations by combining the ZZ transform with the Adomian polynomials. The following is the study's outline: Preliminary information is provided in Section 2, the developed approach is presented in Section 3, the method is applied to the FPDE system in Section 4, and the closing observations are enclosed in Section 5.

## II. Preliminaries:

### [XIII] $\mathcal{ZZ}$ Transform:

Consider the following set X as:

$$\mathrm{X} = \{f(t): \exists \mathcal{M}, v_1, \nu_2 > 0, |f(t)| < \mathcal{M} e^{\frac{|t|}{\nu_i}}, \text{if}, t \in (-1)^i \times [0,\infty)\}$$

Next, the $\mathcal{ZZ}$ transform of $f(t) \in X$ may be expressed as follows:

$$\mathcal{ZZ}[f(t)] = \mathcal{ZZ}(u,s) = s\int_0^\infty f(ut)e^{-st}dt.$$

### [XIII] Inverse $\mathcal{ZZ}$ transform:

Inverse $\mathcal{ZZ}$ transform can be defined as:

$$\mathcal{ZZ}^{-1}[\mathcal{ZZ}(u,s)] = f(t), \quad \text{for } t \geq 0.$$

Or

$$f(t) = \mathcal{ZZ}^{-1}[\mathcal{ZZ}(u,s)] = \frac{1}{2\pi\iota}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{s} e^{st}\, \mathcal{ZZ}(u,s)ds.$$

In the complex plane, i.e.,(s=x+iy), this integral is considered along s= α. Here, s and u are the $\mathcal{ZZ}$ transform variables, while α is a real constant.

**Definition 1:**

The Caputo time derivative regarding t of order β>0 of y(x, t) in $H^1$(a, b) is defined as [XXII]:

$$\mathcal{D}_t^{\beta} y(x,t) = \begin{cases} \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\zeta)^{n-\beta-1}\frac{\partial^n y(x,\zeta)}{\partial \zeta^n} d\zeta, & \text{if}, \ n-1<\beta<n, n\in N \\ \frac{\partial^n y(x,\zeta)}{\partial \zeta^n}, & if, \ \beta = n \end{cases} \quad (1)$$

**Definition 2:**

The Sumudu Transform [VIII] of $\mathcal{D}_t^{\beta} y(x,t)$ regarding t of order β >0:

$$\mathcal{S}\left[\mathcal{D}_t^{\beta} y(x,t)\right] = u^{-\beta}\bar{f}(x,u) - \sum_{k=0}^{n-1} u^{-\beta+k}\mathcal{D}_t^{\beta} y(x,t)|_{t=0}, \quad (2)$$

where $\bar{f}(x,u)$ is Sumudu Transform of y(x, t).

**Definition 3:** Mittag-Leffler function [VI]

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)}, z, \beta \in C, \text{Real}(\alpha) > 0. \quad (3)$$

**Definition 4:**

The Sumudu transform [V] of $t^m$ , i.e.

$$\mathcal{S}\left(\frac{t^m}{m!}\right) = u^m, m = 0,1,2, \ldots$$

And

$$\mathcal{S}\left(\frac{t^{\beta}}{\Gamma(\beta+1)}\right) = u^{\beta}, \beta > -1. \quad (4)$$

**Basic Properties of $\mathcal{ZZ}$ Transform** [XIII]**:**

**Lemma 1.** If $\mathcal{ZZ}$ (u, s) and $\mathcal{S}$ (u) is the $\mathcal{ZZ}$ transform and Sumudu transform of f(t) in X then $\mathcal{ZZ}$ (u, s) = $\mathcal{S}$ $(\frac{u}{s})$.

**Proof.** The $\mathcal{ZZ}$ transform of f(t) is

$$\mathcal{ZZ}[f(t)] = \mathcal{ZZ}(u,s) = s\int_0^{\infty} f(ut)e^{-st}dt$$

put ut= w, the above integral becomes

$$\mathcal{ZZ}[f(t)] = s\int_0^\infty f(w)e^{\frac{-sw}{u}}\frac{dw}{u}$$

$$= \frac{s}{u}\int_0^\infty f(w)e^{\frac{-sw}{u}}\,dw$$

$$\left(\frac{s}{u}\right)F\left(\frac{s}{u}\right)$$

where F is the Laplace transform of f(t).

We know the relation between Laplace and Sumudu Transform (see [V])

$$\mathcal{S}(v) = \left(\frac{1}{v}\right)F\left(\frac{1}{v}\right)$$

Thus

$$\mathcal{ZZ}\ (u,s) = \left(\frac{s}{u}\right)F\left(\frac{s}{u}\right) = \left(\frac{s}{u}\right)\frac{\mathcal{S}\left(\frac{u}{s}\right)}{\left(\frac{s}{u}\right)} = \mathcal{S}\left(\frac{u}{s}\right)$$

Hence proved.

**Theorem 1.** The $\mathcal{ZZ}$ Transform of f(t)= $t^{\alpha}$ is given by the following formula

$$\mathcal{ZZ}[t^{\alpha}] = \mathcal{ZZ}\ (u,s) = \Gamma(\alpha+1)\left(\frac{u}{s}\right)^{\alpha}, \alpha \geq 0 \qquad (5)$$

**Proof.** Use Lemma 1 and Definition 4.

**Theorem 2.** The $\mathcal{ZZ}$ transform of Caputo Fractional partial derivative is defined as

$$\mathcal{ZZ}\left[\mathcal{D}_t^{\beta}y(x,t)\right] = \left(\frac{s}{u}\right)^{\beta}Y(x,v,s) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\beta-k}\mathcal{D}_t^{k}\, y(x,t)|_{t=0}. \qquad (6)$$

**Proof.** Use Lemma 1 and Definition 2.

## III. Methodology

In this section, a detailed analysis of the algorithm has been provided. Let the system of non-linear fractional partial differential equations in Caputo form as follows:

$$\mathcal{D}_t^{\beta_i} y_i(x_i,t) + \mathcal{R}y_i(x_i,t) + \mathcal{N}y_i(x_i,t) = h_i(x,t), \beta_i > 0, i = 1,2 \dots, n$$

$$y_i^k(x_i,t)|_{t=0} = a_{i_k}, k = 0,1,2 \dots n-1, i = 1,2 \dots n, n-1 < \beta_i \leq n \quad (7)$$

Here $\mathcal{D}_t^{\beta_i}$ denotes the Caputo derivative and $\mathcal{N}y_i$ is recognized as the nonlinear part of the given FPDE while $\mathcal{R}y_i$ denotes other linear operator terms. The $h_i(x,t)$ is another function that belongs to X.

Applying $\mathcal{ZZ}$ Transform to (7), we obtain:

$$\mathcal{ZZ}\left[\mathcal{D}_t^{\beta_i} y_i(x_i,t) + \mathcal{R}y_i(x_i,t) + \mathcal{N}y_i(x_i,t)\right] = \mathcal{ZZ}[h_i(x,t)]$$

$$\mathcal{ZZ}\left[\mathcal{D}_t^{\beta_i} y_i(x_i,t)\right] + \mathcal{ZZ}[Ry_i(x_i,t)] + \mathcal{ZZ}[\mathcal{N}y_i(x,t)] = \mathcal{ZZ}[h_i(x,t)]$$

$$\left(\frac{s}{u}\right)^{\beta_i}(Y_i(x_i,s,u) - a) + \mathcal{ZZ}[\mathcal{R}y_i(x_i,t)] + \mathcal{ZZ}[\mathcal{N}y_i(x_i,t)] = \mathcal{ZZ}[h_i(x,t)].$$

Where,

$$a = \sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\beta-k} \mathcal{D}_t^k y_i(x,t)|_{t=0}$$

$$Y_i(x_i,s,u) = a + \left(\frac{u}{s}\right)^{\beta_i}\mathcal{ZZ}[h_i(x,t)] - \left(\frac{u}{s}\right)^{\beta_i}\mathcal{ZZ}[\mathcal{R}y_i(x_i,t)] - \left(\frac{u}{s}\right)^{\beta_i}\mathcal{ZZ}[\mathcal{N}y_i(x_it)]$$

Taking inverse $\mathcal{ZZ}$ Transform we obtain

$$y_i(x_i,t) = \mathcal{ZZ}^{-1}(a) + \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\beta_i}\mathcal{ZZ}[h_i(x_i,t)]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\beta_i}\mathcal{ZZ}[\mathcal{R}y_i(x_i,t)]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\beta_i}\mathcal{ZZ}[\mathcal{N}y_i(x_i,t)]\right] \quad (8)$$

Assume the solution is in series form as $y_i(x_i,t) = \sum_{j=0}^{\infty} y_{ij}(x_i,t)$ and decompose $\mathcal{N}y_i(x_i,t)$ by Adomian polynomials [I] as given below.

$$\mathcal{N}y_i(x_i,t) = \sum_{m=0}^{\infty} A_{im}$$

$A_{i0} = \mathcal{N}(y_{i0})$ and $A_{im}$ is called Adomian polynomials (APs) computed as:

$A_{i0} = \mathcal{N}(y_{i0}),$

$A_{i1} = \mathcal{N}'(y_{i0})(y_{i1}),$

$$A_{i2} = \mathcal{N}'(y_{i0})y_{i2} + \frac{y_{i1}^2}{2!}\mathcal{N}''(y_{i0})$$

$$A_3 = \mathcal{N}'(y_{i0})y_{i3} + \mathcal{N}''(y_{i0})y_{i1}y_{i2} + \frac{v_1^3}{3!}\mathcal{N}'''(y_{i0}), \dots = \dots \dots,$$

$$A_{im} = \frac{1}{m!}\frac{d^m}{d\lambda^m}\,\mathcal{N}\left(\sum_{l=0}^{m}\lambda^l\, y_{il}\right)_{\lambda=0}$$

The following recurrence relation was observed,

$$y_{i0}(x,t) = \mathcal{ZZ}^{-1}(a)$$

$$y_{im}(x_i,t) = \mathcal{ZZ}^{-1}\left[\left(\frac{v}{s}\right)^{\beta_i}\mathcal{ZZ}[h_i(t)]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{v}{s}\right)^{\beta_i}\mathcal{ZZ}[\mathcal{R}y_{i(m-1)}(x,t)]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{v}{s}\right)^{\beta_i}\mathcal{ZZ}[A_{i(m-1)}(x,t)]\right], m = \qquad (9)$$

The answer is derived as the sum of the series components, i.e., by using the aforementioned relation.

$$y_i(x_i,t) = y_{i0} + y_{i1} + y_{i2} + \cdots = \lim_{m\to\infty}\sum_{j=0}^{m} y_{ij}(x_i,t)$$

To see a convergence analysis of the Adomian decomposition method see [XVIII].

## IV. Applications

**Example 1.** Let the system of nonlinear FPDE as

$$\mathcal{D}_\tau^\alpha v(\xi,\tau) + wv_\xi(\xi,\tau) + v(\xi,\tau) = 1, 0 < \alpha \le 1,$$

$$\mathcal{D}_\tau^\beta w(\xi,\tau) - vw_\xi(\xi,\tau) - w(\xi,\tau) = 1, 0 < \beta \le 1, \qquad (10)$$

with initial conditions

$$v(x,0) = e^{\xi},$$

$$w(x,0) = e^{-\xi}.$$

Applying $\mathcal{ZZ}$ transform to equation (10) we get

$$\mathcal{ZZ}[\mathcal{D}_\tau^\alpha v(\xi,\tau)] + \mathcal{ZZ}\left[wv_\xi(\xi,\tau)\right] + \mathcal{ZZ}[v(\xi,\tau)] = \mathcal{ZZ}[1],$$

$$\mathcal{ZZ}\left[\mathcal{D}_\tau^\beta w(\xi,\tau)\right] - \mathcal{ZZ}\left[vw_\xi(\xi,\tau)\right] - \mathcal{ZZ}[w(\xi,\tau)] = \mathcal{ZZ}[1],$$

$$\left(\frac{s}{u}\right)^\alpha V(\xi,u,s) - \left(\frac{s}{u}\right)^\alpha v(\xi,0) + \mathcal{ZZ}\left[w(v)_\xi\right] + \mathcal{ZZ}[v] = 1$$

$$\left(\frac{s}{u}\right)^\beta W(\xi,u,s) - \left(\frac{s}{u}\right)^\beta v(\xi,0) - \mathcal{ZZ}\left[v(w)_\xi\right] - \mathcal{ZZ}[w] = 1$$

$$V(\xi,u,s)=e^{\xi}+\left(\frac{u}{s}\right)^{\alpha}-\left(\frac{u}{s}\right)^{\alpha}ZZ\left[w(v)_{\xi}\right]-\left(\frac{u}{s}\right)^{\alpha}ZZ[v]$$

$$W(\xi,u,s)=e^{-\xi}+\left(\frac{u}{s}\right)^{\beta}+\left(\frac{s}{u}\right)^{\beta}ZZ\left[v(w)_{\xi}\right]+\left(\frac{s}{u}\right)^{\beta}ZZ[w]$$

Applying inverse $ZZ$ transform we get

$$v(\xi,\tau)=e^{\xi}+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}-ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}ZZ\left[w(v)_{\xi}\right]\right]-ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}ZZ[v]\right]$$

$$w(\xi,\tau)=e^{-\xi}+\frac{\tau^{\beta}}{\Gamma(\beta+1)}+ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta}ZZ\left\{v(w)_{\xi}\right\}\right]+ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta}ZZ\{w\}\right]$$

Assume that the solution takes the form of infinite series components. The recurrence formula is then produced by breaking down the nonlinear sections using Adomian polynomials.

$$v_0(\xi,t)=e^{\xi}+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}$$

$$w_0(\xi,t)=e^{-\xi}+\frac{\tau^{\beta}}{\Gamma(\beta+1)}$$

$$v_{n+1}(\xi,\tau)=-ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}ZZ[A_n]\right]-ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha}ZZ[v_{n-1}]\right]$$

$$w_{n+1}(\xi,\tau)=ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta}ZZ[B_n]\right]+ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta}ZZ[w_{n-1}]\right],n=1,2\,.... \quad (11)$$

where $A_n, B_n$ are Adomian polynomials for nonlinear terms $N_1=w(v)_{\xi}, N_2=v(w)_{\xi}$ respectively, and are calculated below.

$$A_m=\frac{1}{m!}\frac{d^m}{d\lambda^m}N_1\left(\left(\sum_{k=0}^{m}\lambda^k w_k\right)\left(\sum_{k=0}^{m}\lambda^k (v_k)_{\xi}\right)\right)|_{\lambda=0}$$

$$B_m=\frac{1}{m!}\frac{d^m}{d\lambda^m}N_2\left(\left(\sum_{k=0}^{m}\lambda^k v_k\right)\left(\sum_{k=0}^{m}\lambda^k (w_k)_{\xi}\right)\right)|_{\lambda=0} \quad (12)$$

A few polynomials are calculated as

$$A_0=w_0(v_0)_{\xi}$$

$$A_1=w_1(v_0)_{\xi}+w_0(v_1)_{\xi}$$

$$A_2=w_2(v_0)_{\xi}+w_1(v_1)_{\xi}+w_0(v_2)_{\xi}\,...$$

Similarly,

$$B_0 = v_0(w_0)_\xi$$
$$B_1 = v_1(w_0)_\xi + v_0(w_1)_\xi$$
$$B_2 = v_2(w_0)_\xi + v_1(w_1)_\xi + v_0(w_2)_\xi \ldots$$

Therefore, calculating the few terms as below

$$v_1(\xi,\tau) = -ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[A_0]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[v_0]\right]$$

$$w_1(\xi,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[B_0]\right] + ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[w_0]\right]$$

In solving the above, we get

$$v_1(\xi,\tau) = -\frac{(1+e^{\xi})\tau^{\alpha}}{\Gamma(\alpha+1)} - \frac{(e^{\xi})\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$w_1(\xi,\tau) = \frac{(-1+e^{-\xi})\tau^{\beta}}{\Gamma(\beta+1)} - \frac{(e^{-\xi})\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\tau^{2\beta}}{\Gamma(2\beta+1)}$$

Now

$$v_2(\xi,\tau) = -ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[A_1]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[v_1]\right]$$

$$w_2(\xi,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[B_1]\right] + ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[w_1]\right]$$

Again, on solving, we have

$$\begin{aligned} v_2(\xi,\tau) = {} & \frac{(2+e^{\xi})\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(e^{\xi}-1)\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + (1+2e^{\xi})\frac{\tau^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \\ & + \left(\frac{\Gamma(\alpha+\beta+1)e^{\xi}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right)\frac{\tau^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \\ & + \left(\frac{\Gamma(\alpha+2\beta+1)e^{\xi}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)}\right)\tau^{2\alpha+2\beta} \\ & - \frac{(e^{\xi})}{\Gamma(\alpha+2\beta+1)}\tau^{\alpha+2\beta} + \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} \end{aligned}$$

$$w_2(\xi,\tau) = \frac{(-2+e^{-\xi})\tau^{2\beta}}{\Gamma(\beta+1)} + \frac{(e^{-\xi}+1)\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + (2-e^{-\xi})\frac{\tau^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}$$
$$+\left(\frac{\Gamma(\alpha+\beta+1)e^{-\xi}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right)\frac{\tau^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}$$
$$+\left(\frac{\Gamma(2\alpha+\beta+1)e^{-\xi}}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)}\right)t^{2\alpha+2\beta}$$
$$+\frac{(e^{-\xi})}{\Gamma(2\alpha+\beta+1)}\tau^{2\alpha+\beta} + \frac{\tau^{3\beta}}{\Gamma(3\beta+1)}$$

Continuing like this to obtain the other components.

Therefore, we have

$$v(\xi,\tau) = v_0(\xi,\tau) + v_1(\xi,\tau) + v_2(\xi,\tau) + \cdots = \lim_{m\to\infty}\sum_{i=0}^{m} v_m(\xi,\tau)$$

$$w(\xi,\tau) = w_0(\xi,\tau) + w_1(\xi,\tau) + w_2(\xi,\tau) + \cdots = \lim_{m\to\infty}\sum_{i=0}^{m} w_m(\xi,\tau)$$

Thus

$$v(\xi,\tau) = e^{\xi} - \frac{e^{\xi}\tau}{\Gamma(\alpha+1)} + \frac{(1+e^{\xi})\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + (1+2e^{\xi})\frac{\tau^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}$$
$$+\left(\frac{\Gamma(\alpha+\beta+1)e^{\xi}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right)\frac{\tau^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}$$
$$+\left(\frac{\Gamma(\alpha+2\beta+1)e^{\xi}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)}\right)\tau^{2\alpha+2\beta}$$
$$-\frac{e^{\xi}}{\Gamma(\alpha+2\beta+1)}\tau^{\alpha+2\beta} + \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots.$$

$$w(\xi,\tau) = e^{-\xi} + \frac{e^{-\xi}\tau^{\beta}}{\Gamma(\beta+1)} + \frac{(-1+e^{-\xi})\tau^{2\beta}}{\Gamma(\beta+1)} + \frac{e^{-\xi}\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$
$$+(2-e^{-\xi})\frac{\tau^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \left(\frac{\Gamma(\alpha+\beta+1)e^{-\xi}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right)\frac{\tau^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}$$
$$+\left(\frac{\Gamma(2\alpha+\beta+1)e^{-\xi}}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(2\alpha+2\beta+1)}\right)\tau^{2\alpha+2\beta}$$
$$+\frac{(e^{-\xi})}{\Gamma(2\alpha+\beta+1)}\tau^{2\alpha+\beta} + \frac{\tau^{3\beta}}{\Gamma(3\beta+1)} + \cdots.$$

In particular for $\alpha = \beta$=1 in the series solution become

$$v(\xi,\tau) = e^{\xi}\left(1-\tau+\frac{\tau^2}{2!}-\frac{\tau^3}{3!}\right)+\cdots = e^{\xi-\tau}$$

$$w(\xi,\tau) = e^{-\xi}\left(1+\tau+\frac{\tau^2}{2!}+\frac{\tau^3}{3!}\right)+\cdots = e^{-\xi+\tau}$$

The above solution matches the solution found in [IX].

**Example 2.** Let the non-linear system of FPDE as follows

$$\mathcal{D}_{\tau}^{\alpha} v(\xi,\eta,\tau) = -v - p_{\xi}w_{\eta} + p_{\eta}w_{\xi}, 0<\alpha\le 1,$$

$$\mathcal{D}_{\tau}^{\beta} p(\xi,\eta,\tau) = p, 0<\beta\le 1,$$

$$\mathcal{D}_{\tau}^{\delta} w(\xi,\eta,\tau) = w - v_{\xi}w_{\xi} - v_{\eta}w_{\eta}, 0<\delta\le 1, \tag{13}$$

Subject to initial conditions

$$v(\xi,\eta,0) = \xi+\eta,$$

$$p(\xi,\eta,0) = 1+\xi-\eta,$$

$$w(\xi,\eta,0) = -\xi+\eta.$$

Applying $\mathcal{ZZ}$ transformation to (13), we have

$$\mathcal{ZZ}[\mathcal{D}_{\tau}^{\alpha} v(\xi,\eta,\tau)] = -\mathcal{ZZ}[v] - \mathcal{ZZ}[p_{\xi}w_{\eta}] + \mathcal{ZZ}[p_{\eta}w_{\xi}],$$

$$\mathcal{ZZ}\left[\mathcal{D}_{\tau}^{\beta} p(\xi,\eta,\tau)\right] = \mathcal{ZZ}[p]$$

$$\mathcal{ZZ}[\mathcal{D}_{\tau}^{\delta} w(\xi,\eta,\tau)] = \mathcal{ZZ}[w] - \mathcal{ZZ}[v_{\xi}w_{\xi}] - \mathcal{ZZ}[v_{\eta}w_{\eta}],$$

$$\left(\frac{s}{u}\right)^{\alpha} V(\xi,\eta,u,s) - \left(\frac{s}{u}\right)^{\alpha} v(x,0) = -\mathcal{ZZ}[v] - \mathcal{ZZ}[p_{\xi}w_{\eta}] + \mathcal{ZZ}[p_{\eta}w_{\xi}],$$

$$\left(\frac{s}{u}\right)^{\beta} P(\xi,\eta,u,s) - \left(\frac{s}{u}\right)^{\beta} p(x,0) = \mathcal{Z}[p]$$

$$\left(\frac{s}{u}\right)^{\delta} W(\xi,\eta,u,s) - \left(\frac{s}{u}\right)^{\delta} w(x,0) = \mathcal{ZZ}[w] - \mathcal{ZZ}[v_{\xi}w_{\xi}] - \mathcal{ZZ}[v_{\eta}w_{\eta}],$$

$$V(\xi,\eta,u,s) = \xi+\eta - \left(\frac{u}{s}\right)^{\alpha}\mathcal{ZZ}[v] - \left(\frac{u}{s}\right)^{\alpha}\mathcal{ZZ}[p_{\xi}w_{\eta}] + \left(\frac{u}{s}\right)^{\alpha}\mathcal{ZZ}[p_{\eta}w_{\xi}],$$

$$P(\xi,\eta,u,s) = 1+\xi-\eta + \left(\frac{u}{s}\right)^{\beta}\mathcal{ZZ}[p]$$

$$W(\xi,\eta,u,s) = -\xi+\eta + \left(\frac{u}{s}\right)^{\delta}\mathcal{ZZ}[w] - \left(\frac{u}{s}\right)^{\delta}\mathcal{ZZ}[v_{\xi}w_{\xi}] - \left(\frac{u}{s}\right)^{\delta}\mathcal{ZZ}[v_{\eta}w_{\eta}],$$

Applying inverse $\mathcal{ZZ}$ transform, we get

$$v(\xi,\eta,\tau) = \xi + \eta - \mathcal{ZZ}^{-1}\left\{\left(\frac{u}{s}\right)^{\alpha} \mathcal{ZZ}[v]\right\} - \mathcal{ZZ}^{-1}\left\{\left(\frac{u}{s}\right)^{\alpha} \mathcal{ZZ}[N_1]\right\} + \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{ZZ}[\mathcal{N}_2]\right],$$

$$p(\xi,\eta,\tau) = 1 + \xi - y + \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{ZZ}[p]\right]$$

$$w(\xi,\eta,\tau) = -\xi + \eta + \mathcal{ZZ}^{-1}\left\{\left(\frac{u}{s}\right)^{\delta} \mathcal{ZZ}[w]\right\} - \mathcal{ZZ}^{-1}\left\{\left(\frac{u}{s}\right)^{\delta} \mathcal{ZZ}[\mathcal{N}_3]\right\} - \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\delta} \mathcal{ZZ}[\mathcal{N}_4]\right],$$

where, $N_1, N_2, N_3, N_4$, are nonlinear terms $p_\xi w_\eta$ , $p_\eta w_\xi$ , $v_\xi w_\xi$ , $v_\eta w_\eta$ respectively.

After breaking down the nonlinear terms using Adomian polynomials and allowing the solutions to take the form of infinite series components, the following recurrence formula was discovered.

$$v_0(\xi,\eta,\tau) = \xi + \eta$$
$$p_0(\xi,\eta,\tau) = 1 + \xi - \eta$$
$$w_0(\xi,\eta,\tau) = -\xi + \eta$$
$$v_{n+1}(\xi,\eta,\tau) = -\mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} [v_n]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{ZZ}[A_n]\right] + \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} \mathcal{ZZ}[B_n]\right],$$
$$p_{n+1}(\xi,\eta,\tau) = \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\beta} \mathcal{ZZ}[p]\right] \tag{14}$$
$$w_{n+1}(\xi,\eta,\tau) = \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\delta} \mathcal{ZZ}[w]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\delta} \mathcal{ZZ}[C_n]\right] - \mathcal{ZZ}^{-1}\left[\left(\frac{u}{s}\right)^{\delta} \mathcal{ZZ}[D_n]\right], n = 1,2 \ldots.$$

where $A_n, B_n, C_n, D_n$ are Adomian polynomials for nonlinear terms $N_1 = p_\xi w_\eta$, $N_2 = p_\eta w_\xi$, $N_3 = v_\xi w_\xi$ , $N_4 = v_\eta w_\eta$ , and are calculated by

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m} N_1 \left(\left(\sum_{k=0}^{m} \lambda^k (p_k)_\xi\right)\left(\sum_{k=0}^{m} \lambda^k (w_k)_\eta\right)\right) |_{\lambda=0}$$

$$B_m = \frac{1}{m!}\frac{d^m}{d\lambda^m} N_2 \left(\left(\sum_{k=0}^{m} \lambda^k (p_k)_\eta\right)\left(\sum_{k=0}^{m} \lambda^k (w_k)_\xi\right)\right)|_{\lambda=0}$$

$$C_m = \frac{1}{m!}\frac{d^m}{d\lambda^m} N_1 \left(\left(\sum_{k=0}^{m} \lambda^k (v_k)_\xi\right)\left(\sum_{k=0}^{m} \lambda^k (w_k)_\xi\right)\right)|_{\lambda=0}$$

$$D_m = \frac{1}{m!}\frac{d^m}{d\lambda^m} N_2 \left(\left(\sum_{k=0}^{m} \lambda^k (v_k)_\eta\right)\left(\sum_{k=0}^{m} \lambda^k (w_k)_\eta\right)\right)|_{\lambda=0} \qquad (15)$$

A Few polynomials are calculated below.

$$A_0 = (p_0)_\xi (w_0)_\eta$$

$$A_1 = (p_1)_\xi (w_0)_\eta + (p_0)_\xi (w_1)_\eta$$

$$A_2 = (\mathrm{p}_2)_\xi (\mathrm{w}_0)_\xi + (\mathrm{p}_1)_\xi (\mathrm{w}_1)_\eta + (\mathrm{p}_0)_\xi (\mathrm{w}_2)_\eta$$

$$A_3 = \ldots.$$

$$B_0 = (p_0)_\eta (w_0)_\xi$$

$$B_1 = (p_1)_\eta (w_0)_\xi + (p_0)_\eta (w_1)_\xi$$

$$B_2 = (p_2)_\eta (w_0)_\xi + (p_1)_\eta (w_1)_\xi + (p_0)_\eta (w_2)_\xi$$

$$B_3 = \ldots$$

$$C_0 = (v_0)_\xi (w_0)_\xi$$

$$C_1 = (v_1)_\xi (w_0)_\xi + (v_0)_\xi (w_1)_\xi$$

$$C_2 = (v_2)_\xi (w_0)_\xi + (v_1)_\xi (w_1)_\xi + (v_0)_\xi (w_2)_\xi$$

$$C_3 = \ldots$$

$$D_0 = (v_0)_\eta (w_0)_\eta$$

$$D_1 = (v_1)_\eta (w_0)_\eta + (v_0)_y (w_1)_\eta$$

$$D_2 = (v_2)_\eta (w_0)_\eta + (v_1)_\eta (w_1)_\eta + (v_0)_\eta (w_2)_\eta$$

$$D_3 = \ldots.$$

The few components are calculated below.

$$v_1(\xi,\eta,\tau) = -ZZ^{-1}\left\{\left(\frac{u}{s}\right)^{\alpha} ZZ[v_0]\right\} - ZZ^{-1}\left\{\left(\frac{u}{s}\right)^{\alpha} ZZ[A_0]\right\} + ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[B_0]\right],$$

$$p_1(\xi,\eta,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[p_0]\right]$$

$$w_1(\xi,\eta,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ[w_0]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ[C_0]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ[D_0]\right]$$

$$v_1(\xi,\eta,\tau) = -ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[\xi+\eta]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ\left[(p_0)_{\xi}(w_0)_{\eta}\right]\right] + ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ\left[(p_0)_{\eta}(w_0)_{\xi}\right]\right]$$

$$p_1(\xi,\eta,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[1+\xi-\eta]\right]$$

$$w_1(\xi,\eta,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ[-\xi+\eta]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ\left[(v_0)_{\xi}(w_0)_{\xi}\right]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ\left[(v_0)_{\eta}(w_0)_{\eta}\right]\right]$$

On solving, we get

$$v_1(\xi,\eta,\tau) = -(\xi+\eta)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}$$

$$p_1(\xi,\eta,\tau) = (1+\xi-\eta)\frac{\tau^{\beta}}{\Gamma(\beta+1)}$$

$$w_1(\xi,\eta,\tau) = (-\xi+\eta)\frac{\tau^{\delta}}{\Gamma(\delta+1)}$$

Similarly,

$$v_2(\xi,\eta,\tau) = -ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[v_1]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[A_1]\right] + ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\alpha} ZZ[B_1]\right],$$

$$p_2(\xi,\eta,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\beta} ZZ[p_1]\right]$$

$$w_2(\xi,\eta,\tau) = ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ[w_1]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} ZZ[C_1]\right] - ZZ^{-1}\left[\left(\frac{u}{s}\right)^{\delta} Z[D_1]\right]$$

In solving the above, we have

$$v_2(\xi,\eta,\tau) = (\xi+\eta)\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p_2(\xi,\eta,\tau) = (1+\xi-\eta)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}$$

$$w_2(\xi,\eta,\tau) = (-\xi+\eta)\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}$$

Continuing like this, to obtain the other components, therefore,

$$v(\xi,\eta,\tau) = v_0(\xi,\eta,\tau) + v_1(\xi,\eta,\tau) + v_2(\xi,\eta,\tau) + \cdots = \lim_{m\to\infty}\sum_{i=0}^{m} v_m(\xi,\eta,\tau)$$

$$p(\xi,\eta,\tau) = p_0(\xi,\eta,\tau) + p_1(\xi,\eta,\tau) + p_2(\xi,\eta,\tau) + \cdots = \lim_{m\to\infty}\sum_{i=0}^{m} p_m(\xi,\eta,\tau)$$

$$w(\xi,\eta,\tau) = w_0(\xi,\eta,\tau) + w_1(\xi,\eta,\tau) + w_2(\xi,\eta,\tau) + \cdots = \lim_{m\to\infty}\sum_{i=0}^{m} w_m(\xi,\eta,\tau)$$

Hence

$$v(\xi,\eta,\tau) = (\xi+\eta)\left(1-\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right) = (\xi+\eta)E_\alpha(-\tau^\alpha)$$

$$p(\xi,\eta,\tau) = (1+\xi-\eta)\left(1+\frac{\tau^{\beta}}{\Gamma(\beta+1)}+\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}+\cdots\right) = (1+\xi-\eta)E_\alpha(\tau^\beta)$$

$$w(\xi,\eta,\tau) = (-\xi+\eta)\left(1+\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}+\cdots\right) = (-\xi+\eta)E_\alpha(-t^\alpha) \quad (16)$$

In particular, for $\alpha = \beta = \delta = 1$ in (16), the solution becomes

$$v(\xi,\eta,\tau) = (\xi+\eta)e^{-\tau}$$

$$p(\xi,\eta,\tau) = (1+\xi-\eta)e^{\tau}$$

$$w(\xi,\eta,\tau) = (-\xi+\eta)e^{\tau}$$

The above solution matches with the solution obtained in [IX].

## V. Conclusion

This article uses Adomian polynomials to adjoin the ZZ transform to solve the system of nonlinear fractional differential equations. For the solution of nonlinear linked fractional partial differential equations, this approach seems to be quite helpful. This approach yields an answer that is very consistent with the solution found using other conventional methods. As a result, this method provides an effective way to handle non-linear systems of FPDE with simplicity. In the future, this method may be expanded to solve higher-order coupled fractional differential equations.

## Conflict of Interest :

There was no relevant conflict of interest regarding this paper.

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