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k-ZUMKELLER LABELING OF CERTAIN GRAPHS

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Abstract

Let G be any graph. Then a one-one function $f: V \rightarrow \mathbb{N}$ is said to be a k-Zumkeller labeling of G if the induced function $f^*: E \rightarrow \mathbb{N}$ defined by $f^*(xy) = f(x)f(y)$ satisfies the following conditions:

(i) For every $xy \in E$, $f^*(xy)$ is a Zumkeller number.

(ii) $|f^*(E)| = k$, where $|f^*(E)|$ denotes the number of distinct Zumkeller numbers on the edges of G.

In this paper, we prove the existence of k-Zumkeller labeling for certain graphs like tadpole, banana, friendship, and firecracker graphs.

Keywords: Zumkeller number, banana graph, friendship graph, firecracker graph, tadpole graph, graph labeling.

I. Introduction

The first half of the 18th century saw the introduction of Graph Theory, following the solution of the Konigsberg Bridge problem in 1736. Since then, Graph theory has emerged as a powerful tool in the field of Mathematical research for its ability to represent any physical problem involving the arrangement of objects. A graph is a collection of vertices and edges between them. Ever since it broke into the mainstream of Mathematical Research, Graph Theory has found application in diverse fields ranging from Biochemistry, Architecture, Psychology, Economics, Linguistics, Sociology, Electrical Engineering and Computer Science to Operations Research.

II. Literature Survey

Alex Rosa is credited with the introduction of graph labeling in the year 1966 in the paper [X]. Graph labeling finds application in coding theory, astronomy, circuit design, radar, and communication network addressing.

A positive integer n is called a perfect number if $\sum_{d|n,d\in\mathbb{N}} d = 2n$. Zumkeller numbers, introduced by R.H. Zumkeller in 2003, are generalized perfect numbers. Clark et. al [XI], in 2008, introduced various conjectures and results on Zumkeller numbers. In the year 2013, Y. Peng and K.P.S. Bhaskara Rao[XII] also established numerous results on Zumkeller numbers.

Zumkeller labeling was introduced by Balamurugan using Zumkeller numbers, defined as a one-one function $f: V \to \mathbb{N}$ such that the induced function $f^*: E \to \mathbb{N}$ defined by $f^*(xy) = f(x)f(y)$ is a Zumkeller number. Concepts like strongly multiplicative Zumkeller labeling [I] and k-Zumkeller labeling [VI] of graphs have also been studied. I. Cahit [VIII] introduced the concept of cordial labeling.

Zumkeller's cordial labeling of graphs [IV] and their existence in paths, cycles, and star graphs were also introduced by Balamurugan.

In this paper, we investigate and establish the existence of Zumkeller labeling for certain classes of graphs including tadpole, banana, friendship, and firecracker.

III. Definition and Properties of Zumkeller numbers

A positive integer n is said to be a Zumkeller number if all the positive factors of n can be partitioned into two disjoint sets such that the sum of the factors in one set is equal to the sum of the factors in the other. [II]

For example, the positive integer 20 is a Zumkeller number since the factors of 20 can be split into two sets $A = \{2, 4, 5, 10\}$ and $B = \{1, 20\}$ such that the sum of the numbers in A is equal to the sum of the numbers in B.

Let G be any graph. An injective function $f: V \to \mathbb{N}$ is said to be a k-Zumkeller labeling [VI] of G if the induced function $f^*: E \to \mathbb{N}$ defined by $f^*(xy) = f(x)f(y)$ satisfies the following conditions:

(i) For every $xy \in E$, $f^*(xy)$ is a Zumkeller number.

(ii) $|f^*(E)| = k$, where $|f^*(E)|$ denotes the number of distinct Zumkeller numbers on

the edges of G.

Following are some of the properties of Zumkeller numbers:

1) Let the prime factorization of an even Zumkeller number n be $2^k.{p_1}^{k_1}.{p_2}^{k_2} \dots {p_m}^{k_m}$. Then at least one of the $k_i^{'}s$ must be an odd number.
2) If n is a Zumkeller number and p is a prime with $(n, p) = 1$, then np^l is a Zumkeller number for any positive integer l.

3) Let n be a Zumkeller number and $p_1^{k_1}.p_2^{k_2}\dots p_m^{k_m}$ be the prime factorization of n. Then for any positive integers $l_1, l_2, \dots, l_m$, $p_1^{k_1+l_1(k_1+1)}.p_2^{k_2+l_2(k_2+1)}\dots p_m^{k_m+l_m(k_m+1)}$ is also a Zumkeller number.
4) For any prime $p \neq 2$ and a positive integer k with $p \leq 2^{k+1}-1$, $2^k p$ is a Zumkeller number.

The Tadpole Graph $T_{m,n}$ is a graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

A Banana Tree is a graph obtained by connecting one leaf of each of n copies of a k-star graph with a single root vertex that is distinct from all the stars.

A graph constructed by joining n copies of the cycle graph C_3 with a common vertex is called a Friendship Graph and is denoted by F_n.

A graph obtained by the concatenation of n k-stars by linking one leaf from each is called a Firecracker Graph and is denoted by $F_{m,n}$.

IV. Main Results

Theorem 4.1: The tadpole graph $T_{m,n}$ is a 6-Zumkeller graph for any $m \equiv 0(mod2)$, where $m, n \in \mathbb{N}$.

Proof: Let $V = \{v_i : 1 \leq i \leq m\} \cup \{u_j : 1 \leq j \leq n\}$ be the vertex set of the graph $T_{m,n}$, where $V_1 = \{v_i : 1 \leq i \leq m\}$ denotes the vertex set of the cycle C_m and $U_1 = \{u_j : 1 \leq j \leq n\}$ that of the path P_n. Let $v_1 \in V_1$ be adjacent to $u_1 \in U_1$.

Let $E = \{v_i v_{i+1} : 1 \leq i \leq m-1\} \cup \{v_m v_1\} \cup \{u_j u_{j+1} : 1 \leq j \leq n-1\} \cup \{v_1 u_1\}$ be the edge set of $T_{m,n}$.

We consider the following two cases:

Case 1: When n is odd.

We define a one-one function $f: V \to \mathbb{N}$ as follows:

$f(v_i) = 2^{\frac{i+1}{2}}$, $f(v_{i+1}) = p_1 . 2^{\frac{m-i+1}{2}}$, for $i = 1, 3, 5, \dots, m-1$ where $p_1 \neq 2$ is prime such that $p_1 < 10$.

$f(u_j) = p_2 . 2^{\frac{n-j+2}{2}}$, $f(u_{j+1}) = 2^{\frac{m+i+1}{2}}$, $j = 1, 3, 5, \dots, n-1$ where $p_2 \neq 2$ is prime such that $p_2 < 10$.

We define another function $f^*: E \to \mathbb{N}$ as follows:

$f^*(x) = f(x)f(y)$, for any $xy \in E$.

We claim that the edges of $T_{m,n}$ are labeled with Zumkeller numbers.

1) For the cycle C_m, $f^*(v_i v_{i+1}) = f(v_i)f(v_{i+1}) = 2^{\frac{i+1}{2}} . p_1 . 2^{\frac{m-i+1}{2}} = p_1 . 2^{\frac{m+2}{2}}$, which is a Zumkeller number and is constant for all $1 \leq i \leq m$.

2) $f^*(v_{i+1}v_{i+2}) = f(v_{i+1})f(v_{i+2}) = p_1.2^{\frac{m-i+1}{2}}.2^{\frac{i+2+1}{2}} = p_1.2^{\frac{m+4}{2}}$, which is a Zumkeller number and is constant for all $1 \le i \le m$ and $1 \le j \le n-2$.
3) $f^*(v_m v_1) = f(v_m)f(v_1) = p_1.2^{\frac{m-(m-1)+1}{2}}.2^1 = p_1.2^1.2^1 = p_1.2^2$, which is again a Zumkeller number and a constant.
4) For the bridge $\{v_1, u_1\}$, $f^*(v_1u_1) = f(v_1)f(u_1) = 2^1.p_2.2^{\frac{n+1}{2}} = p_2.2^{\frac{n+3}{2}}$, which is a Zumkeller number and a constant.
5) For the path P_n, $f^*(u_ju_{j+1}) = f(u_j)f(u_{j+1}) = p_2.2^{\frac{n-j+2}{2}}.2^{\frac{m+j+1}{2}} = p_2.2^{\frac{m+n+3}{2}}$ which is again a Zumkeller number and is constant for all $1 \le j \le n$.
6) $f^*(u_{j+1}u_{j+2}) = f(u_{j+1})f(u_{j+2}) = 2^{\frac{m+j+1}{2}}.p_2.2^{\frac{n-(j+2)+2}{2}} = p_2.2^{\frac{m+n+1}{2}}$ which is again a Zumkeller number and is constant for all $1 \le j \le n$.

From (1)-(6), it is clear that $T_{m,n}$ receives 6 distinct Zumkeller numbers. Therefore $T_{m,n}$ is a 6-Zumkeller graph.

Case 2: When n is even.

In this case, $f(v_1), f(v_i), f(v_{i+1})$ and $f(u_{j+1})$ are defined as in Case 1, while $f(u_j)$ is defined as $f(u_j) = p_2.2^{\frac{n-j+1}{2}}$.
Here the Zumkeller labeling of the edges of the cycle C_n can be shown in Case 1.
For the bridge $\{v_1u_1\}$, $f^*(v_1u_1) = f(v_1)f(u_1) = 2^1.p_2.2^{\frac{n}{2}} = p_2.2^{\frac{n+2}{2}}$ which is a Zumkeller number and is constant.
Again, $f^*(u_ju_{j+1}) = f(u_j)f(u_{j+1}) = p_2.2^{\frac{n-j+1}{2}}.2^{\frac{m+j+1}{2}} = p_2.2^{\frac{m+n+2}{2}}$ which is again a Zumkeller number and is constant for all $1 \le j \le n$.
$f^*(u_{j+1}u_{j+2}) = f(u_{j+1})f(u_{j+2}) = 2^{\frac{m+j+1}{2}}.p_2.2^{\frac{n-(j+2)+1}{2}} = p_2.2^{\frac{m+n}{2}}$ which is again a Zumkeller number and is constant for all $1 \le j \le n$.
Just like in the previous case, $T_{m,n}$ is a 6-Zumkeller graph.

Hence the graph $T_{m,n}$ is a 6-Zumkeller graph for any $m \equiv 0(mod2)$.

Theorem 4.2: The tadpole graph $T_{m,n}$ is a 7-Zumkeller graph for any $m \equiv 1(mod2)$, where $m, n \in \mathbb{N}$.

Proof: The vertex set V and the edge set E of $T_{m,n}$ are defined as in theorem 3.1.

Here also we consider the following cases:

Case 1: When n is odd.

We define a one-one function $f: V \to \mathbb{N}$ as follows:

$$f(v_i) = 2^{\frac{i+1}{2}}, \qquad f(v_{i+1}) = p_1.2^{\frac{m-i+2}{2}}, \qquad f(v_m) = 2p_2$$

for $i = 1, 3, 5, \dots, m-1$, where $p_1 \ne p_2 \ne 2$ are primes such that $p_1, p_2 < 10$.

$$f(u_j) = p_2.2^{\frac{n-j+2}{2}} \qquad f(u_{j+1}) = 2^{\frac{m+j}{2}} \text{ for } j = 1, 3, 5, \dots, n-1.$$

The function $f^*: E \to \mathbb{N}$ is defined as in theorem 3.1. We claim that the edges are labeled with Zumkeller numbers.

1) For the cycle C_m, $f^*(v_i v_{i+1}) = f(v_i)f(v_{i+1}) = 2^{\frac{i+1}{2}}.p_1.2^{\frac{m-i+2}{2}} = p_1.2^{\frac{m+3}{2}}$ which is a Zumkeller number and is constant.
2) $f^*(v_{i+1}v_{i+2}) = f(v_{i+1})f(v_{i+2}) = p_1.2^{\frac{m-i+2}{2}}.2^{\frac{i+2+1}{2}} = p_1.2^{\frac{m+5}{2}}$ which is a Zumkeller number and is constant.
3) $f^*(v_{m-1}v_m) = f(v_{m-1})f(v_m) = p_1.2^{\frac{m-(m-2)+2}{2}}.2p_2 = p_1.p_2.2^3$ which is a Zumkeller number and is clearly constant.
4) Also, $f^*(v_1 v_m) = 2^1.2p_2 = 2^2 p_2$ which is also a Zumkeller number and a constant.
5) For the bridge $\{v_1 u_1\}$, $f^*(v_1 u_1) = f(v_1)f(u_1) = 2^1.p_2.2^{\frac{n+1}{2}} = p_2.2^{\frac{n+3}{2}}$ which is a Zumkeller number and is constant.
6) For the path $P_{n,}$ $f^*(u_j u_{j+1}) = f(u_j)f(u_{j+1}) = p_2.2^{\frac{n-j+2}{2}}.2^{\frac{m+j}{2}} = p_2.2^{\frac{m+n+2}{2}}$ which is again a Zumkeller number and is constant.
7) $f^*(u_{j+1}u_{j+2}) = f(u_{j+1})f(u_{j+2}) = 2^{\frac{m+j}{2}}.p_2.2^{\frac{n-(j+2)+2}{2}} = p_2.2^{\frac{m+n}{2}}$ which is again a Zumkeller number and is constant.

From (1)-(7), we find that $T_{m,n}$ receives 7 distinct Zumkeller numbers. So $T_{m,n}$ is a 7-Zumkeller graph.

Case 2: When n is even.

Here also $f(v_1), f(v_i), f(v_{i+1})$ and $f(u_{j+1})$ are defined exactly as in Case 1, while $f(u_j)$ is defined as $f(u_j) = p_2.2^{\frac{n-j+1}{2}}$.

The Zumkeller labeling of C_n is similar as in Case 1.

1) For the bridge $\{v_1 u_1\}$, $f^*(v_1 u_1) = f(v_1)f(u_1) = 2^1.p_2.2^{\frac{n}{2}} = p_2.2^{\frac{n+2}{2}}$ which is a Zumkeller number and is clearly constant.
2) Again, $f^*(u_j u_{j+1}) = f(u_j)f(u_{j+1}) = p_2.2^{\frac{n-j+1}{2}}.2^{\frac{m+j}{2}} = p_2.2^{\frac{m+n+1}{2}}$ which is again a Zumkeller number and is constant for all $1 \le j \le n$.
3) $f^*(u_{j+1}u_{j+2}) = f(u_{j+1})f(u_{j+2}) = 2^{\frac{m+j}{2}}.p_2.2^{\frac{n-(j+2)+1}{2}} = p_2.2^{\frac{m+n-1}{2}}$ which is again a Zumkeller number and is constant for all $1 \le j \le n$.

Just like in the previous case, $T_{m,n}$ receives 7 distinct Zumkeller numbers.

Hence $T_{m,n}$ is a 7-Zumkeller graph for $m \equiv 1(mod\ 2)$.

Example 4.1: A Zumkeller labeling of $T_{6,3}$ is given in Figure 1 and that of $T_{9,4}$ is given in Figure 2.

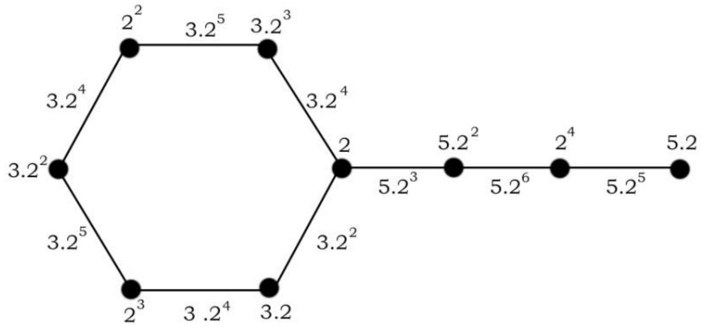

Fig. 1. Zumkeller labeling of $T_{6,3}$

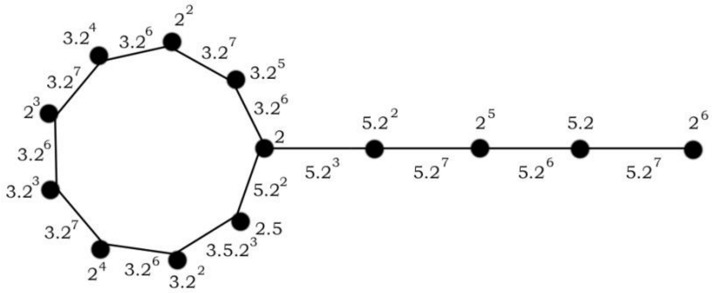

Fig. 2. Zumkeller labeling of $T_{9,4}$

Theorem 4.3: The banana tree $B_{m,n}$ is a $(m+n-1)-$ Zumkeller labeling, where $m, n \in \mathbb{N}$.

Proof: Let $V = x_0 \cup \{u_i : 1 \le i \le m\} \cup \{v_i : 1 \le i \le m\} \cup \{w_{i,j} : 1 \le i \le m, 1 \le j \le n-2\}$ and $E = \{x_0 u_i : 1 \le i \le m\} \cup \{u_i v_i : 1 \le i \le m\} \cup \{v_i w_{i,j} : 1 \le i \le m, 1 \le j \le n-2\}$ be the vertex set and edge set respectively of the graph $B_{m,n}$.

We define a one-one function $f : V \to \mathbb{N}$ as follows:

$$f(x_0) = 1$$

$$f(u_i) = p.2^{1+(i-1)(n-2)}, \qquad f(v_i) = 2^{1+(n-2)(m-i)},$$

$$f(w_{1,j}) = p.2^{m+(n-2)+2-j}, \quad j = 1, 2, \dots, n-2$$

$$(w_{2,j}) = p.2^{m+2(n-2)+2-j}, \quad j = 1, 2, \dots, n-2$$

$$(w_{3,j}) = p.2^{m+3(n-2)+2-j}, \quad j = 1, 2, \dots, n-2$$

In general, $(w_{i,j}) = p.2^{m+i(n-2)+2-j}, \quad j = 1, 2, \dots, n-2$

where $1 \le i \le m$ and $p \ne 2$ is prime such that $p < 10$.

We define another function $f^* : E \to \mathbb{N}$ as $f^*(xy) = f(x)f(y)$, for some $xy \in E$.
Now,

1) $f^*(x_o u_i) = f(x_o)f(u_i) = 1.p.2^{1+(i-1)(n-2)} = p.2^{1+(i-1)(n-2)}$, which is a Zumkeller number.
2) $f^*(u_i v_i) = f(u_i)f(v_i) = p.2^{1+(i-1)(n-2)}.2^{1+(n-2)(m-i)} = p.2^{n(m-1)+2(2-m)}$, which is again a Zumkeller number and is constant.
3) $f^*(v_i w_{i,j}) = f(v_i)f(w_{i,j}) = 2^{1+(n-2)(m-i)}.p.2^{m+i(n-2)+2-j} = p.2^{m(n-1)+3-j}$, which is again a Zumkeller number for $1 \le j \le n-2$.
4) $f^*(v_{i+1} w_{i+1,j}) = f(v_{i+1})f(w_{i+1,j}) = 2^{1+(n-2)(m-i-1)}.p.2^{m+(i+1)(n-2)+2-j} = p.2^{m(n-1)+3-j}$, which is a Zumkeller number for $1 \le j \le n-2$.

From (1)-(4), it can be seen that $B_{m,n}$ receives $m+1+(n-2)$, i.e. $m+n-1$ distinct Zumkeller numbers.

Therefore the banana tree $B_{m,n}$ is a $(m+n-1)$ −Zumkeller number.

Example 3.2: Figure 3 shows the Zumkeller labeling of the graph $B_{3,7}$. Here, $m = 3, n = 7$ and we take $p = 3$. The graph admits 9 Zumkeller numbers.

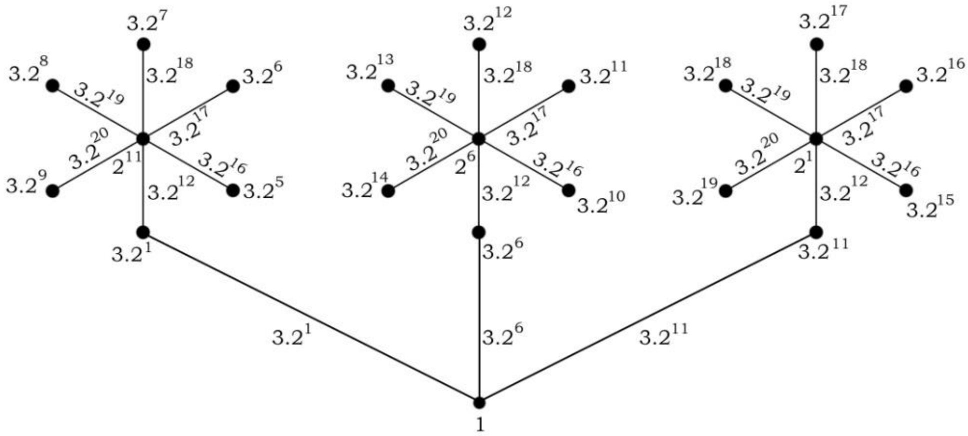

Fig. 2. Zumkeller labeling of $B_{3,7}$

Theorem 4.4: The friendship graph F_n admits a $(2n + 1)$-Zumkeller labeling for all $n \ge 1$, where $n \in \mathbb{N}$.

Proof: Let $V = v_0 \cup \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\}$ and $E = \{v_0 u_i : 1 \le i \le n\} \cup \{v_0 v_i : 1 \le i \le n\} \cup \{u_i v_i : 1 \le i \le n\}$ be the vertex set and edge set respectively of F_n.

The function $f : V \to \mathbb{N}$ is defined as

$$f(v_0) = p,$$

$$f(u_i) = 2^i, \quad f(v_i) = p.2^{n-i+1},$$

where $p \neq 2$ is prime and $1 \le i \le n$.

The function $f^*: E \rightarrow \mathbb{N}$ is defined as

$f^*(xy) = f(x)f(y)$, for any $xy \in E$.

1) $f^*(v_0u_i) = f(v_0)f(u_i) = p.2^i$, which is a Zumkeller number for all $1 \leq i \leq n$.
2) $f^*(v_0v_i) = f(v_0)f(v_i) = p.p.2^{n-i+1} = p^2.2^{n-i+1}$, which is a Zumkeller number for all $1 \leq i \leq n$.
3) $f^*(u_iv_i) = f(u_i)f(v_i) = 2^i.p.2^{n-i+1} = p.2^{n+1}$, which is a Zumkeller number and is constant for all $1 \leq i \leq n$.

From (1), (2), and (3), it can be seen that the graph F_n receives $n + n + 1$, ie. $2n + 1$ Zumkeller numbers.

Therefore the friendship graph F_n admits a (2n + 1) -Zumkeller labeling for all $n \geq 1$.
Example 4.3: Figure 4 shows the 9-Zumkeller labeling of the Friendship Graph F_4. Here we choose $p = 3$.

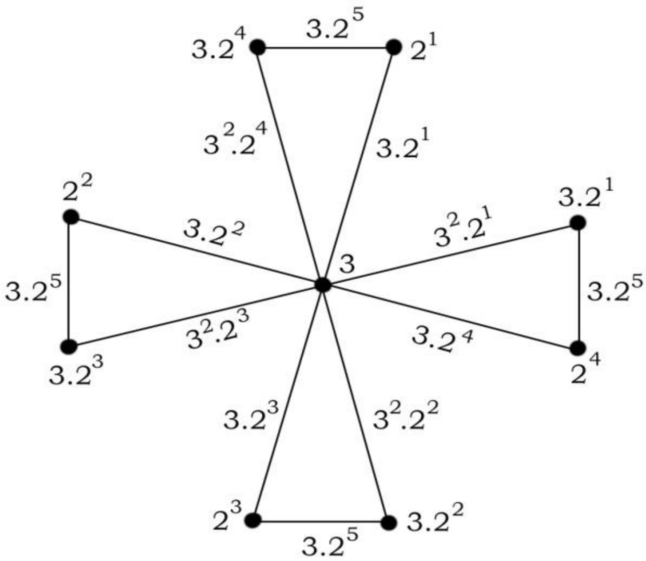

Fig. 4. 9-Zumkeller labeling of the Friendship Graph F_4

Theorem 4.5: The Firecracker graph $F_{m,n}$ is a $(n + 1) -$ Zumkeller graph for any $m, n > 2$, where $m, n \in \mathbb{N}$.

Proof: Let $V = \{u_i, v_i : 1 \leq i \leq m\} \cup \{w_{i,j} : 1 \leq i \leq m, \ 1 \leq j \leq n - 2\}$ and $E = \{u_iu_{i+1} : 1 \leq i \leq m - 1\} \cup \{u_iv_i : 1 \leq i \leq m\} \cup \{v_iw_j : 1 \leq i \leq m, \ 1 \leq j \leq n - 2\}$ be the vertex set and edge set respectively of $F_{m,n}$.

Here we consider two cases:

Case 1: When $m \geq n$

We define the injective function $f: V \to \mathbb{N}$ as follows:

$$f(u_i) = 2^i \ , \qquad f(u_{i+1}) = p.2^{m-i},$$
$$f(v_i) = p.2^{m-i+1}, \quad f(v_{i+1}) = 2^{i+1},$$
$$f(w_{i,j}) = 2^{m+i-1+j}, \quad j = 1, 2, \dots, n-2$$
$$f(w_{i+1,j}) = p.2^{2m-i-1+j}, \quad j = 1, 2, \dots, n-2$$

for $i = 1, 3, 5, \dots, m-1$ where $p \neq 2$ is prime such that $p < 10$.

We define another function $f^*: E \to \mathbb{N}$ as

$$f^*(xy) = f(x)f(y), \quad \text{for some } xy \in E$$

We now proceed to prove our claim that the edges of these graphs can be labeled with Zumkeller numbers.

1) $f^*(u_i u_{i+1}) = f(u_i)f(u_{i+1}) = 2^i.p.2^{m-i} = p.2^m$, which is a Zumkeller number and is constant.
2) $f^*(u_{i+1}u_{i+2}) = f(u_{i+1})f(u_{i+2}) = p.2^{m-i}.2^{i+2} = p.2^{m+2}$, which is a Zumkeller number and is constant.
3) $f^*(u_i v_i) = f(u_i)f(v_i) = 2^i.p.2^{m-i+1} = p.2^{m+1}$, which is a Zumkeller number and is constant.
4) $f^*(u_{i+1}v_{i+1}) = f(u_{i+1})f(v_{i+1}) = p.2^{m-i}.2^{i+1} = p.2^{m+1}$, which is a Zumkeller number and is constant.
5) $f^*(v_i w_{i,j}) = f(v_i)f(w_{i,j}) = p.2^{m-i+1}.2^{m+i-1+j} = p.2^{2m+j}$, which is a Zumkeller number for $1 \le j \le n-2$.
6) $f^*(v_{i+1}w_{i+1,j}) = f(v_{i+1})f(w_{i+1,j}) = 2^{i+1}.p.2^{2m-i-1+j} = p.2^{2m+j}$, which is a Zumkeller number for $1 \le j \le n-2$.

From (1)-(6), it can be seen that $F_{m,n}$ receives $2 + 1 + (n-2)$, i.e. $n+1$ Zumkeller numbers.

Therefore the firecracker graph $F_{m,n}$ is a Zumkeller graph for $m \ge n$.

Case 2: When $m < n$.

In this case, we define the function $f: V \to \mathbb{N}$ as:

$$f(u_i) = 2^i, \quad f(u_{i+1}) = p.2^{m-i},$$

$$f(v_i) = p.2^{m-i+1}, \ f(v_{i+1}) = 2^{i+1},$$

$$f(w_{i,j}) = 2^{j(m+1)+(i-1)}, \quad j = 1, 2, \dots, n-2$$

$$f(w_{i+1,j}) = p.2^{j(m+1)+(m-i-1)}, \quad j = 1, 2, \dots, n-2$$

for $i = 1, 3, 5, \dots, m-1$ where $p \neq 2$ is prime such that $p < 10$

while the function $f^*: E \to \mathbb{N}$ on the edges remain unchanged.

1) $f^*(u_i u_{i+1}) = 2^i.p.2^{m-i} = p.2^m$, which is a Zumkeller number and is constant.
2) $f^*(u_{i+1}u_{i+2}) = f(u_{i+1})f(u_{i+2}) = p.2^{m-i}.2^{i+2} = p.2^{m+2}$, which is a Zumkeller number and is constant.
3) $f^*(u_i v_i) = f(u_i)f(v_i) = 2^i.p.2^{m-i+1} = p.2^{m+1}$, which is a Zumkeller number and is constant.
4) $f^*(u_{i+1}v_{i+1}) = f(u_{i+1})f(v_{i+1}) = p.2^{m-i}.2^{i+1} = p.2^{m+1}$, which is a Zumkeller number and is constant.
5) $f^*(v_i w_{i,j}) = f(v_i)f(w_{i,j}) = p.2^{m-i+1}.2^{j(m+1)+(i-1)} = p.2^{j(m+1)+m}$, which is a Zumkeller number for $1 \le j \le n-2$.
6) $f^*(v_{i+1}w_{i+1,j}) = f(v_{i+1})f(w_{i+1,j}) = 2^{i+1}.p.2^{j(m+1)+(m-i-1)} = p.2^{j(m+1)+m}$, which is a Zumkeller number for $1 \le j \le n-2$.

From (1)-(6), it can again be seen that $F_{m,n}$ receives $2+1+(n-2)$, i.e. $n+1$ Zumkeller numbers.

So the firecracker graph is a $(n+1)$ −Zumkeller graph for $m < n$.

Example 4.4: Figure 5 shows the 5-Zumkeller labeling of the graph $F_{4,4}$. Here we choose $p = 3$.

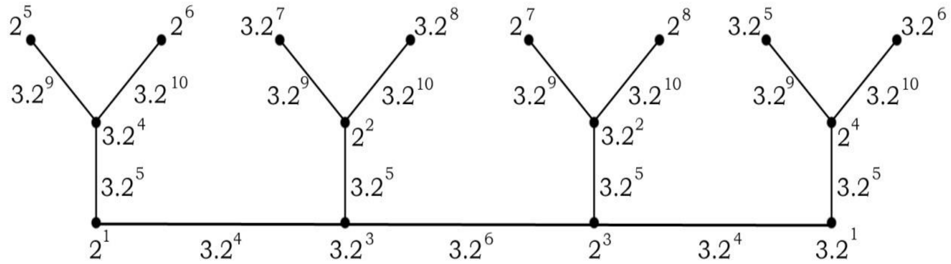

Fig. 52. Zumkeller labeling of $F_{4,4}$

Example 4.5: Figure 6 shows the 5-Zumkeller labeling of the graph $F_{3,4}$. Here also, we choose $p = 3$.

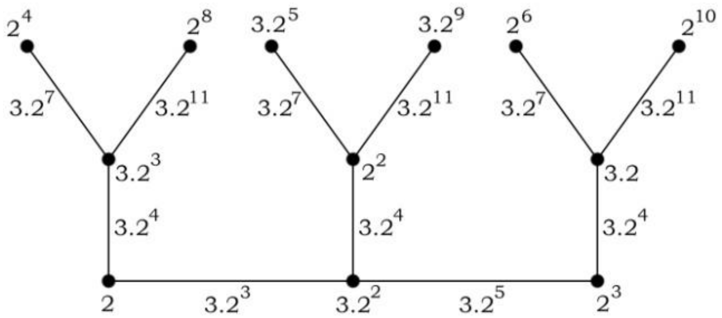

Fig. 6. Zumkeller labeling of $F_{3,4}$

VI. Conclusion

In this article, we have established the existence of k-Zumkeller labeling of different classes of graphs including tadpole graphs, banana graphs, friendship graphs, and firecracker graphs. While the value of 'k' for the tadpole graph $T_{m,n}$ is either 6 or 7 depending on the parity of m and independent of n, the values of 'k' for firecracker graphs and banana graphs depend on both m and n. We also found that the value of 'k' for a friendship graph F_n is always odd regardless of the parity of n.

While this article seeks to contribute to the existing literature on graph labeling, there are still avenues for further research. The exploration of other classes of graphs and the investigation of more complex k-Zumkeller properties remain open challenges.

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Conflict of Interest

The author declares that there was no conflict of interest regarding this paper.

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